Computational Geometry

Voronoi Diagrams and Delaunay Triangulations

Outline

- Motivation
- Definitions
- Algorithm

Data Retrieval

For many database retrieval systems, the objects in the database are represented by fixed dimensional vectors. Given a query, its vector representation is computed.
Data Retrieval (Brute Force)

We could compare the query vector against every vector in the database and return the closest.

This would have complexity:
\[ O(nd) \]
\[ n := \#\text{objects in d.b.} \]
\[ d := \text{dim. of vectors} \]

Data Retrieval (Goal)

In a pre-processing step, partition space, so that when a query vector is presented, it is easy to determine who its nearest neighbor is.

Data Interpolation

Given a discrete sampling of a scalar function

For each point in space, we can find the closest sample point and use its value.

The reconstruction is piecewise constant
1. Triangulate the samples.
2. For each point, interpolate the values at the vertices of the containing triangle.

To do this we need to be able to generate a "good" triangulation of the sample points.

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Given points \( P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^n \):

The Voronoi Problem

Claim: \( p_i \in R_i \),

Proof: \( p_i \) is closer to itself than to any other point.

Claim: The regions \( R_i \) are convex.

Proof: Define \( P_{ij} \) to be the half space passing through the mid-point of \( \overline{p_ip_j} \) and perpendicular to the line segment \( \overline{p_ip_j} \).
The Voronoi Problem

Claim: The regions $R_i$ are convex.
Proof: Define $P_{ij}$ to be the half space passing through the mid-point of $\overline{p-ip_j}$ and perpendicular to the line segment $\overline{p_ip_j}$.

\[ P_{ij} = \{ p \in E^d \mid p - p_i \parallel p - p_j \} \]

\[ R_i = \bigcap_{j \neq i} P_{ij} \]

The Delaunay Problem

Given points $P = \{p_1, \ldots, p_n\} \subset E^n$:

Questions:
- Does a Delaunay triangulation exist?
- Is the triangulation unique?

Dual Graphs

The dual of a planar graph is a graph:
- With a node for every face in the graph,
- And an arc between two nodes whose corresponding faces shared an edge.
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– Is the triangulation unique?

Answer:
– Yes, the Voronoi diagram and the Delaunay triangulation are duals.

Duality (Voronoi to Delaunay)

Given a set of points, and given its Voronoi diagram,

Duality (Voronoi to Delaunay)

Given a set of points, and given its Voronoi diagram, create an edge between any two vertices whose Voronoi cells are adjacent.

Duality (Voronoi to Delaunay)

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Every Delaunay triangle will be associated to one Voronoi branch point.

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Every Delaunay triangle will be associated to one Voronoi branch point. This branch point is equidistant to the vertices of the triangle and is closer to these points than to others.
**Duality (Voronoi to Delaunay)**

**Circum-Circle Property:**
Every Delaunay triangle will be associated to one Voronoi branch point.
This branch point is equidistant to the vertices of the triangle and is closer to these points than to others.
There is a circle centered at the branch point, circumscribing the triangle, and not containing other points.

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**Duality (Delaunay to Voronoi)**

Given a set of points,

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Given a set of points, and given its Delaunay triangulation, compute the circum-centers of the triangles, and connect those of adjacent triangles.

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**Duality (Delaunay to Voronoi)**

Given a set of points, and given its Delaunay triangulation, compute the circum-centers of the triangle, and connect those of adjacent triangles.

For triangles on the convex hull add edges through the center of the hull edges.
Duality (Delaunay to Voronoi)

Given a set of points, and given its Delaunay triangulation, compute the circum-centers of the triangle, and connect those of adjacent triangles.

Note:
The circum-center need not fall within the Delaunay triangle.

Voronoi Diagrams / Medial Axes

Recall:
The skeleton/medial-axis of a shape is the set of points that are equidistant from at least two point and are no closer to any other points.

These are closely related to the Voronoi diagram.

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Convex Hull Projection

Given points \( P = \{p_1, \ldots, p_n\} \subset \mathbb{E}^n \):
Convex Hull Projection

Given points \( P = \{p_1, \ldots, p_n\} \subset \mathbb{E}^n \):
1. Map the points into \( \mathbb{E}^{n+1} \) via:
   \((x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, x_{n+1})\)
2. Compute the convex hull in \( \mathbb{E}^{n+1} \).
3. (Downward-facing) triangles of the hull in \( \mathbb{E}^{n+1} \) are the Delaunay triangles in \( \mathbb{E}^n \).

Key Ideas:
1. The intersection of a paraboloid with a plane will always project to a circle.
2. Sliding the paraboloid along the y-axis will change the radius but not the center.