(25 points) Prove that the following languages are not regular:

(a) $L_1 = \{0^n1^m0^{n+m}|m, n > 0\}$

Proof. Suppose for sake of contradiction that $L_1$ is regular. Then $L_1$ is pumpable, so $\exists$ a pumping length $p$ such that if $s \in L_1$ and $|s| \geq p$ then we can divide $s = xyz$ where $\forall i \geq 0, xy^iz \in L_1$, $|y| > 0$, and $|xy| \leq p$.

Let $p$ be the pumping length and consider $s = 0^p1^p0^{2p}$. Clearly $s \in L_1$. Since $|xy| < p$, we have that $y$ must only contain $0$s. Then $xy^2z = 0^{p+|y|}1^p0^{2p} \notin L_1$ since $|y| > 0$. This exhausts all cases of the split $xyz$, so $L_1$ is not pumpable, which implies by contrapositive that $L_1$ is not regular. \qed

(b) $L_2 = \{w|w \in \{0, 1\}^* \text{ and } w \text{ is not a palindrome}\}$

Assume for sake of contradiction that $L_2$ is regular. Then $(L_2)^c$ (the complement of $L_2$) must be regular and therefore pumpable, so $\exists$ a pumping length $p$ such that if $s \in (L_2)^c$ and $|s| \geq p$ then we can divide $s = xyz$ where $\forall i \geq 0, xy^iz \in L_2$, $|y| > 0$, and $|xy| \leq p$.

Let $s = 0^p1^p$. $s$ is a palindrome. If were to split $s$ into $xyz$, then $y$ must contain only $0$s (since $|xy| < p$) then $xy^2z = 0^{p+|y|}1^p0^{2p} \notin L_2$ since $|y| > 0$. This exhausts all cases of the split $xyz$, so $(L_2)^c$ is not regular. Since regular languages are closed under complement, this means $L_2$ is not regular.

2

(25 points)

(a) Let $B = \{1^k \alpha | \alpha \in \{0, 1\}^* \text{ and } \alpha \text{ contains at least } k \text{ } 1\text{s, for any } k \geq 1\}$.
Is $B$ a regular language? Prove your answer.

$B$ is regular. We claim that $B$ is equivalent to the following language:

$$B' = \{ 1y | y \in \{0, 1\}^* \text{ and } y \text{ contains at least one } 1 \}$$

To see this, first notice that by definition we have that $B' \subseteq B$. Now we show that $B \subseteq B'$.

For any string $w \in B$ where $w = 1^k y$ and $y$ contains at least $k$ 1s, if $k = 1$ then of course $w \in B'$.

Otherwise we have $k \geq 2$. This means $w = 11w'y$, so if we let $y' = 1wy$ then $w = 1y'$ where $y'$ has at least one 1. Therefore $w \in B$ and thus $B = B'$.

Since $B'$ can be represented by the regular expression $R = 1\{0, 1\}^*1\{0, 1\}^*$, $B$ is a regular language.

(b) Let $C = \{ 1^k y | y \in \{0, 1\}^* \text{ and } y \text{ contains at most } k 1s, \text{ for any } k \geq 1 \}$

Is $C$ a regular language? Prove your answer.

No, $C$ is not regular. We can prove this by contradiction.

Assume for sake of contradiction that $C$ is regular. Then by the pumping lemma $\exists$ a pumping length $p$ where any $s$ in $C$ such that $|s| \geq p$ can be split into $xyz$ where $\forall i \geq 0$, $xy^iz \in B$, $|y| > 0$, and $|xy| \leq p$.

Consider the string $1^p01^p$. This string is in $C$ and has length $2p + 1 > p$. However, since $|y| > 0$ and $|xy| \leq p$, we have that $y$ must contain only 1s. Thus if we pump down, e.g. take $xy^0z$ then we get $xy^0z = 1^{p-|y|}01^p \notin C$ since $|y| > 0$. Therefore $C$ is not regular.

3

(25 points) Consider the language $F = \{ a^ib^jc^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k \}$, where $\Sigma = \{a, b, c\}$.

(a) Show that $F$ is not regular. **Hint:** Assume $F$ is regular, use the closure properties of regular languages to convert $F$ into another language, and show that language is not regular.

(b) Show that $F$ acts like a regular language in the pumping lemma. In other words, give a pumping length $p$ and demonstrate that $F$ satisfies the three conditions of the pumping lemma for this value of $p$.

(c) Explain why parts (a) and (b) do not contradict the pumping lemma.

**Solution.** (a) Assume $F$ is regular, consider $F \cap \{ ab^j c^k | j, k \geq 0 \} = \{ ab^i c^i | i \geq 0 \}$. Let’s denote this language as $F'$.

Note that $\{ ab^j c^k | j, k \geq 0 \} = ab^*c^*$ is regular. So $F'$ is regular since the class of regular languages is closed under intersections.

Let $p'$ be the pumping length for $F'$. Consider $ab^{p'}c^{p'}$. By the pumping lemma if viewing this string as $xyz$ then $y$ can be $b^t$ or $ab^t$ or $a$ for some $1 \leq t \leq p' - 1$. Consider $xy^2z$. If $y$ is $b^t$ then
xy^2z has more b’s than c’s. Otherwise if y is ab^i or a then xy^2z has more than one a. This means that xy^2z \notin F' and thus F' is not regular. So F is not regular either.

(b) Let p = 2. By the pumping lemma, consider every string s \in F with |s| \geq 2 and the partition s = xyz with |y| \geq 1, |xy| \leq p = 2. We have three cases.

- If the number of a’s in S is odd, then it’s possible that y = a, x = \epsilon. So xy^iz \in F for every i \in \mathbb{N}. This is because if s has one a, then s = ab^i c^i and it doesn’t matter how the number of a’s changes, the string is always in F. Otherwise the number of a’s in s is at least 3, so after pumping the number of a’s is at least 2, and the new string is in F regardless of the part of b^j c^k.

- If the number of a’s in s is even (excluding 0), then it’s possible that y = aa, x = \epsilon. Thus xy^iz \in F for every i \in \mathbb{N} because after pumping the number of a’s is still even, therefore the new string is in F regardless of the part of b^j c^k.

- If there is no a in s, then y can be b if there are some b’s in the string, or c if there is no b in the string. For either case, xy^iz \in F for every i \in \mathbb{N}.

So for all cases, the pumping lemma is met.

(c) A language that does not meet the pumping lemma is not regular. But a language that meets the pumping lemma may not be regular. In other words, every regular language must satisfy the pumping lemma, but there can also be non-regular languages that satisfy the pumping lemma.

4

(25 points) Give a context-free grammar that generates the following language, where the alphabet \Sigma is \{0, 1\}: \{w | w is not empty and starts and ends with the same symbol. \}

\[
S \rightarrow 0 \mid 1 \mid 0A0 \mid 1A1 \\
A \rightarrow A0 \mid A1 \mid \epsilon
\]

Note: the singletons 0 and 1 satisfy the condition of the language, and the grammar rule A \rightarrow \ldots essentially generates \Sigma^*.