LP Solutions as Metrics:

Previously: trying to solve problem on metric space

Today: interpret LP solution as metric space!

Warmup: min s-t cut

Input: - $G=(V,E)$ (undirected)

- costs $c: E \to \mathbb{R}^+$
- source $s \in V$, sink $t \in V$

Feasible: $A \subseteq E$ s.t. $G \setminus A$ has no s-t path

Objective: $\min \sum_{e \in A} c(e)$

Def: $P_{u,v} = \{\text{all } u\text{-}v \text{ paths in } G\}$

LP relaxation:

$$\min \sum_{e \in E} c(e) x_e$$

$$\text{s.t.} \quad \sum_{e \in P} x_e \geq 1 \quad \forall P \in P_{s,t}$$

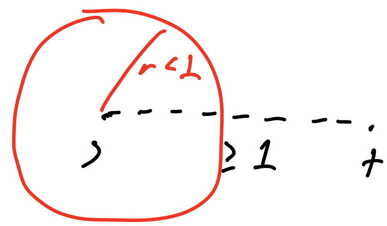
$$0 \leq x_e \leq 1 \quad \forall e \in E$$

Thm: LP can be solved in polytime

Pf:

Separation oracle: given x, is there a $P \in \mathcal{P}_{s,t}$ with $\sum_{e \in P} x_e < 1$?

Shortest path!



Let x be an LP solution. How to round?

Intuition: from separation. Think of x as lengths

Def: $d(u)$ = shortest path distance from s to u under edge lengths x

Def: $B(s,r) = \{v \in V : d(v) \le r\}$

Def: For $S \subseteq V$, let $\delta(S) = E(S, \bar{S}) = \{e \in E : e \cap S \neq \emptyset$ and $e \cap \bar{S} \neq \emptyset\}$

Rounding Alg (x an LP solution)

- Choose r uniformly at random in $(0,1)$
- $S = B(s,r)$
- return $A = \delta(S)$

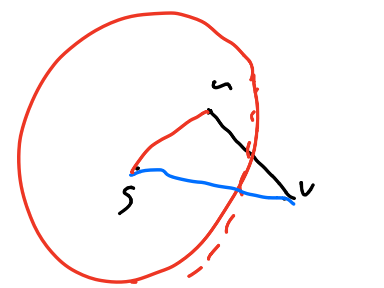
Claim: A feasible with probability 1

Pf: For all choices of r, $s \in S$, $t \notin S$ (since $d(t) \geq 1$)

$\Rightarrow A$ a feasible s-t cut

Claim: $\Pr[e \in A] \leq x_e \quad \forall e \in E$

Pf: Let $e = \{u, v\}$.



WLOG, $d(u) \leq d(v)$

$e \in A$ iff $\quad d(u) \leq r < d(v)$

$\Rightarrow \Pr[e \in A] = \Pr[d(u) \leq r < d(v)]$

$$= \frac{d(v) - d(u)}{1} \leq x_e$$

$d(v) \leq d(u) + x_e$

$$\Rightarrow E[c(A)] = E\Big[\sum_{e \in E} c(e) \cdot \mathbb{1}[e \in A]\Big] \leq \sum_{e \in E} c(e) E[\mathbb{1}[e \in A]]$$

$$\leq \sum_{e \in A} c(e) x_e = LP \leq OPT$$

Randomized alg for s-t mincut!

Deterministic:

- After solving LP to get x, $\leq n$ different cuts alg might return

- Try each, take best.
  - $\Rightarrow$ best has cost $\leq$ expectation, $\Rightarrow c(A) \leq OPT$
    - exact algorithm!

