1. (20 points) Let $G$ be a $d$-regular graph with adjacency matrix $A$.

   (a) Show that $A$ has no eigenvalue larger than $d$ in absolute value.

   (b) Show that the multiplicity of the $d$ eigenvalue is $k$ if and only if $G$ has $k$ connected components.

2. (20 points) Given a graph $G = (V, E)$ with $|V| = n$, recall the cover time of a random walk on $G$ is defined as $cov = \max_{u \in V} E_u C$, where $C$ is the first time the random walk visits all vertices of $G$. Recall that the maximum hitting time is defined as $h_{max} = \max_{u,v \in V} E_u H_v$. Show that

   $$cov = O(h_{max} \log n).$$

   **Hint:** Break the random walk into intervals of length $O(h_{max})$, and use Markov’s inequality for each interval.

3. (20 points) Let $G$ be a complete graph on $n$ vertices. Compute the cover time for a random walk on $G$ exactly.

4. (20 points) An $(n, k, d)$ code is said to be a perfect code if for every $v \in \mathbb{F}_q^n$ there is exactly one codeword $w$ such that $\Delta(v, w) < d/2$. The Hamming code is a $(n = 2^\ell - 1, k = 2^\ell - \ell - 1, d = 3)$ binary code for any integer $\ell > 1$. Show that the Hamming code is a perfect code.

5. (20 points) A group of $n$ players first agree on a strategy and then play the following game. A hat is placed on each player’s head that is red or blue, each with probability half and independent of the other hats. Each player can see the hats of all other players, but not his own. Next, each player is given the option to guess the color of his hat; he may either make no guess or guess “red” or “blue”. When making his guess, a player may not see other players’ guesses. The players win if at least one player guesses and all guesses are correct.

   Show a strategy for the players which achieves success probability $1 - 1/(n+1)$ if $n = 2^t - 1$ for some integer $t \geq 2$.

   **Hint:** Use the previous problem. Think about directing the edges of the $n$-dimensional hypercube. It’s helpful to first show how to achieve probability $3/4$ with $n = 3$. 
