1 Asymptotic Notation (40 points)

For each of the following statements explain if it is true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is $\log_e$.

(a) $100(n \log^4 n + \frac{1}{2}n^2) = O(n^2)$

True. Let $n_0 = 2^{32}$. We have $\log_4 n_0 = 2^{20}$. Then for $n \geq n_0$, we have $2^{20} \leq \log_4 n \leq n$. Thus for $n \geq n_0$, we get that

$$100(n \log^4 n + \frac{1}{2}n^2) \leq 100(n^2 + \frac{1}{2}n^2) = 150n^2.$$ 

Thus, if we choose $c = 150$ and $n_0 = 2^{32}$, these functions satisfy the definition of big $O$.

(b) $2^n = \Theta(2^{(n/2)})$

False. We claim that $2^n \not\in O(2^{n/2})$ and so $2^n \not\in \Theta(2^{n/2})$. To see this, suppose for contradiction that there exist constants $c, n_0 > 0$ such that $2^n \leq c \cdot 2^{n/2}$ for all $n \geq n_0$. Then basic algebra implies that $\frac{2^n}{2^{n/2}} = 2^{n/2} \leq c$ for all $n \geq n_0$, and thus $n/2 \leq \log c$ for all $n \geq n_0$. But this is a contradiction, since $n/2 > \log c$ for all $n > 2 \log c$. Thus no such $c, n_0$ exist, and thus $2^n \not\in O(2^{n/2})$.

(c) $\log(n^{6 \log n}) = \Theta \left( (\log n^{1/3})^2 \right)$

True. In order to prove that $\log(n^{6 \log n}) = \Theta((\log n^{1/3})^2)$, we must prove that $\log(n^{6 \log n}) = O((\log n^{1/3})^2)$ and $\log(n^{6 \log n}) = \Omega((\log n^{1/3})^2)$. We know that

$$\log(n^{6 \log n}) = 6 \log n \log n = 6(\log n)^2,$$

and that

$$\log(n^{1/3})^2 = \left(\frac{1}{3} \log n\right)^2 = \frac{1}{9}(\log n)^2.$$ 

Thus for all $n > 1$, we have that $\log(n^{6 \log n}) \geq (\log n^{1/3})^2$ and hence $n_0 = 1, c = 1$ satisfy the condition that $\log(n^{6 \log n}) = \Omega((\log n^{1/3})^2)$. 

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On the other hand, we also have that $\log(n^{6\log n}) = 54 \cdot (\log n^{1/3})^2$ for all $n \geq 1$. Therefore, if we choose $c = 54$ and $n_0 = 1$, the function satisfies the definition of $\log(n^{6\log n}) = O((\log n^{1/3})^2)$

(d) $3^n = \Theta((3.1)^n)$

**False.** Recall that $3^n = \Theta((3.1)^n)$ if both $3^n = O((3.1)^n)$ and $3^n = \Omega((3.1)^n)$. Thus, in order to prove that $3^n$ is not $\Theta((3.1)^n)$, we need to show at least one of these conditions does not hold. It happens that $3^n$ is in fact $O((3.1)^n)$ (though we will not prove that here). So let’s prove by contradiction that $3^n$ is not $\Omega((3.1)^n)$.

Suppose $3^n = \Omega((3.1)^n)$. That is, suppose there exist some constants $c, n_0 > 0$ such that, for all $n > n_0$, $3^n \geq c(3.1)^n$. Rearranging the terms in the inequality, we find that $c \leq (\frac{3}{3.1})^n$ for all $n > n_0$. But clearly this is impossible, because $(\frac{3}{3.1})^n$ is less than $c$ when $n > \log_{3/3.1} c$ (we take the logarithm of both sides of the inequality). Thus we have a contradiction, and hence $3^n$ cannot be $\Omega((3.1)^n)$.

(e) $\sqrt{n} + \cos n = O(\sqrt{n})$

**True.** We need to show that there exists constants $c$ and $n_0$ such that $\sqrt{n} + \cos n \leq cn$ for all $n > n_0$. We know that $\cos n \leq 1$, and hence $\sqrt{n} + \cos n \leq \sqrt{n} + 1$ for all $n > 1$. If we set $n_0 = 1$, then for all $n > n_0$ we have that $\sqrt{n} + \cos n \leq \sqrt{n} + 1 \leq \sqrt{n} + \frac{3}{2} \sqrt{n} = 2\sqrt{n}$. Thus, if we choose $c = 2$ and $n_0 = 1$, these functions satisfy the definition of big $O$.

(f) Let $f, g$ be positive functions. Then $f(n) + g(n) = O(\max(f(n), g(n)))$

**True.** In order to prove that $f(n) + g(n) = O(\max(f(n), g(n)))$, we need to show that there exists constant $c, n_0 > 0$ such that $f(n) + g(n) \leq c \cdot \max(f(n), g(n))$ for all $n > n_0$. We know that for all $n > 1$, both $f(n) \leq \max(f(n), g(n))$ and $g(n) \leq \max(f(n), g(n))$ and hence,

$$f(n) + g(n) \leq 2 \cdot \max(f(n), g(n))$$

because $f(n), g(n)$ are positive functions. Thus, if we choose $c = 2$ and $n_0 = 1$, then these functions satisfy the definition of big $O$.

