Multicut:

Input:
- \( G = (V, E) \)
- \( c : E \rightarrow \mathbb{R}^+ \)
- \( k \) pairs \((s_1, t_1), \ldots, (s_k, t_k)\)

Feasible solution: \( F \subseteq E \) s.t. \( \forall i \in \{1, \ldots, k\} \) there is a \( s_i - t_i \) path in \( G \setminus F \)

Objective: \( \min C(F) = \sum_{e \in F} c(e) \)

Last time: \( O(\log n) \)-approx

Today: \( O(\log k) \)-approx

\( \mathcal{P}_i = \{ s_i - t_i \text{ paths} \} \)

LP:
\[
\begin{align*}
\min \quad & \sum_{e \in E} c(e) x_e \\
\text{s.t.} \quad & \sum_{e \in P_i} x_e \geq 1 \quad \forall i \in \{1, \ldots, k\}, \forall e \in P_i \\
& 0 \leq x_e \leq 1 \quad \forall e \in E
\end{align*}
\]
Solving LP: Ellipsoid separation oracle = shortest path

Let $x$ optimal LP solution

$$V^* = \sum_{e \in E} c(e) x_e = LP$$

$d: V \times V \to \mathbb{R}_{\geq 0}$ shortest path metric using $x$ as edge lengths

$\exists d(s_i, t_i) \geq 1$ by LP (but no bound on $d(s_i, t_j)$)

$\delta(s) = E(s, s_j)$

Metaphor: $x_e = \text{length of edge } e$

Think of $c(e)$ as "cross-section area" of $e$

$\Rightarrow c(e) x_e = \text{"volume" of } e.$

$V^* = \text{total volume of } G$
Def: (Normalized) Volume:

$$V(s_i, r) = \frac{V^*}{k} + \sum_{e \in \mathcal{E}} c(e)x_e + \sum_{e \in \mathcal{E} : \mathcal{E} \not\subseteq B(s_i, r)} (c(e)(r - d(s_i, v)))$$

Lemma (Region Growing): For all $i \in \mathcal{C}(k)$, in polytime we can find some $0 \leq r < \frac{1}{2}$ s.t.

$$\left( \delta(B(s_i, r)) \right) \leq 2 \ln(k+1) \cdot V(s_i, r)$$

Prove lemma later. Assuming it, running alg:

$$F = \emptyset$$

for $i = 1$ to $k$

if ($s_i$ is connected in $G \setminus F$)

Let $r_i$ be radius from lemma for $i$
\[ F = F \cup \delta(B(s_i, r_i)) \]
Remove \( B(s_i, r_i) \) and incident edges from graph

Note: Lemma applied to changing graph

**Theorem:** Return feasible solution

**Proof:** \( V_i \) component of \( G \setminus F \) containing \( s_i \) is \( B(s_i, r_i) \) for some \( r_i < \frac{1}{2} \).

\( B_i \neq \emptyset \) and \( d(s_i, t_i) \geq 1 \)

\( \Rightarrow \{s_i, t_i\} \) not connected in \( G \setminus F \)

**Theorem:** \( c(F) \leq 4 \ln(k+1) \cdot V^* \leq 4 \ln(k+1) \cdot OPT \)

**Proof:** Definitions:
- \( B_i = B(s_i, r_i) \) in iteration \( i \)
  (If ball not constructed in iteration \( i \), \( B_i \) already disconnected, \( B_i = \emptyset \))
  (Ball constructed in iteration \( i \); not necessarily \( B(s_i, r_i) \) in original graph)
- \( F_i \): edges added to cut in iteration \( i = \delta(B(s, v_i)) = \delta(B_i) \)
  in iteration \( i \) graph

\[ \exists F = \bigcup_{i=1}^{k} F_i \quad F_i \cap F_j = \emptyset \quad \forall i \neq j \in \{1, k\} \]

- \( V_i \): volume of edges removed in iteration \( i \)

\[ = \sum_{e \in \{u, v_i\} : u, v \in B_i} c(e) x_e + \sum_{e \in F_i} c(e) x_e \]

