

## 14.1 Preliminaries

This is a trying time for all of us. If you have any issues or concerns, feel free to get in touch. As a graduate class, the most important thing is that this course should be informative and fun. This isn't undergrad algorithms, where I'm teaching material that it is crucial that all computer scientists know. So I'm going to keep assigning homeworks like usual, and your final project will go on as usual, but I don't want you to stress out about this class.

Speaking of your final project, now would be a good time to start exploring possibilities. If you can find something related to your research, then that's awesome. Otherwise, consider looking through other parts of the textbooks, or some of the references, or the proceedings of STOC, FOCS, and SODA (the major theory conferences) and EC, WINE, and SAGT (the major AGT conferences). When you have an idea for a project, let me know and I'll make sure that it's reasonable. Feel free to work in small groups, if you want.

### 14.1.1 Online Teaching

I'm going to try to keep teaching basically like usual, but with Zoom instead of in person. We'll see how it works – hopefully it won't be too much worse than in-person lectures. But as you all know by now, I like to encourage participation in the class when possible, since otherwise it's easy for a lecture to just be me writing down a bunch of equations. Participation might be harder in Zoom, but let's all do our best to basically pretend that we're in lecture together.

In terms of technicalities, in order to prevent stray interfering audio I'm going to initially have everyone muted but me. If you want to ask a question or interject (please do!), there should be a “raise hand” button in your zoom (you might need to have “manage participants” up). Then I'll unmute you and we can discuss.

## 14.2 Connection Game: Introduction

Now we're going to talk about a different game known as the *connection game*, which is one of the simplest “network formation games”. Like with most of the other game we've talked about, we care about the game because it is interesting and important, but also to demonstrate a new concept or type of analysis. We'll use the connection game to talk about the price of *stability* (rather than anarchy) and to introduce a new type of equilibrium known as a *strong Nash*.

Let's start by defining the game.

- Directed graph  $G = (V, E)$ .

- Edge costs  $c_e \in \mathbb{R}_{\geq 0}$  for each  $e \in E$ .
- $k$  players, player  $i$  has a source/sink pair  $(s_i, t_i) \in V \times V$
- Strategies for player  $i$  are  $\mathcal{P}_i = \{s_i \rightarrow t_i \text{ paths}\}$ . Let  $S = \mathcal{P}_1 \times \dots \times \mathcal{P}_k$ .
- The cost of every edge is split among the edges using it. More formally:
  - Let  $s = (P_1, P_2, \dots, P_k) \in S$
  - For each  $e \in E$ , let  $k_e(s) = |\{i : e \in P_i\}|$ .
  - Then we define player costs as  $C_i(s) = \sum_{e \in P_i} \frac{c_e}{k_e(s)}$ .
- Our global cost will be  $\text{cost}(s) = \sum_{i=1}^k C_i(s) = \sum_{e \in \cup_{i=1}^k P_i} c_e$ .

It's pretty easy to see that the price of anarchy can be quite bad. Consider the following simple example: there are two nodes  $s$  and  $t$ , and all players have source  $s$  and sink  $t$ . There are two edges from  $s$  to  $t$ , the top with cost 1 and the bottom with cost  $k$ . Then OPT is for all players to use the top edge, for a global cost of 1. But if all players use the bottom edge then they all split its cost, so they all pay 1, and thus none of them have incentive to deviate to the top edge. Thus all players using the bottom edge is a Nash with global cost  $k$ . So the price of anarchy is at least  $k$ .

### 14.3 Price of Stability

Even though the price of anarchy is bad, what about the price of stability? Recall that the price of stability is the ratio of the *best* Nash to OPT, rather than the worst Nash. As a side note, this means that as we expand our notion of equilibrium (from pure Nash, to mixed, to correlated, to coarse correlated) the price of stability might get better, and it never gets worse (the opposite of the price of anarchy). So, in particular, if we can show that there's a good pure Nash, then we'll have bounded the price of stability for all notions of equilibrium which include pure Nash.

