1 Multicut in Trees (50 points)

Consider the multicut problem in trees. In this problem, we are given a tree $T = (V, E)$, $k$ pairs $(s_i, t_i)$ of vertices, and edge costs $c : E \rightarrow \mathbb{R}^+$. A feasible solution is a set $F \subseteq E$ such that for all $i \in [k]$, $s_i$ and $t_i$ are in different connected components of $T - F$. The objective is to minimize the total edge cost $\sum_{e \in F} c(e)$.

Let $P_i$ be the unique path between $s_i$ and $t_i$ in $T$. Then we can write an integer linear programming formulation of this problem:

$$\min \sum_{e \in E} c(e)x_e$$

subject to $\sum_{e \in P_i} x_e \geq 1 \quad \forall i \in [k]$  

$x_e \in \{0, 1\} \quad \forall e \in E$

(a) (25 points) Write the dual of the LP relaxation of the above ILP

Suppose that we root the tree at an arbitrary vertex $r$. Let $\text{depth}(v)$ be the number of edges on the path from $v$ to $r$. Let $\text{lca}(s_i, t_i)$ be the vertex $v$ on the path from $s_i$ to $t_i$ whose depth is minimum. Suppose that we use the primal-dual method to solve this problem, where the dual variable that we increase in each iteration corresponds to the violated (primal) constraint that maximized $\text{depth}(\text{lca}(s_i, t_i))$. After all primal constraints are satisfied, we do a “reverse cleanup” stage like in Steiner Forest, where we look at the edges we added in reverse order and remove them if we can do so while still having a feasible solution.

(b) (25 points) Prove that this is a 2-approximation. Hint: consider a path $P_i$ where the dual variable is nonzero. How many edges in the final solution can be on the path from $s_i$ to $\text{lca}(s_i, t_i)$, and how many can be on the path from $t_i$ to $\text{lca}(s_i, t_i)$?

2 Partition into 0-Trees (50 points)

Consider the following problem. The input is a graph $G = (V, E)$, a number $k$ where $|V|$ is an integer multiple of $k$, and costs $c : E \rightarrow \mathbb{R}^+$. A feasible solution is a set of edges $F \subseteq E$ so that $H = (V, F)$ is a forest and every tree $T$ in $H$ has $0 \pmod k$ vertices. The objective is to minimize $\sum_{e \in F} c(e)$. 


(a) (25 points) Write an LP relaxation for this problem, and write the dual LP. Hint: consider a
set $S \subseteq V$ with $|S| \not\equiv 0 \pmod{k}$. What do you know about any feasible solution and $\delta(S)$?

(b) (25 points) Design a primal-dual 2-approximation for this problem (12 points for the algo-
rithm, 13 points for the proof that it is a 2-approximation). Hint: consider the Steiner Forest
algorithm and analysis.