Complex Wavelets

Arnab Ghoshal

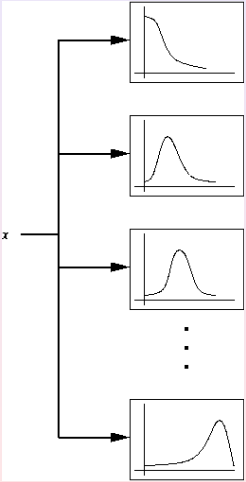
600.658 - Seminar on Shape Analysis and Retrieval

Outline

1. Discrete Wavelet Transform
   - Basics of DWT
   - Advantages and Limitations
2. Dual-Tree Complex Wavelet Transform
   - The Hilbert Transform Connection
   - Hilbert Transform Pairs of Wavelet Bases
3. Results
   - 1-D Signals
   - 2-D Signals

Outline

How do Electrical Engineers Dream of Wavelets?



Filter banks

- Use a set of filters to analyze the frequency content of a signal
- Lots of redundant information!
- Can we do better?

Decimated filter banks

- Subsample the frequency bands
- Same length as that of the original signal
- Theoretically possible to design filters $(h, g, \tilde{h}, \tilde{g})$ which will give perfect reconstruction

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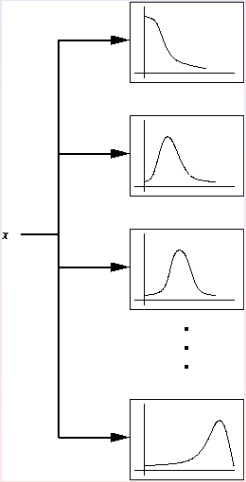

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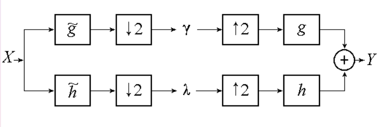

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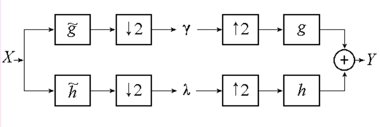

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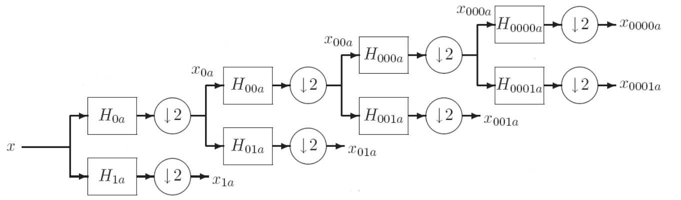
Fully Decimated Wavelet Tree



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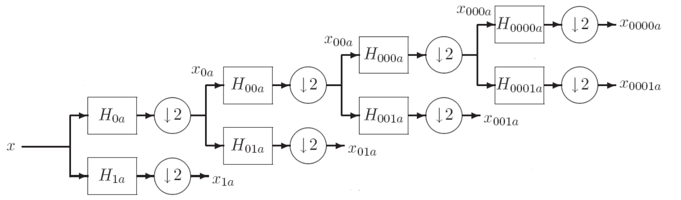

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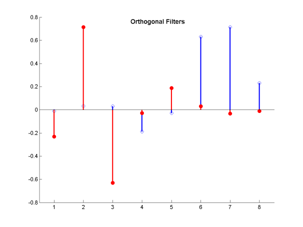

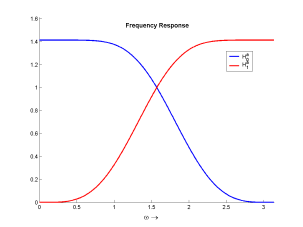

Wavelet Bases

Orthogonal filter-bank

- Even length filters (length $= M + 1$)
- Shift orthogonal: $\sum_n h_0^a(n) h_0^a(n+2k) = \delta(k)$
- Alternating flip: $h_1^a(n) = (-1)^n h_0^a(M-n)$

Scaling and wavelet functions

- *Scaling* function: $\phi^a(t) = \sqrt{2} \sum_n h_0^a(n) \phi^a(2t-n)$
- *Wavelet* function: $\psi^a(t) = \sqrt{2} \sum_n h_1^a(n) \phi^a(2t-n)$