Consider the following example. There are  $k$  players, each one of which has their own source  $s_i$ . There is an edge from each  $s_i$  to a global sink  $t$ , with the cost of this edge being  $1/i$ . There is one other vertex  $v$ , and there is an edge of cost 0 from each  $s_i$  to  $v$ . Finally, there is an edge of cost  $(1 + \epsilon)$  from  $v$  to  $t$ .

OPT in this game is for every player to go from their source to  $v$  (along a zero-cost edge) and then to  $t$ , for a total cost of  $1 + \epsilon$ . But in this solution player  $k$  is paying  $(1 + \epsilon)/k$ , while if they switch to the edge  $(s_k, t)$  they would pay  $1/k$ . Once they've switched, the same logic will hold for player  $k - 1$ , etc. So it's not too hard to see that the unique pure Nash is when every player uses their direct edge. More formally, suppose that there is some set  $A \subseteq [k]$  of players that are using the path through  $v$  in some strategy profile  $s$ . Let  $i$  be the maximum value in  $A$ . Then  $C_i(s) = \frac{1+\epsilon}{|A|} \geq \frac{1+\epsilon}{i}$ . Hence  $s$  is not a pure Nash, since player  $i$  has incentive to switch.

So the only pure Nash is when all players use their direct edge. The total cost of this Nash is  $\sum_{i=1}^k \frac{1}{i} = H_k = \Theta(\ln k)$ . Hence the price of stability in this example is only  $H_k$  (the  $k$ 'th harmonic number). This implies that in general, the price of stability in the connection game is  $\Omega(\ln n)$ .

It turns out that this is the *exactly* the worst case, even up to constants!

**Theorem 14.3.1.** *The price of stability in any connection game is at most  $H_k$ .*

Before we prove this theorem, it will be useful for us to prove that this is actually a potential game. This is useful not only because it implies that there is a pure Nash, but because the potential function itself will be useful when we prove Theorem 14.3.1. We'll prove that the following function is a potential function (where recall that  $H_0 = 0$ ):

$$\begin{aligned}\Psi_e(s) &= c_e \cdot H_{k_e(s)} & \forall e \in E \\ \Psi(s) &= \sum_{e \in E} \Psi_e(s)\end{aligned}$$

Recall that in order to be a potential function, we need to prove that the change in the potential function when (exactly) one player deviates is equal to the change in cost of the player who deviates. So let's prove this.

**Lemma 14.3.2.** *Let  $s = (P_1, P_2, \dots, P_k) \in S$ , and let  $P'_i \in \mathcal{P}_i$ . Then*

$$\Psi(s) - \Psi(s_{-i}, P'_i) = C_i(s) - C_i(s_{-i}, P'_i)$$

*Proof.*

$$\begin{aligned}\Psi(s) - \Psi(s_{-i}, P'_i) &= \sum_{e \in E} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) \\ &= \sum_{e \in P_i \setminus P'_i} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) + \sum_{e \in P'_i \setminus P_i} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) \\ &\quad + \sum_{e \in P_i \cap P'_i} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) + \sum_{e \in E \setminus (P_i \cup P'_i)} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) \\ &= \sum_{e \in P_i \setminus P'_i} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) + \sum_{e \in P'_i \setminus P_i} (\Psi_e(s) - \Psi_e(s_{-i}, P'_i)) \\ &= \sum_{e \in P_i \setminus P'_i} (c_e H_{k_e(s)} - c_e H_{k_e(s)-1}) + \sum_{e \in P'_i \setminus P_i} (c_e H_{k_e(s)} - c_e H_{k_e(s)+1}) \\ &= \sum_{e \in P_i \setminus P'_i} \frac{c_e}{k_e(s)} - \sum_{e \in P'_i \setminus P_i} \frac{c_e}{k_e(s) + 1} \\ &= \sum_{e \in P_i \setminus P'_i} \frac{c_e}{k_e(s)} - \sum_{e \in P'_i \setminus P_i} \frac{c_e}{k_e(s_{-i}, P'_i)} \\ &= \sum_{e \in P_i} \frac{c_e}{k_e(s)} - \sum_{e \in P'_i} \frac{c_e}{k_e(s_{-i}, P'_i)} \\ &= C_i(s) - C_i(s_{-i}, P'_i) \quad \square\end{aligned}$$

Thus the connection game is a potential game, and so in particular any strategy profile that minimizes  $\Psi$  must be a pure Nash equilibrium (if any player had incentive to deviate then that deviation would result in a strategy profile which makes  $\Psi$  smaller).

