600.469 / 600.669 Approximation Algorithms **Lecturer:** Michael Dinitz
Topic: Chernoff bounds: Minimizing Congestion and 3-Coloring **Date:** 3/12/15
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12.1 Minimizing Congestion

- Valid instances : Graph $G = (V, E)$, pairs of vertices $\{s_i, t_i\} \subseteq V$, $i \in [k]$.
- Feasible solutions : Paths $p_1, \dots, p_k$ such that $p_i \in \mathcal{P}_i$. Here $\mathcal{P}_i$ denotes all the possible paths from s_i to t_i.
- Objective function : Let $cong(e) = |\{i \mid e \in p_i, i \in [k]\}|$. In other words, given a feasible solution of paths, $cong(e)$ is the number of paths that use e. Our objective is minimizing $\max_{e \in E} cong(e)$.
- Linear program : Consider linear program (**MIN-CONG**) below.

$$
\begin{array}{llll}
\text{minimize:} & W & & (\textbf{MIN-CONG}) \\
\text{subject to:} & \displaystyle\sum_{p \in \mathcal{P}_i} x_p = 1 & \text{for each } i \in [k] & (12.1.1) \\
& \displaystyle\sum_{i=1}^{k} \sum_{p \in \mathcal{P}_i : e \in p} \leq W & \text{for each edge } e \in E & (12.1.2) \\
& 0 \leq x_p \leq 1 & \text{for each } p \in \mathcal{P}_i, i \in [k] &
\end{array}
$$

- Algorithm : As always, first solve the LP to get the optimal solution $(\{x_p^*\}, W^*)$ of the linear program. For each $i \in [k]$, we choose a path $p \in \mathcal{P}_i$ from the distribution defined by $\{x_p^*\}_{p \in \mathcal{P}_i}$.

We define several random variables:

- Y_e : Number of chosen paths using edge e.
- Y_e^i : $Y_e^i = \begin{cases} 1, & \text{if path chosen for } i \text{ uses } e \\ 0, & \text{otherwise} \end{cases}$
- Z_p^i : $Z_p^i = \begin{cases} 1, & \text{if path } p \text{ is chosen for pair of endpoints } i \\ 0, & \text{otherwise} \end{cases}$

We know that $Y_e = \sum_{i=1}^{k} Y_e^i$ and that $Y_e^i = \sum_{p \in \mathcal{P}_i, e \in p} Z_p^i$, and thus $Y_e = \sum_{i=1}^{k} \sum_{p \in \mathcal{P}_i, e \in p} Z_p^i$. Therefore $\mathbb{E}[e] \leq W^*$ for each $e \in E$ (we did this analysis in the previous class).

We will use LP rounding for this problem. However, we have to be careful, since in fact this LP can be a very bad relaxation. This is formalized in the next theorem.

Theorem 12.1.1 *The integrality gap of this linear program (***MIN-CONG***) is at least $n-2$.*

Proof: Consider the graph $G=(V,E)$ with $V=\{s_1,t_1,v_1,v_2,\ldots,v_{n-2}\}$, $E=\{(u,v)\mid u\in\{s_1,t_1\}, v\in V\backslash\{s_1,t_1\}\}$ and $k=1$. Then for each path p of the form $s_1-v_i-t_1$ we can set $x_p^*=\frac{1}{n-2}$ and get a feasible fractional solution with $W^*=\frac{1}{n-2}$. But the optimal solution for the original problem is clearly 1, since *some* path must be chosen. Therefore the integrality gap is at least $n-2$. ■

However, this result is derived from the observation that $\mathbb{E}[Y_e]=\frac{1}{n-2}$ for all $e\in E$. In this instance, actually $\mathbb{E}[\max_{e\in E} Y_e]$ is still 1, and it's the same as the optimal solution.

Now we will use Chernoff Bounds to estimate the expectation of the maximum edge congestion.

Theorem 12.1.2 *Let $X_1,\ldots,X_n$ be independent random variables distributed over $[0,1]$. Let $X=\sum_{i=1}^n X_i$, then*

- $\forall 0<\varepsilon<1$,
$$\Pr[X>(1+\varepsilon)\mathbb{E}[X]]\le e^{-\frac{\varepsilon^2}{3}\mathbb{E}[X]}$$
$$\Pr[X<(1-\varepsilon)\mathbb{E}[X]]\le e^{-\frac{\varepsilon^2}{2}\mathbb{E}[X]}$$
- $\forall t>2e\mathbb{E}[X]$,
$$\Pr[X>t]\le 2^{-t}$$

We know that $Y_e=\sum_{i=1}^k Y_e^i$ and that $Y_e^i\in[0,1]$ are independent. So we can use Chernoff. Consider 2 cases of W^*:

- **Case** 1: $W^*\ge 1$. In this case, we have that $\forall e\in E$, $\Pr[Y_e>3\log n\cdot W^*]\le 2^{-3\log n\cdot W^*}\le\frac{1}{n^3}$. This is because $3\log n\cdot W^*\ge 3\log n\cdot\mathbb{E}[Y_e]\ge 2e\cdot\mathbb{E}[Y_e]$, so we can apply the second form of Chernoff.
- **Case** 2: $W^*<1$. In this case, we have that $\forall e\in E$, $\Pr[Y_e>3\log n]\le 2^{-3\log n}\le\frac{1}{n^3}$ (again by the second form of Chernoff).

Thus for every edge e, we get that $\Pr[Y_e>3\log n\cdot\max(W^*,1)]\le\frac{1}{n^3}$. Using a union bound over all edges, we have $\Pr[\max_{e\in E} Y_e>3\log n\max(W^*,1)]\le\frac{1}{n}$.