Caveat

- Wavelets need not be orthogonal!
- In fact, JPEG2000 uses 5/3 and 9/7 bi-orthogonal filters

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Fundamental Properties of Wavelets

Advantages

- Non-redundant orthonormal bases, perfect reconstruction
- Multiresolution decomposition, attractive for object matching
- Fast $O(n)$ algorithms with short filters, compact support

Limitations

- Lack of adaptivity
- Poor frequency resolution
- Shift variance
  - Wavelet coefficients behave unpredictably when the signal is shifted
  - DWT lacks phase information

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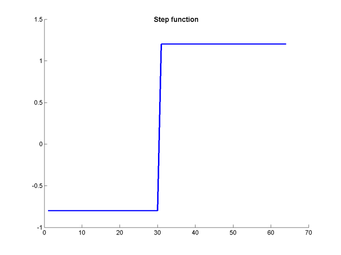
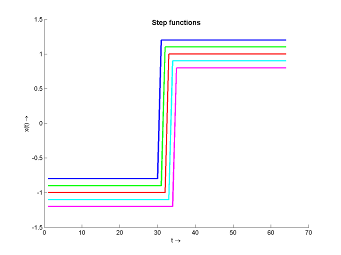
Shift Variance of DWT





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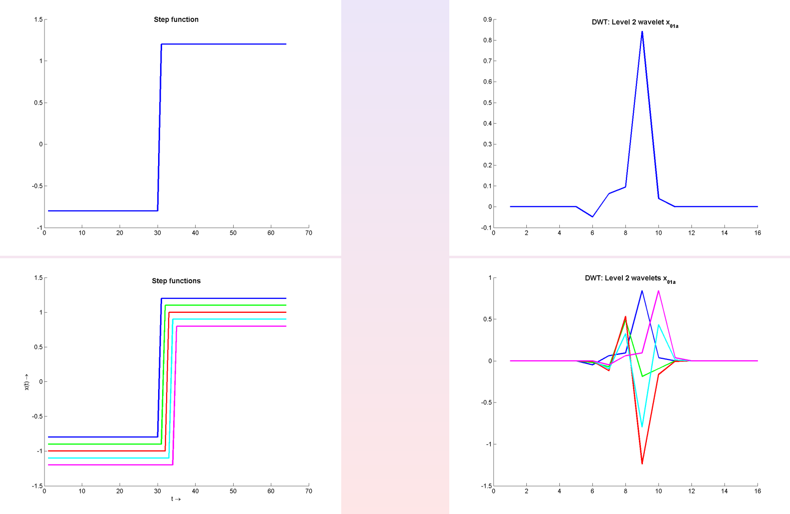

Outline

Frequency Ananlysis of Real Signals

- Every real signal has equal amounts of positive and negative frequency components
  - $\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$
  - $\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2}$
- **Hilbert transform:** $y(t) = \mathcal{H}\{x(t)\}$, iff
  - $Y(\omega) = -jX(\omega), \omega > 0$ and $Y(\omega) = jX(\omega), \omega < 0$
  - $y(\theta) = x(\theta - \frac{\pi}{2}), \theta > 0$ and $y(\theta) = x(\theta + \frac{\pi}{2}), \theta < 0$
- The signal $z(t) = x(t) + j\mathcal{H}\{x(t)\}$ has no negative frequencies (*analytic signal*)
- **Fourier Transform:** Real sin and cos form a *Hilbert transform pair*!

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Hilbert Transform Pairs of Wavelet Bases

- We want to find $\psi^b(t) = \mathcal{H}\{\psi^a(t)\}$
- Very hard to find $\psi^b(t)$ having finite support
- Design both the wavelets *simultaneously* [2]
  - Assume $H_0^b(\omega) = H_0^a(\omega)e^{-j\theta(\omega)}$
  - Hilbert transform condition on wavelets and scaling functions is satisfied by $H_0^b(\omega) = H_0^a(\omega)e^{-j\frac{\omega}{2}} \Rightarrow h_0^b(n) = h_0^a(n - \frac{1}{2})$
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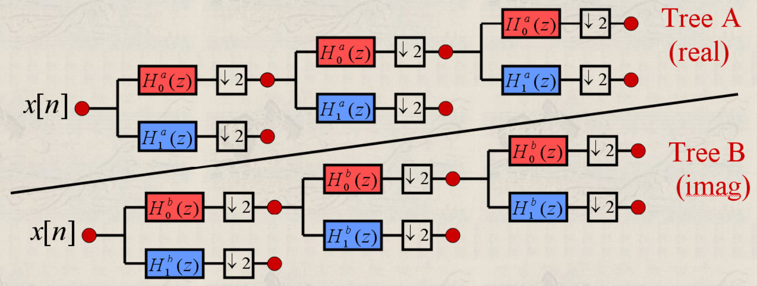
1-D Dual-Tree Wavelets