One important thing to note about this lemma: it holds even if we replace  $P'_i$  by  $\emptyset$ . In other words, the change in the potential is equal to  $C_i(s) - 0$  if we “pretend” to allow player  $i$  to deviate to not play a path at all (which of course isn’t actually allowed in the game).

Let’s relate this potential function to our global cost function.

**Lemma 14.3.3.**  $cost(s) \leq \Psi(s) \leq H_k \cdot cost(s)$  for all  $s \in S$ .

*Proof.*

$$cost(s) = \sum_{e \in \cup_{i=1}^k P_i} c_e \leq \sum_{e \in E} c_e H_{ke}(s) = \Psi(s) = \sum_{e \in \cup_{i=1}^k P_i} c_e H_{ke}(s) \leq H_k \sum_{e \in \cup_{i=1}^k P_i} c_e = H_k \cdot cost(s) \quad \square$$

Now we can finally prove Theorem 14.3.1. Let  $s$  be a global minimizer of  $\Psi$  (and thus a pure Nash). Let  $s^*$  be the optimum (the global minimizer of cost). Then

$$cost(s) \leq \Psi(s) \leq \Psi(s^*) \leq H_k \cdot cost(s^*)$$

## 14.4 Strong Nash

What can we do if still want to analyze worst-case equilibria? How can we get around the bad example for the price of anarchy? One way would be to even further restrict our notion of equilibrium to a subset of the pure Nash, and hope that this subset doesn’t include any bad equilibria. Of course, we don’t want an arbitrary subset, but one which makes some kind of game-theoretic sense.

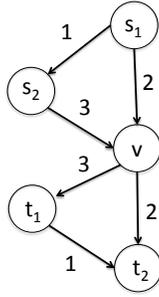
To get some intuition, let’s think about what kind of extra criterion would get rid of the bad example we started with. Intuitively, the bad Nash is extremely “unstable”: while no one wants to individually deviate to the top edge, if *two* players deviated to the top edge then they would each pay  $1/2$ , and so would be happier. In other words, if we allow “coalitions” of size two, then this bad Nash equilibrium is actually not an equilibrium (while the very good Nash equilibrium where everyone uses the bottom edge is still an equilibrium). Let’s try to formalize this intuition.

**Definition 14.4.1.** Let  $s \in S$ . The strategies  $s'_A \in \prod_{i \in A} S_i$  are a beneficial deviation for  $A \subseteq [k]$  if  $C_i(s_{-A}, s'_A) \leq C_i(s)$  for all  $i \in A$ , with strict inequality for at least one  $i \in A$ .

**Definition 14.4.2.**  $s \in S$  is a strong Nash if there are no beneficial deviations for any  $A \subseteq [k]$ .

Note that any strong Nash is also a pure Nash, since a pure Nash can be thought of as the special case where we only care about beneficial deviations for sets (coalitions) of size 1.

Unfortunately, strong Nash equilibria do not always exist, even in the connection game. Consider the following example.



Each player has two possible strategies: a 2-hop path and a 3-hop path. Let's consider the different strategy profiles and show that none of them are a strong Nash.

- If both players use their 2-hop paths then their paths do not overlap, so they each have a cost of five. But if they both deviated to their three-hop paths then they would be overlapping on  $(s_2, v)$  and  $(v, t_1)$ , so they each would have cost four.
- If they both use their 3-hop paths then they would both pay 4. But then if player 1 deviated to their 2-hop path they would instead of  $2 + (3/2) = 3.5$ . So player 1 has incentive to deviate.
- If one of them is using a 2-hop path and the other is using a 3-hop path, then whoever is using the 2-hop path has cost 3.5 and whoever is using the 3-hop path has cost 5.5. So the player using a 3-hop path has incentive to deviate to the 2-hop path (to have cost 4).