Therefore, with high probability, $\max_{e\in E}(cong(e))\le 3\log n\cdot\max(W^*,1)\le 3\log n\cdot OPT$. Hence it is an $O(\log n)$-approximation.

Remark:

If we use a better Chernoff bound, the approximation rate can be improved to $O(\frac{\log n}{\log\log n})$.

We can also get better result when W^* is big. $\forall 0\le\varepsilon\le 1$, if $W^*\ge\frac{6e}{\varepsilon^2}\ln n$, then Consider 2 cases of each $\mathbb{E}[Y_e]$:

Case 1: $\mathbb{E}[Y_e]\ge\frac{3}{\varepsilon^2}\ln n$

In this case, we have that $\Pr[Y_e>(1+\varepsilon)\mathbb{E}[Y_e]]\le e^{-\frac{\varepsilon^2}{3}\mathbb{E}[Y_e]}\le\frac{1}{n^3}$.

Case 2: $\mathbb{E}[Y_e] < \frac{3}{\varepsilon^2} \ln n$

In this case, we have that $W^* \geq 2e\mathbb{E}[Y_e]$, thus $\Pr[Y_e > W^*] \leq 2^{-W^*} \leq \frac{1}{n^3}$.

Hence with high probability, $\max_{e \in E}(cong(e)) \leq (1+\varepsilon)OPT$, so this algorithm is an $(1+\varepsilon)$-approximation.

12.2 3-Coloring on Dense 3-Colorable Graphs

This is not an approximation algorithms problem per se, but is a nice application of Chernoff bounds for a problem which is very related to approximation algorithms.

- Input : Constant $\delta > 0$, 3-colorable graph $G = (V, E)$ with $degree(v) \geq \delta n$ for each $v \in V$.
- Output : 3-coloring of G.
- Algorithm : Create a sample set $S \subseteq V$ by adding each $v \in V$ with probability $\frac{3C \ln n}{\delta n}$ with sufficient large constant C. Let $N(S)$ denote vertices with at least one neighbor in S. The algorithm *fails* if $|S| > \frac{6C \ln n}{\delta}$ or $N(S) \cup S \neq V$. If it does not fail, then try all possible coloring of S. For each possible coloring of S, convert the problem of coloring $N(S)$ to a 2SAT problem (which will be explained later). Solve this 2SAT problem.

Lemma 12.2.1 $\Pr[|S| > \frac{6C \ln n}{\delta}] \leq n^{-\frac{C}{\delta}}$.

Proof:

$$\Pr[|S| > \frac{6C \ln n}{\delta}] = \Pr[|S| > 2\mathbb{E}[|S|]] \leq e^{-\frac{1}{3}\frac{3C \ln n}{\delta}} = n^{-\frac{C}{\delta}},$$

where we use the first version of the Chernoff bound with $\epsilon = 1$. ■

Lemma 12.2.2 *For each $v \notin S$, $\Pr[v \notin N(S)] \leq n^{-3C}$.*

Proof:

$$\Pr[v \notin N(S)] \leq \left(1 - \frac{3C \ln n}{\delta n}\right)^{|N(v)|} \leq e^{-\frac{3C \ln n |N(v)|}{\delta n}} \leq e^{-3C \ln n} = n^{-3C},$$

where we use the fact than $|N(v)| \geq \delta n$ by assumption. ■

Lemma 12.2.3 *With probability $\geq 1 - \frac{2}{n^C}$, $|S| \leq \frac{6C \ln n}{\delta}$ and $N(S) \cup S = V$ (i.e. the algorithm does not fail).*

Proof: Use union bound, the probability is less or equal to

$$n^{-\frac{C}{\delta}} + n \cdot n^{-3C} \leq \frac{1}{n^C} + \frac{1}{n^{3C-1}} \leq \frac{2}{n^C}$$

■

If $|S| \leq 6C \ln n\delta$ and $N(S) \cup S = V$, fix a valid 3-coloring of S where $\forall u \in S, n(u) \in \{0, 1, 2\}$ (such a coloring must exist since we assumed at the start that the entire graph is 3-colorable).

For each $v \in V \backslash S$, we consider 3 cases:

Case 1: v adjacent to 3 nodes in S with different colors

In this case, v can not be colored by any color, so we try the next coloring of S (this coloring cannot be valid).

Case 2: v adjacent to 2 node in S with different colors

In this case, v can only be colored by one specific color. So we color v with that color.

Case 3: v adjacent to only one color in S

In this case, v can be colored by two specific colors.

Now we start converting the problem of coloring $N(S)$ to a 2SAT problem. Let $m(v) \in \{0, 1, 2\}$ be the color which is forbidden to v (since its neighbor(s) in S have that color).

We define variable $x(v) = \begin{cases} True, & \text{if } v \text{ is colored } m(v) + 1 \pmod 3 \\ False, & \text{if } v \text{ is colored } m(v) + 2 \pmod 3 \end{cases}$.

Because all the vertices have 2 possible colors, $x(v)$ is well defined.

Consider any edge $(u, v) \in E$ such that $u, v \notin S$. If u, v have the same possible colors, there are 4 ways to color them, 2 of these are bad colorings. If u, v have different possible colors, there are 4 ways to color them, 1 of these is bad coloring. For each bad coloring, we write a clause forbidding that coloring. For example: If $x(u) = True$ and $x(v) = False$ is a bad coloring, we add clause $(\overline{x(u)} \vee x(v))$.

With all of these clauses added, valid solutions to the 2SAT instance correspond to valid colorings (and vice versa). Because 2SAT can be solved in polynomial time, trying to extend any given coloring of S to the rest of the graph can be done in polynomial time. Moreover, since with high probability $|S| \leq \frac{6C \ln n}{\delta}$, there are at most a polynomial number of colorings ($3^{|S|}$). Therefore with high probability, this algorithm solves the problem in polynomial time.