1 MSTs and light edges (33 points)

Let $G = (V, E)$ be an undirected graph, and let $w : E \rightarrow \mathbb{R}^+$ be a positive edge weighting (edge weights are not necessarily distinct). Recall that a light edge for a cut $(S, V \setminus S)$ is an edge crossing the cut (one endpoint in $S$ and one endpoint not in $S$) with weight at most the weight of any edge crossing the cut.

(a) (11 points) Let $T$ be an MST. Prove that every edge $e \in T$ is a light edge for some cut.

(b) (11 points) Let $(S, V \setminus S)$ be a cut such that there is a unique light edge $e$ for the cut. Prove that $e$ must be in every MST.

(c) (11 points) Prove that the minimum spanning tree is unique if, for every cut in the graph, there is a unique light edge for the cut.

2 Clustering (33 points)

Consider the following problem. You are given a graph $G = (V, E)$ and a length function $\ell : E \rightarrow \mathbb{R}_{\geq 0}$. You should output a clustering (i.e., a partition) $(C_1, C_2, \ldots, C_k)$ of $V$ into $k$ clusters which maximizes the minimum distance between clusters. More formally, you should maximize $\min_{i,j \in [k], i \neq j} d(C_i, C_j)$, where $d(C_i, C_j) = \min_{x \in C_i, y \in C_j} d(x, y)$ and where $d(x, y)$ is the shortest-path distance (under length function $\ell$). Design an algorithm for this problem which runs in $O(m \log n)$ time, and prove correctness and running time.

3 Matroids (34 points)

(a) (17 points) Let $G = (V, E)$ be a connected undirected graph. Let $\mathcal{I} = \{I \subseteq E : (V, E \setminus I) \text{ is connected}\}$. Prove that $(E, \mathcal{I})$ is a matroid.

(b) (17 points) Let $U$ be a finite set and let $U_1, U_2, \ldots, U_k$ be a partition of $U$ into nonempty disjoint subsets (where $k \geq 2$). Let $r_1, r_2, \ldots, r_k$ be positive integers. Let $\mathcal{I} = \{S \subseteq U : |S \cap U_i| \leq r_i\}$ for all $i \in \{1, 2, \ldots, k\}$. Prove that $(U, \mathcal{I})$ is a matroid.