(g) Let $f, g$ be positive functions, and let $g(n) = \omega(f(n))$. Then $f(n) + g(n) = \Theta(g(n))$

**True.** Recall the definition of little $\omega$: for all constants $c > 0$, there exists a constant $n_0$ such that $g(n) \geq cf(n)$ for all $n > n_0$.

Let $c_1$ be some constant, and let $n_1$ be the corresponding constant such that $g(n) \geq c_1 f(n)$ for all $n > n_1$. Then the following is true for all $n > n_1$:

$$f(n) + g(n) \leq g(n)/c_1 + g(n) = (1/c_1 + 1)g(n)$$

Thus, if we choose $c = 1/c_1 + 1$ and $n_0 = n_1$, these functions satisfy the definition of big $O$. 

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Proving big $\Omega$ is very straightforward. Since $f(n)$ and $g(n)$ are positive functions, we have that for all $n$,

$$f(n) + g(n) \geq g(n)$$

Thus, if we choose $c = 1$ and $n_0 = 1$, then these functions satisfy the definition of big $\Omega$.

(h) $2^{\log n}/2 = \Theta(n)$

False. Observe that $2^{\log n}/2 = (2^{\log n})^{1/2} = n^{1/2}$. It is true that $n^{1/2} = O(n)$, but it is not true that $n^{1/2} = \Omega(n)$. Suppose for contradiction that there exists some $c$ and $n_0$ such that $\sqrt{n} \geq cn$ for all $n > n_0$. Then $\sqrt{n} \leq 1/c$ for all $n > n_0$. But this is impossible for any $n$ where $n^{1/2} > 1/c$, and hence we have a contradiction. Therefore $2^{\log n}/2$ cannot be $\Omega(n)$.

2 Recurrences (32 pts)

Solve the following recurrences, giving your answer in $\Theta$ notation. For each of them you may assume $T(x) = 1$ for $x \leq 5$ (or if it makes the base case easier you may assume $T(x)$ is any other constant for $x$ at most some constant). Justify your answer (formal proof not necessary, but recommended).

(a) $T(n) = n^{1/3}T(n^{2/3}) + n$

Solution. Draw the recursion tree.

We first claim that the total value of each level is exactly $n$. We will prove this by induction on the levels.

For the base case, this is clearly true at the top level. So now suppose that it is true for level $i$, and we are trying to prove it for level $i + 1$. Clearly at each level all nodes have the same size, so suppose that at level $i$ the value of each node is $k$. Then by the inductive hypothesis, there must be exactly $n/k$ nodes. Hence at level $i + 1$ each node has size $k^{1/3}$, and the total number of nodes at level $i + 1$ is $\frac{n}{k} \cdot k^{2/3} = n/k^{1/3}$. Thus the total value at level $i + 1$ is $k^{1/3} \cdot (n/k^{1/3}) = n$. So we have completed the inductive step.

Since the total value at every level is exactly $n$, it remains only to calculate the number of levels. Clearly at level $i$, the value of each node is $n^{(2/3)^i}$. The recursion will stop when this is 5 (or less). Thus we know that $n^{(2/3)^i} \geq 5$. Taking logs of both sides, we get that
\((\frac{3}{2})^i \log n \geq \log 5\), or equivalently that \((\frac{3}{2})^i \leq \log_5 n\). Taking logs again (this time base \(3/2\)), we get that that \(i \leq \log_{3/2} \log_5 n\). Thus the total number of levels is \(\Theta(\log \log n)\).

Putting this together we get that

\[
T(n) = \Theta(n \log \log n)\]

(b) \(T(n) = 8T(n/4) + n\)

**Solution.** We can use the Master Theorem with \(a = 8, b = 4, k = 1\) and \(a > b^k\) to conclude that

\[
T(n) = \Theta(n^{\log_4 8}) = \Theta(n^{3/2})
\]

(c) \(T(n) = T(n - 3) + 5\)

**Solution.** We can solve this using the unrolling method:

\[
T(n) = T(n - 3) + 5
= (T(n - 6) + 5) + 5
= ((T(n - 9) + 5) + 5) + 5
\vdots
= T(n - 3 \left\lfloor \frac{n}{3} \right\rfloor) + \sum_{i=1}^{\left\lfloor \frac{n}{3} \right\rfloor} 5
= 1 + 5 \left\lfloor \frac{n}{3} \right\rfloor
\]

Again, the last step works because \(n - 3 \left\lfloor \frac{n}{3} \right\rfloor \leq 5\).

