Problem

- Given a set of points \( \{p_i\} \ i = 1..n \)
- Define an implicit function \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \), such that:
  \[ f(p_i) = 0, \ i = 1..n \]
- Surface is defined by the zero set of \( f \)

Zero-Everywhere Solution

- \( f(x) = 0, \ \forall x \in \mathbb{R}^3 \) satisfies our initial constraint
- Add off-surface normal points to constrain the solution further

Off-Surface Normal Points

- For each \( p_i \), add two more points defining the inside and outside

Off-Surface Normal Points

- Create Points:
  \[ \text{Op}_i = p_i + \epsilon n_i \]
  \[ \text{Ip}_i = p_i - \epsilon n_i \]
- New constraints:
  \[ f(p_i) = 0, \]
  \[ f(\text{Ip}_i) = -\delta, \]
  \[ f(\text{Op}_i) = +\delta, \ \ i = 1..n \]
Choice of Parameters

- $\delta$ is chosen as the distance of the new point from $p_i$: signed distance function
- $\varepsilon$ must be chosen such that the displacement to the new points doesn't intersect other parts of the surface:

Generating Normals

- Input is a point cloud: no normals
- Generate normals as per [Hoppe92]
  - Estimation from plane fitted to neighbourhood
- Additionally, use knowledge of scanner position to infer ambiguous cases
- If that fails, don’t define off-surface normal points

Interpolation Problem

- We now have an interpolation problem:
- Given a set of unstructured points with values, generate a function that interpolates all input points

BL$^{(2)}$(R$^3$)

- The interpolant is chosen from the “Beppo-Levi space of distributions on R$^3$ with square integral second derivatives”
- Square integral means $\int |f(x)|^2 \, dx < \infty$
- So, the space of functions $\varphi$ in R$^3$ with second derivatives that fall off quickly

Radial Basis Function

- All functions in $\varphi$ have a (rotationally invariant) semi-norm.
  - The semi-norm defines the smoothness of the function
- Paper shows that the functions with small semi-norms are radial basis functions:
  $s(x) = p(x) + \sum_{i} \lambda_i \varphi(x - x_i)$
- The radial basis functions are the most smooth, compactly supported functions in R$^3$
Radial Basis Functions

\[ s(x) = p(x) + \sum_{i=1}^{N} \lambda_i \varphi(x-x_i) \]

- \( p(x) \) is a low degree polynomial
- \( \lambda_i \) are the coefficients
- \( \varphi(x-x_i) \) is a function of the Euclidean distance between two points
  - \( \varphi \) is chosen to fall off with distance. May have compact support

Radial Basis Functions

- RBFs are good for interpolating scattered data
  - A RBF’s linear equations are always invertible at the location of data points (with some conditions)
  - Use RBFs to solve this data interpolation problem

Solving RBF

- Solve a system of linear equations:

\[
\begin{bmatrix}
A & P \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\lambda \\
\phi
\end{bmatrix} =
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

- \( A_{ij} = \varphi(|x_i-x_j|) \)
- \( P_{ij} = p_j(x_i) \)

- Note that \( B \) is symmetric
- Need to solve for \( \lambda, c \)

Solving RBF

- Once \( \lambda \) and \( c \) are known, we can evaluate the signed distance function anywhere
- Can extract zero-set surface via ‘surface following’ – use an input point as a seed

Sparsity and Support

- If \( \varphi \) is chosen with compact support, then \( B \) will be sparse (easier to solve)
  - Since \( \varphi \) will only include nearby points (more local)
- However, compact support limits performance with irregular sampling

Efficiency

- Solving with compact \( \varphi \) is OK (ish)
- Scales with number of input points
  - Practical for up to 1000’s of points
Fast Methods

- Use of approximation
- Treat clusters of far points as a single point
- Fitting accuracy controls how close the approximate RBF is to the actual RBF

Fast Methods

- Reduce number of RDH centers
  - The same RBF may be represented with fewer, well placed centers
- Greedy Algorithm
  - Choose a subset of centers
    - Evaluate residual at all nodes
    - If residual is < fitting accuracy Then Done
    - Else add new centers where residual is large
    - Rinse and Repeat

Noise

- Interpolating methods perform poorly with positional noise
- Add a parameter $\rho$ that controls interpolation vs. smoothness:
  $$\min_{\mathbf{s} \in \mathbb{R}^N} \rho \sum_{i=1}^{N} |s(x_i) - f_i|^2$$
- Paper shows solution $s^*$ changes $B$ to:
  $$\begin{bmatrix} A - N/2 \rho I & P \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$$

Hole Filling

- The input is a partial mesh
- Normals are extracted from mesh
- Reconstruction will fill holes

Contributions

- Apply the fast method of solving
- Applicable to Surface Reconstruction and Hole Filling
- Smooth, Manifold surfaces
- Interpolation vs. Approximation is adjustable
- Kiwis can write SIGGRAPH papers

Figures - Iterations

Figure from Carr
Figures – 594,000 Centers