Thus there is no strong Nash, even though both using their 2-hop paths is a pure Nash.

This is unfortunate. However, we might get lucky and there might exist a strong Nash. And there are some interesting cases where we can guarantee that there is a pure Nash.

**Theorem 14.4.3.** *If  $s_i = s$  and  $t_i = t$  for all  $i \in [k]$  (all players have same source and destination) there there is a strong Nash equilibrium.*

*Proof.* Let  $s$  be a strategy profile in which every player chooses the same shortest  $s \rightarrow t$  path. If any group of players deviate, then they are on a longer path with fewer players to split the cost of the edges, and so will be paying more. Thus there are no beneficial deviations for any coalition.  $\square$

Now let's show that *if* there is a strong Nash, then it must be relatively close to optimal in the connection game.

**Theorem 14.4.4.** *In the connection game, every strong Nash  $s$  has  $cost(s) \leq H_k \cdot cost(s^*)$  for all  $s^* \in S$  (the strong price of anarchy is at most  $H_k$ ).*

*Proof.* Let  $s = (P_1, P_2, \dots, P_k) \in S$  be a strong Nash and let  $s^* = (P_1^*, P_2^*, \dots, P_k^*) \in S$ . Since  $s$  is a strong Nash, there are no beneficial deviations for any coalition, and in particular no coalition wants

to deviate to the strategies that they play in  $s^*$ . So consider the coalition  $A_k = [k]$ . Then there is some  $i_k \in A_k$  such that  $C_{i_k}(s) \leq C_{i_k}(s^*)$ . Let's remove  $i_k$  from the coalition to get  $A_{k-1} = A_k \setminus \{i_k\}$ .

But now  $A_{k-1}$  is a coalition which does not want to deviate to  $s^*$ , so there is some  $i_{k-1} \in A_{k-1}$  so that  $C_{i_{k-1}}(s) \leq C_{i_{k-1}}(s_{-A_{k-1}}, s_{A_{k-1}}^*)$ . Let's remove  $i_{k-1}$  from  $A_{k-1}$  to get  $A_{k-2} = A_{k-1} \setminus \{i_{k-1}\}$ .

We can keep iterating this argument to get an ordering of the players  $i_k, i_{k-1}, i_{k-2}, \dots, i_2, i_1$  so that

$$C_{i_j}(s) \leq C_{i_j}(s_{-A_j}, s_{A_j}^*)$$

for all  $j \in [k]$ . Then we can sum up all of these inequalities to get that

$$\text{cost}(s) = \sum_{i=1}^k C_i(s) = \sum_{j=1}^k C_{i_j}(s) \leq \sum_{j=1}^k C_{i_j}(s_{-A_j}, s_{A_j}^*)$$

Now note that  $C_{i_j}(s_{-A_j}, s_{A_j}^*) \leq C_{i_j}(s_{A_j}^*)$  for all  $j$ , since removing all of the players not in  $A_j$  (rather than having them play their strategies from  $s$ ) means that the costs to all other players can only go up, since there are fewer players to possibly split the cost of the edges they want to buy. Thus we can extend the above inequalities to get

$$\text{cost}(s) \leq \sum_{j=1}^k C_{i_j}(s_{A_j}^*)$$

Now let's use our potential function, but consider the "deviation" to  $\emptyset$  that we discussed earlier. We get

$$\begin{aligned} \text{cost}(s) &\leq \sum_{j=1}^k C_{i_j}(s_{A_j}^*) = \sum_{j=1}^k \left( C_{i_j}(s_{A_j}^*) - C_{i_j}(s_{A_j - i_j}^*, \emptyset) \right) \\ &= \sum_{j=1}^k \left( \Psi(s_{A_j}^*) - \Psi(s_{A_{j-1}}^*) \right) \\ &= \Psi(s^*) - \Psi(s_{\emptyset}^*) = \Psi(s^*) \\ &\leq H_k \cdot \text{cost}(s^*) \end{aligned}$$

□