Based on the unrolling, we can conclude that

\[
T(n) = \Theta(n)
\]

(d) \(T(n) = 3T(n/3) + n \log_3 n\)

**Solution.** We can solve this by drawing out the recursion tree:

Suppose \(n\) is an exact multiple of three. Then, it is easy to see that the tree will have exactly \(\log_3 n - 1\) levels. At the \(i\)-th level, there will be \(3^i\) nodes, each with value \(\frac{n}{3^i} \log_3 \frac{n}{3^i}\). This
gives us the following solution to the recurrence relation:

\[
T(n) = \sum_{i=0}^{\log_3 n-2} 3^i \cdot \frac{n \log_3 n}{3^i} \\
= \sum_{i=0}^{\log_3 n-2} n \log_3 \frac{n}{3^i} \\
= n \sum_{i=0}^{\log_3 n-2} \log_3 \frac{n}{3^i} \\
= n \sum_{i=0}^{\log_3 n-2} (\log_3 n - \log_3 3^i) \\
= n \sum_{i=0}^{\log_3 n-2} (\log_3 n - i) \\
= n \sum_{i=2}^{\log_3 n} i \\
\]

(The last step follows by reversing the direction of the sum.) Based on the standard result that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \), we can conclude that \( T(n) = \Theta(n \log^2 n) \).

3 Basic Proofs (28 pts)

(a) (9 points) There are 13 course assistants for this class. Let us assume that 145 students submit their assignments for this problem set, and each submission is graded by one course assistant. Prove that there is some course assistant who grades at least 12 submissions.

**Solution.** For \( i \in \{1, 2, \ldots, 13\} \), let \( n_i \) be the number of submissions graded by CA \( i \). Then for sake of contradiction, assume that \( n_i < 12 \) for all \( i \). Then since each \( n_i \) is an integer, we get that \( n_i \leq 11 \) for all \( i \). Thus the total number of submissions is

\[
145 = \sum_{i=1}^{13} n_i \leq \sum_{i=1}^{13} 11 = 143.
\]

This is a contradiction, since 145 \( \neq \) 143. Thus there must exist some \( n_i \geq 12 \). Note that this is also called the pigeonhole principle.

(b) (9 points) I have a bucket with 32 balls, 20 of which are white and 12 of which are black. If I draw 9 balls at random from the bucket (all at one time), what is the probability that exactly three of them are white? Prove your answer.

**Solution.** There are \( \binom{32}{9} \) ways of drawing 9 balls from the bucket. There are \( \binom{20}{3} \cdot \binom{12}{6} \) ways of choosing 3 white balls and 6 black balls. Thus the probability of drawing exactly 3 white
balls is
\[
\frac{\binom{20}{3} \cdot \binom{12}{6}}{\binom{32}{9}} = \frac{1053360}{28048800} = \frac{4389}{116870}
\]

(c) (10 points) Prove that \(\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}\) for all \(n \geq 1\). Hint use induction.

\textbf{Solution.} We use induction on \(n\). First we argue that the claim holds for \(n = 1\). This is straightforward since \((-1)^2 \cdot (1) + (-1)^3 \frac{1}{2} = \frac{1}{2}\). No assume that the equality holds for \(i = n\), i.e. we have (by \textit{Inductive Hypothesis}),

\[
\sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k}
\]

Then we show that the equality also holds for \(i = n + 1\) (\textit{Inductive Step}). We have,

\[
\sum_{k=1}^{2(n+1)} (-1)^{k+1} \frac{1}{k} = \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} + (-1)^{2n+2} \frac{1}{2n+1} + (-1)^{2n+3} \frac{1}{2n+2}.
\]

Then using the inductive hypothesis and replacing the sum, we will get:

\[
\sum_{k=1}^{2(n+1)} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} + (-1)^{2n+2} \frac{1}{2n+1} + (-1)^{2n+3} \frac{1}{2n+2}
\]

\[
= \sum_{k=n+1}^{2n} \frac{1}{k} + \left( \frac{1}{2n+1} - \frac{1}{2n+2} \right)
\]

\[
= \left( \sum_{k=n+1}^{2n} \frac{1}{k} - \frac{1}{2n+2} \right) + \frac{1}{2n+1}
\]

\[
= \left( \sum_{k=n+1}^{2n} \frac{1}{k} - \frac{1}{n+1} + \frac{1}{2(n+1)} \right) + \frac{1}{2n+1}
\]

\[
= \left( \sum_{k=n+1}^{2n} \frac{1}{k} - \frac{1}{n+1} + \frac{1}{2(n+1)} \right) + \frac{1}{2n+1}
\]

\[
= \left( \sum_{k=n+1}^{2n} \frac{1}{k} \right) + \frac{1}{n+1} + \frac{1}{2(n+1)}
\]

\[
= \sum_{k=n+1}^{2n+2} \frac{1}{k}
\]

We have thus shown that \(\sum_{k=1}^{2(n+1)} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2(n+1)} \frac{1}{k}\). Which concludes the inductive step, and hence the claim holds.