Problem 1

Answer. The DFA $M = (Q, \Sigma, \delta, p_0, F)$ diagram is as follows.

Here $Q = \{p_0, p_1, p_2, p_3, p_4\}$, $\Sigma = \{0, 1\}$.

The transition function $\delta$ is as follows.

- $\delta(p_0, 0) = p_4, \delta(p_0, 1) = p_1$.
- $\delta(p_1, 0) = p_4, \delta(p_1, 1) = p_2$.
- $\delta(p_2, 0) = p_4, \delta(p_2, 1) = p_3$.
- $\delta(p_3, 0) = p_4, \delta(p_3, 1) = p_4$.
- $\delta(p_4, 0) = p_4, \delta(p_4, 1) = p_4$.

The start state is $p_0$.

The set of accept states is $F = \{p_0, p_1, p_4\}$.

Problem 2

Answer. The DFA $M = (Q, \Sigma, \delta, p_1, F)$ diagram is as follows.

Here $Q = \{p_1, p_2, \ldots, p_8\}$, $\Sigma = \{0, 1\}$.

The transition function $\delta$ is as follows.

- $\delta(p_1, 0) = p_2, \delta(p_1, 1) = p_3$.
- $\delta(p_2, 0) = p_1, \delta(p_2, 1) = p_4$.
- $\delta(p_3, 0) = p_4, \delta(p_3, 1) = p_5$.
- $\delta(p_4, 0) = p_3, \delta(p_4, 1) = p_6$. 

The start state is $p_1$. The accept states form set $F = \{p_1, p_3, p_5, p_6, p_7\}$.

Problem 3

**Answer.** If the language $A$ is regular, then there is a DFA $M = (Q, \Sigma, \delta, p_0, F)$ that recognizes $A$.

Now we construct $M' = (Q', \Sigma, \delta', p', F')$. The state set $Q' = Q \cup \{p'\}$. For every $p, q \in Q$ and every $a \in \Sigma$, if $\delta(p, a) = q$, then let $\delta'(q, a) = p$. Also let $\delta'(p', \epsilon) = q$, for every $q \in F$. Let $F' = \{p_0\}$.

So $M'$ is an NFA, according to the definition of NFA. Next we prove that $M'$ recognizes $A^R$.

For every $w \in A$, by reading $w$, $M$ goes through a path $P = (x_1, x_2, \ldots, x_t)$ from $x_1 = p_0$ to a state $x_t \in F$, for some positive integer $t$. Consider $w^R \in A^R$. By reading $w^R$, $M'$ goes through the path $P' = (p', x_t, \ldots, x_2, x_1)$, from $p'$ to $x_1 = p_0 \in F'$. So $w^R$ is accepted by $M'$.

On the other hand, for every $w$ that can be accepted by $M'$, by reading $w$, $M'$ follows a path $P' = (x_1, x_2, \ldots, x_t)$ for some positive integer $t$, where $x_1 = p', x_t = p_0$ and $x_2 \in F$. As a result, by reading $w^R$, $M$ follows the path $P = (x_t, \ldots, x_2)$ from $x_t = p_0$ to a state $x_2 \in F$. This means $w^R \in A$. So $w \in A^R$.

This proves that $M'$ recognizes $A^R$.

Problem 4

**Answer.** The language $B_n$ is regular as it can be recognized by the following DFA $M$, represented as a diagram.
Problem 5

Answer. If $A$ is regular, there exists a DFA $M$ that recognizes $A$.

Assume $M = (Q, \Sigma, \delta, p_0, F)$. Notice that the only difference between $M$ and $M'$ is the set of accept states.

Now we prove that there exists a set $F' \subseteq Q$ such that $M'$ recognizes $A/B$.

We define $F'$ in the following way.

A state $p \in Q$ is in $F'$ if and only if there exists a string $w \in B$ such that, starting from $p$ and following $w$, $M$ ends at a state $q \in F$.

Next we prove that $M'$ recognizes $A/B$.

For any string $u$ accepted by $M'$, starting at $p_0$, following $u$, $M'$ ends at some $p^* \in F'$. Then according to the definition of $F'$, there exists some $w \in B$ such that starting from $p^*$ and following $w$, $M$ ends at a state $q \in F$. So starting from $p_0$, following $u \circ w$, $M$ ends at $q \in F$. Thus $u \circ w \in A$. As a result, $u \in A/B$.

For any string $u \in A/B$, according to the definition of $A/B$, there is a string $u \circ w \in A$ where $w \in B$. Let the state that $M'$ or $M$ reaches, after reading $u$, be $p^*$. As $u \circ w \in A$, starting from $p^*$ and following $w$, $M$ ends at a state $q \in F$. Also we know that $w \in B$. So $p^* \in F'$ according to the definition of $F'$. Thus $M'$ accepts $u$.

This proves that $M'$ recognizes $A/B$. 

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