1 Potential Games (50 points)

A team game is a game in which all players have the same utility function: \( u_1(s) = \cdots = u_n(s) \) for every outcomes \( s \). In a dummy game, the utility of every player \( i \) is independent of its strategy: \( u_i(s) = u_i(s_{-i}, s'_i) \) for every \( s \in S \) and \( s'_i \in S_i \).

Prove that a game with utilities \( u_1, \ldots, u_n \) is a potential game (i.e., admits a potential function \( \Phi \)) if and only if it is the sum of a team game \( u^t_1, \ldots, u^t_n \) and a dummy game \( u^d_1, \ldots, u^d_n \) (i.e., \( u_i(s) = u^t_i(s) + u^d_i(s) \) for all \( i \) and \( s \)).

**Solution:** Suppose that our game is a potential game, with potential function \( \Phi \). Let \( u^t_i(s) = \Phi(s) \) for all \( i \in [n] \) and \( s \in S \). Note that by definition, the \( u^t_i \)'s form a team game. Let \( u^d_i(s) = u_i(s) - \Phi(s) \). Then clearly \( u_i(s) = u^t_i(s) + u^d_i(s) \) for all \( i \) and \( s \), so we just need to show that the \( u^d_i \)'s form a dummy game. To prove this, by the definition of a potential function we have that

\[
u^d_i(s_{-i}, s'_i) = u_i(s_{-i}, s'_i) - \Phi(s_{-i}, s'_i) = u_i(s) - \Phi(s) = u^d_i(s),\]

and thus the \( u^d_i \)'s form a dummy game.

For the other direction, assume that our game is the sum of a team game and a dummy game. Let \( \Phi(s) = u^t_i(s) \) (which by the definition of a team game is also equal to \( u^t_i(s) \) for all \( i \in [n] \)). We claim that \( \Phi \) is a potential function, thus implying that our game is a potential game. A few simple calculations suffices to prove this:

\[
u_i(s_{-i}, s'_i) - u_i(s) = u^t_i(s_{-i}, s'_i) + u^d_i(s_{-i}, s'_i) - u^t_i(s) - u^d_i(s) \\
= u^t_i(s_{-i}, s'_i) + u^d_i(s) - u^t_i(s) - u^d_i(s) \\
= u^t_i(s_{-i}, s'_i) - u^t_i(s) \\
= \Phi(s_{-i}, s'_i) - \Phi(s)
\]

2 No-Regret and Coarse Correlated Equilibria (50 points)

We proved that if every player uses a no-regret algorithm, then the time-averaged history converges to a coarse correlated equilibrium. Let’s prove almost the converse. Consider a
Let’s assume for now that the nonzero probabilities in \( \sigma \) are rational. Let \( M \) be their GCD. So there exists a function \( f : \{0, 1, \ldots, M - 1\} \to S \) so that \( \frac{1}{M}|\{t \in \{0, 1, \ldots, M - 1\} : f(t) = s\}| = \sigma(s) \). We define the algorithm \( A_i \) as follows:

(a) At time \( t \), play action \( f(t \mod M) \).

(b) If the cost at time \( t \) is not \( C_i(f(t \mod M)) \), then switch the algorithm and for the rest of time use multiplicative weights.

We need to prove that if every player \( i \) uses \( A_i \) then the time-averaged history converges to \( \sigma \) as \( T \) tends to infinity, and that each \( A_i \) is a no-regret algorithm.

Let’s prove first that if every player uses its assigned algorithm, then the second step of the algorithm is never executed by any player (i.e., no player switches to multiplicative weights). We prove this by induction. At time 0 it is clearly true since every player \( j \) will play action \( f(0)_j \) and thus the realized outcome will be \( f(0) \) and so the cost to player \( i \) will be \( C_i(f(0)) \), and thus no player will execute the second step and switch to multiplicative weights. If no player has switched to multiplicative weights by the beginning of time \( t \), then in time \( t \) player \( j \) will play \( f(t \mod M)_j \) and thus the realized outcome will be \( f(t \mod M) \) and so the cost to player \( i \) will be \( C_i(f(t \mod M)) \), and thus the player will not switch to multiplicative weights.

Now we know that if every player uses its assigned algorithm then they never switch to multiplicative weights. Let \( k \) be the largest integer such that \( kM \leq T \). Then for every \( s \in S \), the time-averaged fraction in which it appeared is

\[
\frac{kM}{T} \sigma(s) \leq |\{t \in \{0, 1, \ldots, T - 1\} : f(t \mod M) = s\}| < \frac{(k + 1)M}{T} \sigma(s),
\]

and thus as \( T \) goes to infinity this converges to \( \sigma(s) \).

Now we need to prove that each \( A_i \) is a no-regret algorithm. There are two cases: if \( i \) eventually switches to multiplicative weights, or if doesn’t. If it does switch, then after it switches it is (by definition) a no-regret algorithm, and thus the regret tends to 0 as \( T \) tends towards infinity as required. On the other hand, if it never switches, then since \( C_i(s) \neq C_i(s') \) for all \( i, s, s' \) we know that the time-averaged distribution of play converges to \( \sigma \) which is a CCE, and thus by the definition of a CCE player \( i \) would not be able to do better by switching to any fixed action and thus the algorithm is no-regret.

If the probabilities are irrational, then instead of a cycle of time \( M \) we just need a function \( f : \mathbb{N} \to S \) so that as \( T \) tends towards infinity the fraction of time in which \( f(t) = s \) approaches \( \sigma(s) \) for all \( s \in S \). Such functions clearly exist, so we can just use one of them.