As-Rigid-As-Possible Surface Modeling

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Goal

- A shape modeling framework that supports intuitive and detail-preserving deformation.

[Botsch and Sorkine, 08]
Previous (surface-based) Works

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  – Physically-based methods that directly minimize stretching and/or bending energy
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  – Physically-based methods that directly minimize stretching and/or bending energy
    • Problem: Not so detail-preserving
    • Remedy: Multi-resolution Editing
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  – Physically-based methods that directly minimize stretching and/or bending energy
  – Differential Coordinates Editing methods that modify differential properties instead of spatial coordinate.
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• For the sake of interactive frame-rates, most of previous works were based on linear methods. Two categories:
  – Physically-based methods that directly minimize stretching and/or bending energy
  – Differential Coordinates Editing methods that modify differential properties instead of spatial coordinate.
    • Still suffer from linearization...(interpolating either gradients or rotations)
Observation

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• This motivates us to preserve local rigidity as much as possible.

• The paper proposes to break a surface into overlapping cells and seek to keep the transformation in each cell rigid.
Measurement of Rigidity

• We define each cell covers the triangles incident upon a vertex (i.e. the one-ring neighborhood).

• Given the cell $C_i$ corresponding to vertex $i$ that deforms to $C_i'$, we know there exists a rotation $R_i'$ such that

$$p_i' - p_j' = R_i (p_i - p_j), \quad \forall j \in \mathcal{N}(i)$$

if the transformation of that cell is perfectly rigid.
Measurement of Rigidity

• This leads to an energy function that is minimized when the rigidity is maximized:

\[ E(C_i, C'_i) = \sum_{j \in \mathcal{N}(i)} w_{ij} \| (p'_i - p'_j) - R_i(p_i - p_j) \|^2. \]

• Summing up energy functions from all cells, we obtain a global energy functional:

\[ E(S') = \sum_{i=1}^{n} w_i E(C_i, C'_i) = \]

\[ = \sum_{i=1}^{n} w_i \sum_{j \in \mathcal{N}(i)} w_{ij} \| (p'_i - p'_j) - R_i(p_i - p_j) \|^2 \]
Minimization Scheme

• The energy functional is non-linear and depends only on $p_i'$: the new positions of the vertices.

• A two-stage, alternating scheme is proposed to decrease energy iteratively. This is achieved by separating the optimization of $R_i$ (though it depends on $p_i'$) from the optimization of $p_i'$
Stage 1: Optimization of $R_i$

• Given vertex positions, we would like to find optimal rigid transformations \{R_i\}
• Rewrite the energy function:

\[
\sum_j w_{ij} (e'_{ij} - R_i e_{ij})^T (e'_{ij} - R_i e_{ij}) = \\
= \sum_j w_{ij} (e'_{ij} e'_{ij} - 2 e'_{ij} R_i e_{ij} + e'_{ij} R_i R_i e_{ij}) = \\
= \sum_j w_{ij} (e'_{ij} e'_{ij} - 2 e'_{ij} R_i e_{ij} + e'_{ij} e_{ij}).
\]

• Dropping constants then

\[
\arg\min_{R_i} \sum_j -2w_{ij} e'_{ij} R_i e_{ij} = \arg\max_{R_i} \sum_j w_{ij} e'_{ij} R_i e_{ij} = \\
= \arg\max_{R_i} \text{Tr} \left( \sum_j w_{ij} R_i e_{ij} e'_{ij} \right) = \\
= \arg\max_{R_i} \text{Tr} \left( R_i \sum_j w_{ij} e_{ij} e'_{ij} \right). \\
\]

• \(\text{Tr}(R_i S_i)\) can be maximized by making \(R_i S_i\) symmetric PSD via SVD
Stage 2: Optimization of $P_i'$

- Given $\{R_i\}$, we would like to find optimal vertex positions $P_i'$
• Differentiating the energy function, we have

\[
\frac{\partial E (S')}{\partial p_i'} = \frac{\partial}{\partial p_i'} \left( \sum_{j \in \mathcal{N}(i)} w_{ij} \| (p'_i - p'_j) - R_i (p_i - p_j) \|^2 + \right.
\]

\[
+ \sum_{j \in \mathcal{N}(i)} w_{ji} \| (p'_j - p'_i) - R_j (p_j - p_i) \|^2 \right) = \]

\[
= \sum_{j \in \mathcal{N}(i)} 2 w_{ij} \left( (p'_i - p'_j) - R_i (p_i - p_j) \right) + \]

\[
+ \sum_{j \in \mathcal{N}(i)} -2 w_{ji} \left( (p'_j - p'_i) - R_j (p_j - p_i) \right). \]

• Since the weights are symmetric

\[
\frac{\partial E (S')}{\partial p_i'} = \sum_{j \in \mathcal{N}(i)} 4 w_{ij} \left( (p'_i - p'_j) - \frac{1}{2} (R_i + R_j) (p_i - p_j) \right) \]

• Setting it to zero, we achieve:

\[
\sum_{j \in \mathcal{N}(i)} w_{ij} (p'_i - p'_j) = \sum_{j \in \mathcal{N}(i)} w_{ij} \frac{1}{2} (R_i + R_j) (p_i - p_j) \]
• Differentiating the energy function, we have

\[
\frac{\partial E(S')}{\partial p_i'} = \frac{\partial}{\partial p_i'} \left( \sum_{j \in \mathcal{N}(i)} w_{ij} \| (p_i' - p_j') - R_i (p_i - p_j) \|^2 + \right. \\
\left. + \sum_{j \in \mathcal{N}(i)} w_{ji} \| (p_j' - p_i') - R_j (p_j - p_i) \|^2 \right) = \\
= \sum_{j \in \mathcal{N}(i)} 2w_{ij} ((p_i' - p_j') - R_i (p_i - p_j)) + \\
+ \sum_{j \in \mathcal{N}(i)} -2w_{ji} ((p_j' - p_i') - R_j (p_j - p_i)) .
\]

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\frac{\partial E(S')}{\partial p_i'} = \sum_{j \in \mathcal{N}(i)} 4w_{ij} \left( (p_i' - p_j') - \frac{1}{2} (R_i + R_j) (p_i - p_j) \right)
\]

• Setting it to zero, we achieve: \( L p' = b \)
Discussion

• The first stage involves solving a series of (3x3) SVD problem. The complexity is linear.

• The second stage involves solving a Poisson equation. It is important to note that the system matrix never changes, only the constraint does.

• This suggests pre-factorization of the system. Though expensive to compute, it is an one-time task.
Discussion

• Cholesky factorization is used to decompose the system matrix into two triangle matrices.

• As a result, solving for a new constraint amounts to 3(?) times of back-substitution, which has a quadratic(!?) complexity.
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• Cholesky factorization is used to decompose the system matrix into two triangle matrices.
• As a result, solving for a new constraint amounts to 3(?) times of back-substitution, which has a quadratic(!?) complexity.
  – Note that the matrix is sparse. In practice, it is permuted first to ensure the resulting factorization will be sparse too. So the complexity will be linear in the number of vertices.
Conclusion

• The paper proposes a shape deformation modeling framework that preserves local rigidity as much as possible (and thus detail-preserving).

• The advantages of the approach are
  1. Robustness(?): guaranteed to converge to something...(a good initial guess is important!)
  2. Simplicity: easy to implement
  3. Efficiency: able to pre-factorize the system

• The visual results are comparable to the fully non-linear technique [Botsch et al., 07]