Three-Dimensional $\alpha$ Shapes
Herbert Edelsbrunner and Ernst P. M"ucck
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Presented by Matthew Bolitho

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1 Theory
  - Background
  - Intuition
  - Definition
  - Delaunay Triangulation

2 Implementation
  - $\alpha$-Complexes
  - Edelsbrunner’s Algorithm

3 Applications
  - Properties
  - Surface Reconstruction
Question

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- The $\alpha$-shape
\( \alpha \)-Shapes

- A generalisation of the convex hull
\(\alpha\)-Shapes

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A generalisation of the convex hull

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- As $\alpha \rightarrow \infty$, the $\alpha$-shape is the convex hull
- As $\alpha \rightarrow 0$, the $\alpha$-shape is the point set $S$
**α-Shapes**

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\( \alpha \)-shape may be concave or disjoint
α-Shapes

Figure from Edelsbrunner94

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Three-Dimensional α Shapes
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Presented by Matthew Bolitho Three-Dimensional $\alpha$ Shapes
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- There are places that the scoop is blocked
  - i.e. $|p_i - p_j| < \alpha, i \neq j$
  - At these positions there are at least $d$ points in $P$ touching the scoop.
- An edge of the $\alpha$-shape is defined by connecting those points.
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- Then boundary of the $\alpha$-shape is a collection of these edges
Let $P \subset \mathbb{R}^d$ be a set of $n$ points
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Let $S_\alpha(P)$ be the $\alpha$-shape of $P$ for a given $\alpha$ value.
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*This allows us to ignore special cases*
Let $B_\alpha(p)$ be an open half ball with radius $\alpha$ covering a space $p$ for $0 \leq \alpha \leq \infty$, $p \subset \mathbb{R}^d$. 

α-balls

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Three-Dimensional α Shapes
\( \alpha \)-balls

Let \( B_\alpha(p) \) be an open half ball with radius \( \alpha \) covering a space \( p \) for \( 0 \leq \alpha \leq \infty \), \( p \subset \mathbb{R}^d \)

- \( B_0(p) \) is the point \( p \)
- \( B_\infty(p) \) is the half-space \( p \)
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\(B_\alpha(p)\) is empty if \(p \cap P = 0\)
An \( n \)-simplex is an \( n \)-dimensional analogue of a triangle:
- The 0-simplex is a point
- The 1-simplex is a line
- The 2-simplex is a triangle
- The 3-simplex is a tetrahedron

A \( n \)-simplex has \( n + 1 \) vertices
Let \( T \subset P \), and \( |T| = k + 1 \leq d + 1 \)

The polytope \( \triangle_T = \text{conv}(T) \) has dimension \( k \) and is therefore a \( k \)-simplex
Exposed Simplices

Let $\delta p$ be the surface of $B_\alpha(p)$

A $k$-simplex $\triangle_T$ is said to be exposed if there is an empty $B_\alpha(p)$ where $T \in \delta p$
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A $k$-simplex $\triangle_T$ is said to be exposed if there is an empty $B_\alpha(p)$ where $T \in \delta p$
Building the $\alpha$-shape

- $S_\alpha(P)$ is constructed from all exposed simplices:

$$\delta S_\alpha(P) = \{ \triangle_T | T \subset P, |T| \leq d \text{ and } \triangle_T \text{ is exposed} \}$$
Observations

It is easy to show:

\( \lim_{\alpha \to \infty} S_\alpha(P) \) is the convex hull of \( P \)

\( \lim_{\alpha \to 0} S_\alpha(P) \) is the original set \( P \)
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Claim

\( S(\alpha) \) is a subset of the Delaunay triangulation of \( P \)
Let $DT(P)$ be a set of $k$-simplices $0 \leq k \leq d$ such that
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- For $k = d$, an $\alpha$-ball $B_\alpha(p)$ coincides with the circumsphere of $\triangle_T$
- By definition, this does not contain any other points from $P$, therefore $B_\alpha(p)$ is empty
- Thus the simplices that form the edges of the $d$-simplex are exposed, and form the boundary for some $\alpha$-shape.
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Three-Dimensional $\alpha$ Shapes
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Three-Dimensional $\alpha$ Shapes
Definitions

Let $\sigma_T$ be the radius of the circumsphere of a simplex $\triangle T$
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Let $\mu_T$ be the center of the circumsphere of a simplex $\triangle T$
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Let $\mu_T$ be the center of the circumsphere of a simplex $\triangle_T$

- A simplex $\triangle_T$ from $DT(P)$ is in $C_\alpha(P)$ if either:
  - $\sigma_T < \alpha$ and the $\alpha$-ball at $\mu_T$ is empty
  - $\triangle_T$ is the face of another $\triangle_T$ in $C_\alpha(P)$
Description

- Compute the Delaunay Triangulation
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- Extract the boundary of the $\alpha$-shape from the $\alpha$-complex
- Steps 1 and 2 can be precomputed for a given $P$
Complexity

- Delaunay Triangulation: $O(n \log n)$
- Generate $\alpha$-complex: $O(m \log m)$
- Extract boundary of $\alpha$-shape: $O(m)$ (could be better?)
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In reality, point sets often break these constraints.
General Position Assumption

- We assumed that $P$ was in general position form
- In reality, point sets often break these contraints
- Solution: *Simulation of simplicity*  H. Edelsbrunner and E. P. Mcke, ACM Trans. Graph. 9(1) 1990
Properties of $\alpha$-Shapes

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- $S_{\alpha_1}(P) \subseteq S_{\alpha_2}(P)$ if $\alpha_1 < \alpha_2$
- $S_\alpha(P) \subseteq DT(P)$
Advantages

- $\alpha$-shape reconstructions can have arbitrary topology
- The $\alpha$-shape interpolates the set $P$
Disadvantages

- The choice of $\alpha$-value is non-intuitive
- The reconstruction may not be water tight
- The reconstruction may be disjoint
Given a point set $P$, one can only choose a single $\alpha$ value.
Sampling Density

- Given a point set \( P \), one can only choose a single \( \alpha \) value.
  - The \( \alpha \)-value is determined by the smallest feature in the reconstruction.
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- The $\alpha$-value is determined by the smallest feature in the reconstruction.
- Thus, the $\alpha$-value determines the sampling density everywhere.
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- The $\alpha$-value is determined by the smallest feature in the reconstruction
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$P$ must be uniformly sampled, at a resolution constrained by the locally highest resolution desired.

- What about blending several $\alpha$-shapes together?
- What about defining $\alpha(q)$ for $q \in \mathbb{R}^d$ such that $\alpha(q)$ is proportional to sampling density near $q$