1. (20 points) Show that for every $i$, if $\Sigma_i^p \subseteq \Pi_i^p$ or $\Pi_i^p \subseteq \Sigma_i^p$, then $\Sigma_i^p = \Pi_i^p$.

2. (20 points) Prove that for every $i$, if $\Sigma_i^p = \Pi_i^p$, then the polynomial hierarchy collapses to the $i$th level. Prove also the reverse direction, i.e., if the polynomial hierarchy collapses to the $i$th level, then $\Sigma_i^p = \Pi_i^p$.

3. (20 points) A language $L \subseteq \{0, 1\}^*$ is sparse if there is a polynomial $p$ such that $|L \cap \{0, 1\}^n| \leq p(n)$ for every natural number $n$. Show that every sparse language is in $\mathsf{P/poly}$.

4. (20 points) Let $k \leq n$. Prove that the following family $H_{n,k}$ is a collection of pairwise independent hash functions from $\{0, 1\}^n$ to $\{0, 1\}^k$: Identify $\{0, 1\}$ with the field $GF(2)$. For every $k \times n$ matrix $A$ with entries in $GF(2)$, and $b \in GF(2)^k$, $H_{n,k}$ contains the function $h_{A,b} : GF(2)^n \rightarrow GF(2)^k$ defined as $h_{A,b}(x) = Ax + b$. Note: the field $GF(2)$ is just $\{0, 1\}$, and the operations in $GF(2)$ are just normal operations modulo 2, i.e., $1 + 1 = 0 + 0 = 0, 1 + 0 = 0 + 1 = 1, 1 \times 1 = 1, 1 \times 0 = 0 \times 1 = 0 \times 0 = 0$.

5. (20 points) Prove that for every $\mathsf{AM}[2]$ protocol for a language $L$, if the prover and the verifier repeat the protocol $k$ times in parallel (verifier runs $k$ independent random strings for each message) and the verifier accepts only if all $k$ copies accept, then the probability that the verifier accepts $x \notin L$ is at most $(1/3)^k$. (Note that you cannot assume that the prover is acting independently in each execution).