1 Asymptotic Notation (32 points)

For each of the following statements explain if it true or false and prove your answer. The base of log is 2 unless otherwise specified, and ln is log_e.

(a) $2^n = \Theta(e^n)$
(b) $n \sin n = O(n)$
(c) $20(n \log^2 n + \frac{1}{2}n^2) = \Theta(n^2)$
(d) $3^n = \Omega(3^{n-4})$
(e) $\log(n^{20}) = \Theta(\log(n^{1/10}))$
(f) Let $f, g$ be positive functions. Then $f(n) + g(n) = \Omega(\max(f(n), g(n)))$
(g) Let $f, g$ be positive functions, and let $g(n) = o(f(n))$. Then $f(n) + g(n) = \Theta(f(n))$
(h) $2^{3 \log n} = O(n^2)$

2 Recurrences (33 pts)

Solve the following recurrences, giving your answer in $\Theta$ notation (so prove both an upper bound and a lower bound). For each of them you may assume $T(x) = 1$ for $x \leq 5$ (or if it makes the base case easier you may assume $T(x)$ is any other constant for $x \leq 5$). Justify your answer (formal proof not necessary, but recommended).

(a) $T(n) = 5T(n-3)$
(b) $T(n) = n^{1/4}T(n^{3/4}) + n$
(c) $T(n) = 6T(n/4) + n$
(d) $T(n) = T(n-2) + 10$
(e) $T(n) = 3T(n/3) + n \log_3 n$
3 Basic Proofs and Probability (35 pts)

(a) There are currently 86 students registered for the class. Prove that there are at least 8 students who were all born in the same month.

(b) Let \( x_1, x_2, \ldots, x_n \) be real numbers. Prove that the maximum is at least the arithmetic mean, i.e. prove that

\[
\max_{i=1}^{n}\{x_i\} \geq \frac{1}{n} \sum_{i=1}^{n} x_i.
\]

(c) I have a bucket with 32 balls, 20 of which are white and 12 of which are black. If I draw 9 balls at random from the bucket (all at one time), what is the probability that exactly three of them are white?

(d) Let \( X \) be a nonnegative random variable, and suppose that \( E[X] \) is well defined (i.e., the expectation is not infinite). Prove that \( \Pr[X \geq t] \leq E[X]/t \) for all \( t > 0 \).

(e) An array \( A[1..n] \) contains \( n \) distinct numbers that are randomly ordered, with each permutation of the \( n \) numbers being equally likely. What is the expectation of the index of the maximum element in the array? What is the expectation of the index of the minimum element in the array? Justify your answer.