The Powercrust Algorithm for Surface Reconstruction

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Correctness
• Boundary of a solid
• Close to original surface
• Homeomorphic to original surface

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Tools - Voronoi Diagram
Points closest to each sample form cells.
Cell boundaries have more than one closest sample.
Adjacent cells define adjacent samples.

Delaunay Triangulation
Delaunay triangles connect adjacent samples.
**Delaunay Triangulation**
Delaunay triangles connect adjacent samples.
Voronoi balls centered at Voronoi vertices pass through closest samples.

**3D Voronoi/Delaunay**
Voronoi cells are convex polyhedra.
Voronoi balls pass through 4 samples.
Delaunay tetrahedra.

**Voronoi-based Surface Reconstruction**
Boissonnat, 84
Edelsbrunner and Mucke, 94
Bernardini et al, 98
Amenta and Bern, 98

**Medial Axis**
Think of object surface as infinite set of samples.
Medial axis is set of points with more than one closest sample.

**Medial Axis**
Maximal ball avoiding surface is a medial ball.
Every solid is a union of balls!

**3D Medial Axis**
Medial axis of a surface forms a dual surface.
2D Medial Axis Approximation

Voronoi balls approximate medial balls.

Poles

Problem in 3D:
Not all Voronoi vertices are near medial axis, even when samples are arbitrarily dense.

Poles

For dense surface samples, Voronoi cells are:
• long and skinny,
• perpendicular to surface,
• with ends near the medial axis.

Poles

Subset of Voronoi vertices, the poles, approximate medial axis.

Amenta & Bern, 98

Poles

Poles are Voronoi vertices at opposite ends.

To find: farthest Voronoi vertex from sample, farthest on opposite side.

Sliver tetrahedra

In 3D, some Voronoi vertices are not near medial axis...
**Crust Algorithm**

Surface reconstruction with theoretical guarantees.
Uses poles to find Delaunay triangles eligible for surface.

Amenta, Bern and Kamvysselis, 98

**Improvements on Crust**

Amenta et al, 00: Simpler algorithm, simpler proof, topological guarantees.
Dey and Giesen, 01: Sharp corners and boundaries.
Ramos, 01: \( O(n \lg n) \) algorithm, replacing Delaunay with well-separated pair decomposition.

**Practical Crust Drawbacks**

- Fails when sample is not sufficiently dense: holes in surface, errors at sharp corners.
- Need to select surface from set of eligible triangles. Hard to do in a way that is provably correct, makes nice surface, etc.
Project: Algorithm which is robust, has no post-processing, and is still correct.

**Power Crust**

Idea: Approximate object as union of balls, compute polygonal surface from balls.

**Power Crust**

Compute Voronoi diagram of samples. Select poles to approximate object and its complement by finite unions of balls.

Compute polygonal surface from polar balls using power diagram.
Power Diagram

Power diagram is Voronoi diagram of balls.

Voronoi diagram program can be easily modified to produce power diagrams.

Has polyhedral cells.

Power Diagram

Ball B, center c, radius r
Power Distance from B to point x:

\[ d_{\text{pow}} = d^2(c, x) - r^2 \]

Power Crust

Label power diagram cells inside or outside object (skipping details).

Inside cells form polyhedral solid.

Power Crust

Boundary of solid approximates surface: power crust.

Connect inner poles with adjacent power diagram cells: power shape approximates medial axis.

Power Crust

Robust: Always boundary of a solid.
Simple: No surface extraction or hole-filling steps required.
Correct: Theoretical results relate geometric and topological quality of approximation to quality of sample.

Sampling Requirement

Sample is sufficiently dense when distance from any surface point \( x \) to nearest sample is at most small constant \( r \) times distance to medial axis.
**Sampling Requirement**

Captures intuition that we need dense sampling where curvature is high or where there are nearby features.

**Large balls tangent**

Any large ball (with respect to distance to medial axis) touching sample $s$ has to be nearly tangent to the surface at $s$.