Picture courtesy: Dr. Trac D. Tran (*520.646 Lecture notes*)

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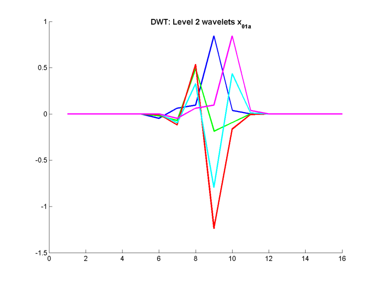
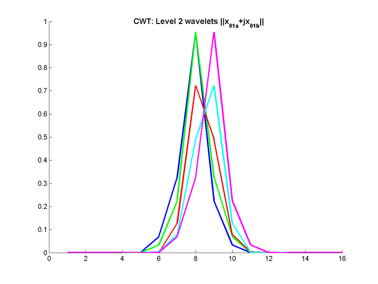
Level 2 Wavelets for Shifted Step Functions

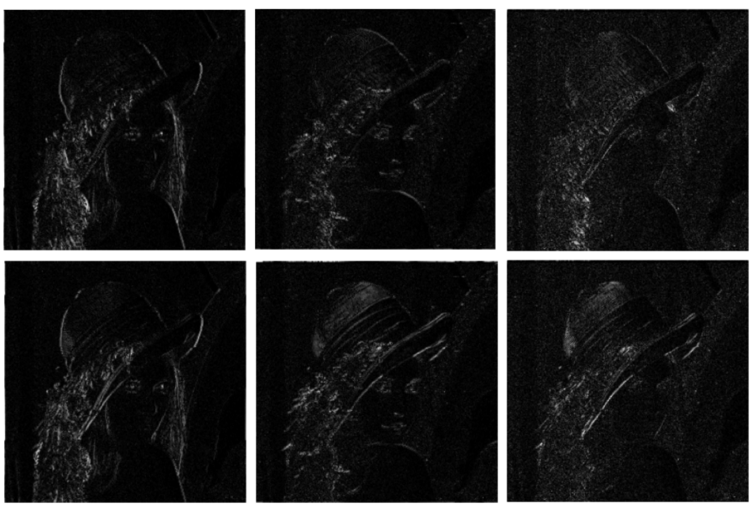




Original Signal

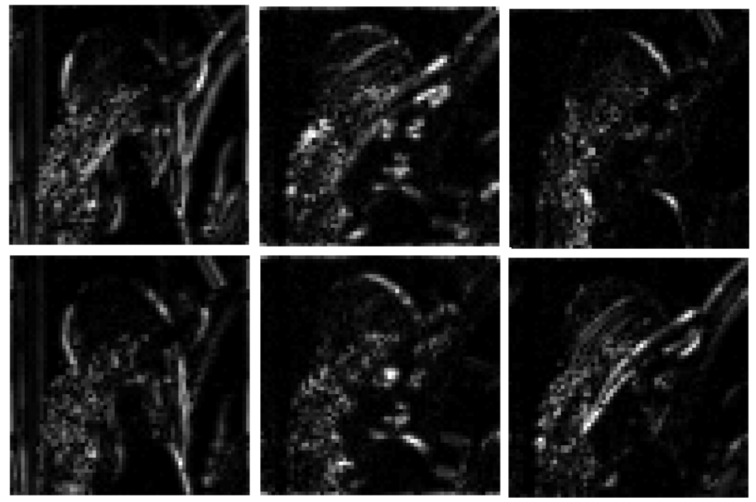
DWT: Level 1

Complex Wavelets: Level 1



DWT: Level 3

Complex Wavelets: Level 3



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