1 Asymptotic Notation (40 points)

For each of the following statements explain if it is true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is $\log_e$.

(a) $100(n \log^4 n + \frac{1}{2} n^2) = O(n^2)$
(b) $2^n = \Theta(2^{(n/2)})$
(c) $\log(n^{6\log n}) = \Theta\left((\log n^{1/3})^2\right)$
(d) $3^n = \Theta((3.1)^n)$
(e) $\sqrt{n} + \cos n = O(\sqrt{n})$

(f) Let $f, g$ be positive functions. Then $f(n) + g(n) = O(\max(f(n), g(n)))$

(g) Let $f, g$ be positive functions, and let $g(n) = \omega(f(n))$. Then $f(n) + g(n) = \Theta(g(n))$

(h) $2^{\log \frac{n}{2}} = \Theta(n)$

2 Recurrences (32 pts)

Solve the following recurrences, giving your answer in $\Theta$ notation. For each of them you may assume $T(x) = 1$ for $x \leq 5$ (or if it makes the base case easier you may assume $T(x)$ is any other constant for $x$ at most some constant). Justify your answer (formal proof not necessary, but recommended).

(a) $T(n) = n^{1/3}T(n^{2/3}) + n$
(b) $T(n) = 8T(n/4) + n$
(c) $T(n) = T(n - 3) + 5$
(d) $T(n) = 3T(n/3) + n \log_3 n$
3 Basic Proofs (28 pts)

(a) (9 points) There are 13 course assistants for this class. Let us assume that 145 students submit their assignments for this problem set, and each submission is graded by one course assistant. Prove that there is some course assistant who grades at least 12 submissions.

(b) (9 points) I have a bucket with 32 balls, 20 of which are white and 12 of which are black. If I draw 9 balls at random from the bucket (all at one time), what is the probability that exactly three of them are white? Prove your answer.

(c) (10 points) Prove that \( \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} \) for all \( n \geq 1 \). Hint: use induction.