Min-Makespan on Identical Parallel Machines

Input: Jobs $J$ ($|J| = n$)
- Machines $M$ ($|M| = k$)
- Processing times $p : J \rightarrow N$

Feasible solution: $\mathcal{I} : J \rightarrow M$

Objective: $\text{min makespan} = \min_{m \in M} \max_{j \in J} \sum_{i : i \in \mathcal{I}(j) = m} p(i)$

Thus: greedy is a 2-approximation.

Want to do better!
Idea: What if we have a “guess” $T$ for $OPT$?

**Def**: A \((1+\epsilon)\)-relaxed decision procedure is an algorithm which, given input and $T$:

1) If $T \geq OPT$, returns a solution with makespan $\leq (1+\epsilon)T$
2) If $T < OPT$, either returns “false” or returns a solution with makespan $\leq (1+\epsilon) \cdot OPT$

Let $P = \sum_{j \in J} p(j)$. Then $1 \leq OPT \leq P$

Idea: do binary search to find value of $T$ s.t.
procedure returns a solution on $T$, “false” on $T-1$

$\implies T \leq OPT$

$\implies$ solution has makespan $\leq (1+\epsilon) \cdot OPT$

Takes $O(\log P)$ iterations: poly-time!

So just want to design such a procedure, given guess $T$.

**Def**: $J_{\text{max}} = \{ j \in J : p(j) \leq \epsilon T^3 \}$

$J_{\text{true}} = \{ j \in J : p(j) > \epsilon T^3 \}$
Then, given schedule for \( T \) with makespan \( \leq (1 + \epsilon) T \), we can find a schedule for \( J \) with makespan \( \leq (1 + \epsilon) \max(T, OPT) \).

**Proof:**

Start with schedule for \( T \).

- Sort jobs on \( T \) in arbitrary order, add job to least-loaded machine.

Consider machine \( m \).

- (Case 1): \( m \) has no small jobs.
  \[ \Rightarrow \text{load} \leq (1 + \epsilon) T \]

- (Case 2): \( m \) has \( \geq 1 \) small job.
  greedy analysis:

  Let \( j \) be last small job assigned to \( m \)
  \[ \Rightarrow p(j) \leq \epsilon \cdot T \]
  \[ \Rightarrow \text{load on } m \text{ just before } j \leq OPT \]
  \[ \Rightarrow \text{total load} \leq OPT + \epsilon T \leq (1 + \epsilon) \max(T, OPT) \]
So just need to schedule $J_{use}$ with makespan $\leq (1+\varepsilon) T$

Let $b = \lceil \frac{I}{b^k} \rceil$, so $\frac{1}{b} \leq \varepsilon$

**Def:** Let $p_i'(j) = \left\lfloor \frac{p_i(j) b^k}{T} \right\rfloor \cdot \frac{T}{b^k}$ ("rounded instance")

$p_j$ is $p_j$ rounded down to multiple of $\frac{T}{b^2}$

$\Rightarrow p_i'(j) \leq p_i(j) \leq p_i'(j) + \frac{T}{b^2}$

$p_i'(j) = k \frac{T}{b^2}$ for some $k \in \{b, b^2, \ldots, b^{2k}\}$ (since $j \in J_{use}$)

For schedule $\bar{J}$, let $m_c(\bar{J})$ be makespan in original, $m_r(\bar{J})$ makespan in rounded.

**Lemma 1:** Let $\bar{J}$ be schedule. Then $m_r(\bar{J}) \leq m(\bar{J})$

$p.j$. 

$p_i'(j) \leq p_i(j)$ $\forall i, j$

**Lemma 2:** Let $\bar{J}$ be schedule. Then $m(\bar{J}) \leq (1+\varepsilon) m_r(\bar{J})$ $\bar{T}$
Let $m$ some machine

\[ p'(i) \geq \frac{T}{b}, \quad |j^{*}(m)| \leq b \]

\[ \sum_{i \in j^{*}(m)} p(i) \leq \sum_{i \in j^{*}(m)} \left( p'(i) + \frac{T}{5^k} \right) \]

\[ = \sum_{i \in j^{*}(m)} p'(i) + \sum_{i \in j^{*}(m)} \frac{T}{5^k} \]

\[ \leq T + \frac{T}{b} \leq T + \varepsilon T = (1 + \varepsilon)T \]

Suppose we had an algorithm to find some $I$ with $m_r(I) \leq T$ if one exists

If $T \geq \text{OPT}$:

\[ \exists I \text{ with } m(I) \leq T \Rightarrow m_r(I) \leq T \quad \text{ (Lemma 1)} \]

\[ \Rightarrow \exists \text{ schedule } w/ \text{ makespan } \leq T \text{ in round } \]

\[ \exists \text{ alg will find some } I' \text{ with } m_r(I') \leq T \]

\[ m(I') \leq (1 + \varepsilon)T \quad \text{ (Lemma 2)} \]
If $T < \text{OPT}$:

If alg returns $\mathcal{I}$, have

\[ m(\mathcal{I}) \leq (1 + \epsilon) T \]  \quad \text{(Lemma 2)}

So just need to find $\mathcal{I}$ with $m(\mathcal{I}) \leq T$ if one exists!

**Def:** A configuration is a tuple $(a_1, a_2, \ldots, a_n)$ such that:

1) Each $a_i \in \{0, 1, \ldots, b\}$

2) $\sum_{i=1}^{b} a_i \cdot i \cdot \frac{T}{b^2} \leq T$

**Idea:** Look at jobs assigned to some machine, let $a_i$ be the with $p'(j) = i \cdot \frac{T}{b^2}$

$\Rightarrow$ get a configuration
Let $\mathcal{C}(T)$ be set of all configurations

$|\mathcal{C}(T)| \leq (b+1)^{b^2-b} = (b)^{b^2}$

**Dynamic Programming:** given (unidentified) jobs, how many machines are necessary?

More formally:

$f(n_1, n_2, \ldots, n_b) = \min \ m \ s.t. \ can \ schedule \ n_i \ jobs

of \ length \ i \cdot \frac{T}{b^2} \ & \ i \ with \ makespan \ \leq T$

$f(\varnothing) = 0$

$f(n_1, \ldots, n_b) =

1 + \min_{\delta \in \mathcal{C}(T)} f(n_1 - n_1, \ldots, n_b - n_b)$

Time table entry: $|\mathcal{C}(T)| = b^{b^2}$

# table entries: $\leq b^{b^2}$
\[ \text{total time} \leq n = O(\beta^3) = O(\beta^{0.75}) \]

Once table filled out, check entry corresponding to \( J_{\text{large}} \) (rounded)

If \( m \geq n \) return \( \exists \) with \( n_x(\exists) \leq T \)

If \( m \geq n \) return \( \exists \) with \( n(\exists) \leq T \)

\( \rightarrow \) return false

If \( \leq m \) return \( \exists \) with \( n_x(\exists) \leq T \)