Multiway Cut:

Input:
- $G = (V, E)$ (undirected)
- costs $c: E \to \mathbb{R}^+$
- $T = \{s_1, s_2, \ldots, s_k\} \subseteq V$

Feasible: $A \subseteq E$ s.t. $G \setminus A$ has no $s_i - s_j$ path $\forall i \neq j \in [k]$

Objective: $\min \sum_{e \in E} c(e)$

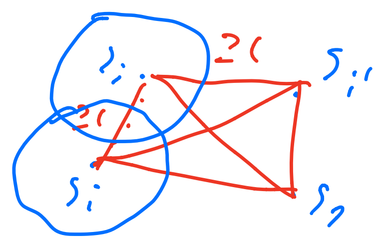
LP:

$$\min \sum_{e \in E} c(e)\, x_e$$

$$\text{s.t.} \quad \sum_{e \in P} x_e \geq 1 \qquad \forall i,j \in [k], i \neq j, \ \forall P \in \mathcal{P}_{s_i, s_j}$$

$$0 \leq x_e \leq 1 \qquad \forall e \in E$$

Interpret x as edge lengths, get shortest-path metric d



Rounding Algorithm:

$A = \emptyset$

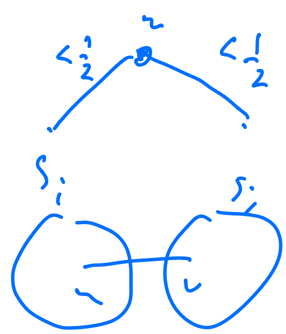
Choose r uniformly at random from $[0, \frac{1}{2})$

for $i = 1$ to k {

$\quad A_i = \delta(B(s_i, r))$

}

return $A = \bigcup_{i=1}^{k} A_i$



Claim: A feasible

Pf: LP constraints $\Rightarrow d(s_i, s_j) \geq 1 \quad \forall i,j \in [k]$

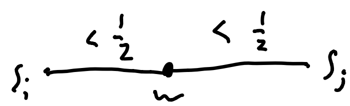
$\Rightarrow$ Every $s_i - s_j$ path uses some edge in A_i

<u>Claim</u>: $Pr[e \in A] \leq 2x_e \quad \forall e \in E$

<u>Pf</u>: Let $C_i = \{w \in V : d(s_i, w) < \frac{1}{2}\}$

Note: Each $w \in V$ in at most one C_i

otherwise:

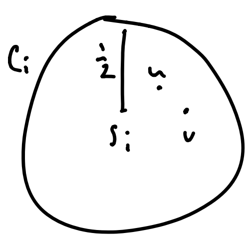


Let $e = \{u, v\} \in E$

WLOG, $d(u,T) \leq d(v,T)$

(u is closer to a terminal than v is)

<u>Case 1</u>: $u, v \in C_i$ for some i



$$Pr[e \in A] \overset{= Pr[e \in A_i] =}{=} Pr[\, d(s_i, u) \leq r < d(s_i, v)]$$

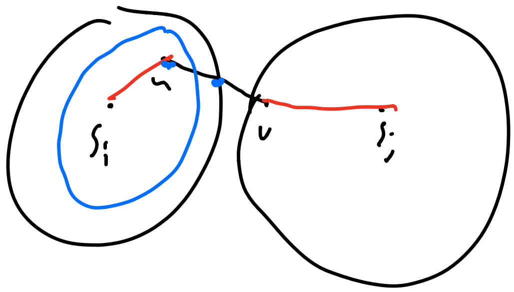
$$= \frac{d(s_i, v) - d(s_i, u)}{\frac{1}{2}} \leq \frac{d(u,v)}{\frac{1}{2}} \leq 2x_e$$

<u>Case 2</u>: No i s.t. $u, v \in C_i$

if neither u nor v in <u>any</u> of the C_i's

$\Rightarrow Pr[e \in A] = 0 \leq 2x_e$

So WLOG $u \in C_i$ for some i



<u>Observation</u>: $e \in A$ iff $r \geq d(s_i, u)$

$d(s_i,u) \leq r < \frac{1}{2}$

$$\Rightarrow Pr[e \in A] = \frac{\frac{1}{2} - d(s_i,u)}{\frac{1}{2}}$$

$$= 2\left(\tfrac{1}{2} - d(s_i,u)\right)$$

$$\leq 2\, d(u,v) \qquad \left(d(u,v) \geq \tfrac{1}{2} - d(s_i,u)\right)$$

$$\leq 2 x_e$$

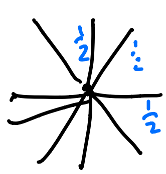
Linearity of Expectations:

$$E[c(A)] = \sum_{e \in E} c(e) \cdot Pr[e \in A] \leq 2 \sum_{e \in E} c(e) \cdot x_e = 2 \cdot LP$$