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**Specifically**

Given an $\varepsilon$-sample from a surface $F$:

- Angle between normal to $F$ at sample $s$ and vector from $s$ to either pole = $O(\varepsilon)$

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**Theoretical Results**

Assume sufficiently dense sampling, smooth surface.

- Power crust approaches object surface linearly as sampling density increases.
- Power crust normals converge to surface normals linearly.
- Power crust is homeomorphic to surface for dense enough samples.

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**Theoretical Results**

- Similar results for union of balls.
- Power shape is homotopy equivalent to solid object.
- Set of poles converges to medial axis, faster in some places than in others.
  - Also Boissonnat and Cazals, 01; and Dey, 02 gives polygonal MA approximation.

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**Results**

Laser range data, power crust, approximate medial axis.
Results
Four laser range scans merged.
Hole deep inside object filled.

Robust
Good reconstruction even with lots of added noise.

Medial Axis Simplification
Simplification of original power shape.
Simplification of power shape of noisy hand.

Sharp Corners
With additional hack, sharp edges can be resolved when they are far enough away from other features.
No need to have samples on the edge!

Approximate Offset Surface
Shrink inner balls.
Grow outer balls.
Compute power crust - always solid output!
Only accurate for small offsets.
**Medial Axis Simplification**

Samples determining noise balls are closer together than noise threshold. Remove noise balls before computing surface.

**Software**

Software, papers, models....

[www.cs.utexas.edu/users/amenta/powercrust](http://www.cs.utexas.edu/users/amenta/powercrust)

**Incremental Constructions con BRIO**

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Sunghee Choi (UT-Austin)  
Günter Rote (Freie Univ. Berlin)

**Randomized Incremental Delaunay Algorithm**

Add points one by one in random order, update triangulation. Simple and optimal.

**Drawback**

Perform great...until!

**Idea**

- Partially randomized insertion order
- Increase locality of reference, especially as data structure gets large
- Retain enough randomness to guarantee optimality
Result

- We give a new ordering called BRIO (biased randomized insertion order) that is still optimal.
- Size of input we could compute increased 500K → 10M.

Biased Randomized Insertion Order (BRIO)

- Choose each point with prob = 1/2.
- Insert chosen points recursively con BRIO.
- Insert the remaining points in arbitrary order.

BRIO

\[ \log n \text{ rounds of insertion} \]

- round \( \log n \)
- round \( \log n - 1 \)
- round \( \log n - 2 \)
- \( \vdots \)
- round 0

Implementation

- Divide points into local cells (oct-tree)
- In each round, visit cells in fixed order and add points in cell together

Analysis

Randomness has two benefits:

- Bound total number of tetrahedra
- Bound time required for locating new points in triangulation
Analysis

Two cases:

Worst-case - size of Delaunay triangulation is $O(n^2)$

Realistic-case - size of Delaunay triangulation is $O(n)$. Assume for any random subset $R$, $DT(R) = O(|R|)$

“Realistic” case

Linear size, linear expected size of intermediate triangulations.

Results

In "realistic case":

Expected total number of tetrahedra = $O(n)$

Expected running time = $O(n \log n)$

Results

In worst case:

Expected total number of tetrahedra = $O(n^2)$

Expected running time = $O(n^2)$

Hull

CGAL Hierarchy

128M RAM, 360 MHz, 4 G virtual memory
Pyramid

More space-efficient but $O\left(n^{1/4}\right)$ point location.

Use smaller memory, slower machine and much larger data. Multiple “Happy buddha”. 4096 kd-cells.

360 MHz
128 M RAM
4 GB Virtual memory

Point Location Hack

• Instead of $O\left(n^{1/4}\right)$ jump-and-walk, just walk from last inserted point.
• As size grows, locality increases, so point location time remains roughly constant.

Pyramid

128 M RAM, 360 MHz, 4 G virtual memory

4 million points in 2 hours
10 million points in 5.5 hours

Pyramid

512 M RAM, 10 million points in 30 min