- \( V \): volume of cut edges but not
  normalization term

\[ \exists V_i \geq V(s, v_i) - \frac{V^*}{k} \quad \text{since } V_i \text{ includes full volume} \]

of cut edges but not
  normalization term

\[ \sum_{i=1}^{k} V_i \leq V^*, \text{since every edge contributes to } V_i \]
  for at most one \( i \)

\[ \exists (F) = \sum_{i=1}^{k} (F_i) \]

\[ \leq 2 \ln(k+1) \sum_{i=1}^{k} V(s, v_i) \]

\[ \leq 2 \ln(k+1) \sum_{i=1}^{k} \left( V_i + \frac{V^*}{k} \right) \]

\[ \leq 2 \ln(k+1) (V^* + V^*) = 4 \ln(k+1) \cdot V^* \]
So just need to prove region growing lemma.

**Lemma (Region Growing):** For all \( i \in \mathcal{E}(k) \), in polytime we can find some \( 0 \leq r < \frac{1}{2} \) s.t.
\[
\left( \delta(\mathcal{B}(s_{i}, r)) \right) \leq 2 \ln(k+1) \cdot U(s_{i}, r)
\]

**Proof:** Simplify notation:
\[
\begin{align*}
C(r) &= \delta(\mathcal{B}(s_{i}, r)) \\
V(r) &= U(s_{i}, r)
\end{align*}
\]
\[
\begin{align*}
C : \mathbb{R}_{0} &\rightarrow \mathbb{R}_{0} \\
V : \mathbb{R}_{0} &\rightarrow \mathbb{R}_{0}
\end{align*}
\]
\[
\exists \text{ WTS can find } r < \frac{1}{2} \text{ s.t. } C(r) \leq 2 \ln(k+1) \cdot V(r)
\]
\[
\exists \frac{C(r)}{V(r)} \leq 2 \ln(k+1)
\]

**Algorithm (For Now):**

\( \text{choose } r \text{ uniformly at random in } (0, \frac{1}{2}) \)

WTS: \( \mathbb{E} \left[ \frac{C(r)}{V(r)} \right] \leq 2 \ln(k+1) \)

**Main idea:** (above) calculus
\[ V(v) = \frac{V^*}{k} + \sum_{e \in E} c(e)x_e + \sum_{e \in E} c(e)(r - d(s_i, v)) \]

Order \( B(s_i, \frac{r}{2}) \): \( s_i = v_0, v_1, v_2, \ldots, v_m \) s.t. \( d(s_i, v_j) = d(s_i, v_{j+1}) \)

Let \( r_j = d(s_i, v_j) \)

If \( r \in (r_j, r_{j+1}) \):

\[
\frac{dV(r)}{dr} = \sum_{e \in E} c(e) \]

\[
= c(v) \]

If \( V(v) \) continuous, differentiable (it's not):

\[
E \left[ \frac{c(v)}{V(v)} \right] = \frac{1}{2} \int_0^{\frac{1}{k}} \frac{c(r)}{V(v)} \, dr \]

\[
= 2 \int_0^{\frac{1}{k}} \frac{1}{V(v)} \frac{dV(v)}{dr} \, dr \]
\[ x = \text{constant} \quad \Rightarrow \quad \int_0^{\frac{1}{2}} \frac{1}{x} \, dx \]

\[ = 2 \int_0^{\frac{1}{2}} \frac{U(x)}{U(x)} \, dU(x) \]

\[ = 2 \left( \ln \left( \frac{U(\frac{1}{2})}{U(0)} \right) \right) \]

\[ = 2 \ln \left( \frac{U(\frac{1}{2})}{U(0)} \right) \]

\[ \leq 2 \ln \left( \frac{U^*}{k} + U^* \right) \]

\[ \leq 2 \ln (k+1) \]

\[ \Rightarrow \text{by mean value theorem for } \frac{c(x)}{U(x)} \leq 2 \ln (k+1) \]

Not continuous or differentiable! See notes/book.
Breakpoints at \( r_i \)'s, linear increasing between breaks.

Actually finding \( r \):

\( r \in (r_{i-1}, r_i) \) i.e. \( c(x) \) same for all \( r \) in interval

\( U(x) \) maximized when just less than \( r_i \)

\( \Rightarrow \frac{c(x)}{U(x)} \) minimized in \( (r_{i-1}, r_i) \) just below \( r_i \).
best choice of a some one, take best.