Tightness: integrality gap

<u>Thm</u>: The integrality gap of the LP is $\geq 2(1 - \frac{1}{k})$

<u>Pf</u>: Star, terminals are leaves



OPT: $n-2$

LP: $\frac{n-1}{2}$

$k = n-1$

$$\Rightarrow \frac{OPT}{LP} \geq \frac{k-1}{\frac{k}{2}} = 2\left(\frac{k-1}{k}\right) = 2\left(1 - \frac{1}{k}\right)$$

Better Algorithm:

Need a better LP relaxation!

Think of opt cut $F \subseteq E$.

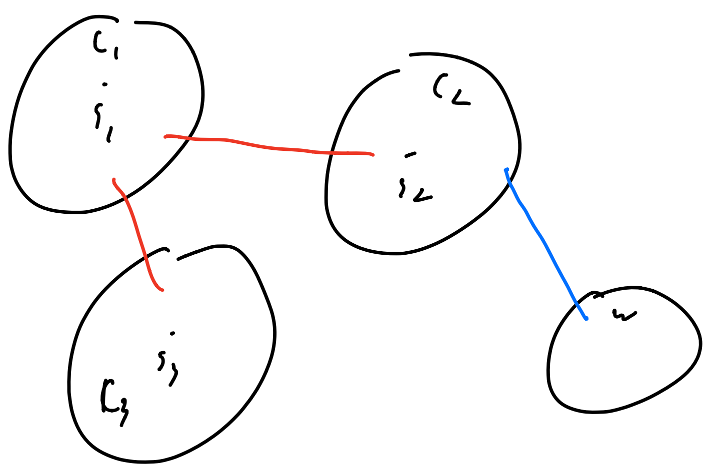
Let $C_i = \{w \in V : w \text{ reachable from } s_i \text{ in } G \setminus F\}$

$\Rightarrow C_i \cap C_j = \emptyset \quad \forall i \neq j \in [k]$

Claim: C_i's form partition

Pf: Need to show $w \in \bigcup_{i=1}^{k} C_i \quad \forall w \in V$

Sps false: WLOG, G connected



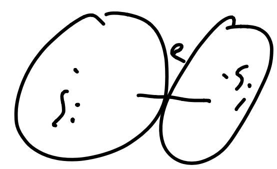
$\Rightarrow$ Different point of view:

find partition $C_1, C_2, \dots, C_k$ of V s.t. $s_i \in C_i$,

cut = edges between parts

New LP:

variables: $x_u^i = \begin{cases} 1 & \text{if } u \in C_i \\ 0 & \text{otherwise} \end{cases}$



$z_e^i = \begin{cases} 1 & \text{if } e \in \delta(C_i) \\ 0 & \text{otherwise} \end{cases}$

$$\min \ \frac{1}{2} \sum_{e \in E} \sum_{i=1}^{k} c(e) z_e^i \qquad \sum_{e \in E} c(e) \left(\frac{1}{2} \sum_{i=1}^{k} z_e^i \right)$$

$$\text{s.t.} \quad \sum_{i=1}^{k} x_u^i = 1 \qquad \forall u \in V$$

$$x_{s_i}^i = 1 \qquad \forall i \in [k]$$

$$z_e^i \geq x_u^i - x_v^i \qquad \forall e = \{u,v\} \in E, \ \forall i \in [k]$$

$$z_e^i \geq x_v^i - x_u^i \qquad \forall e = \{u,v\} \in E, \ \forall i \in [k]$$

$$0 \leq x_u^i \leq 1 \qquad \forall u \in V, \ \forall i \in [k]$$

$$0 \leq z_e^i \leq 1 \qquad \forall e \in E, \ \forall i \in [k]$$