Wheeler graphs, part 2

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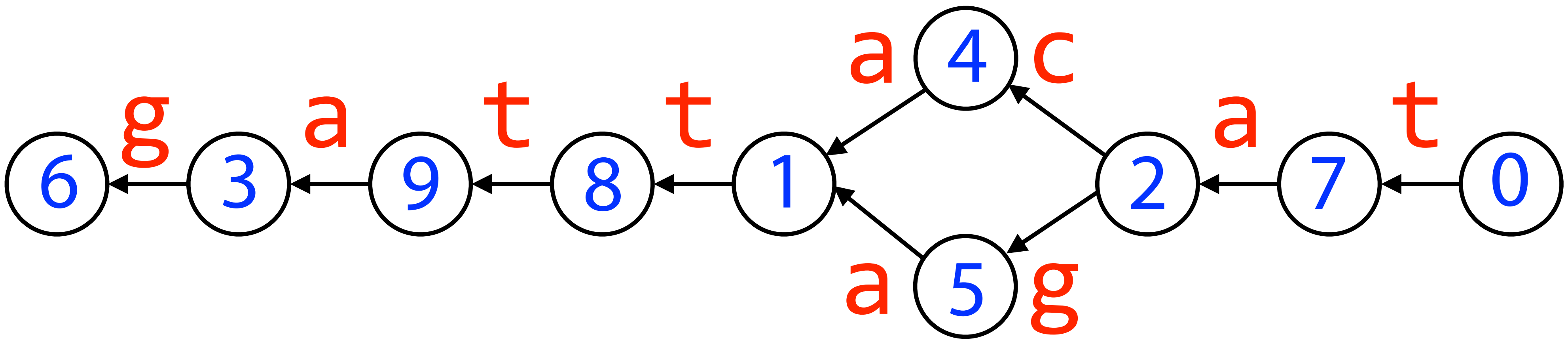
Wheeler graphs



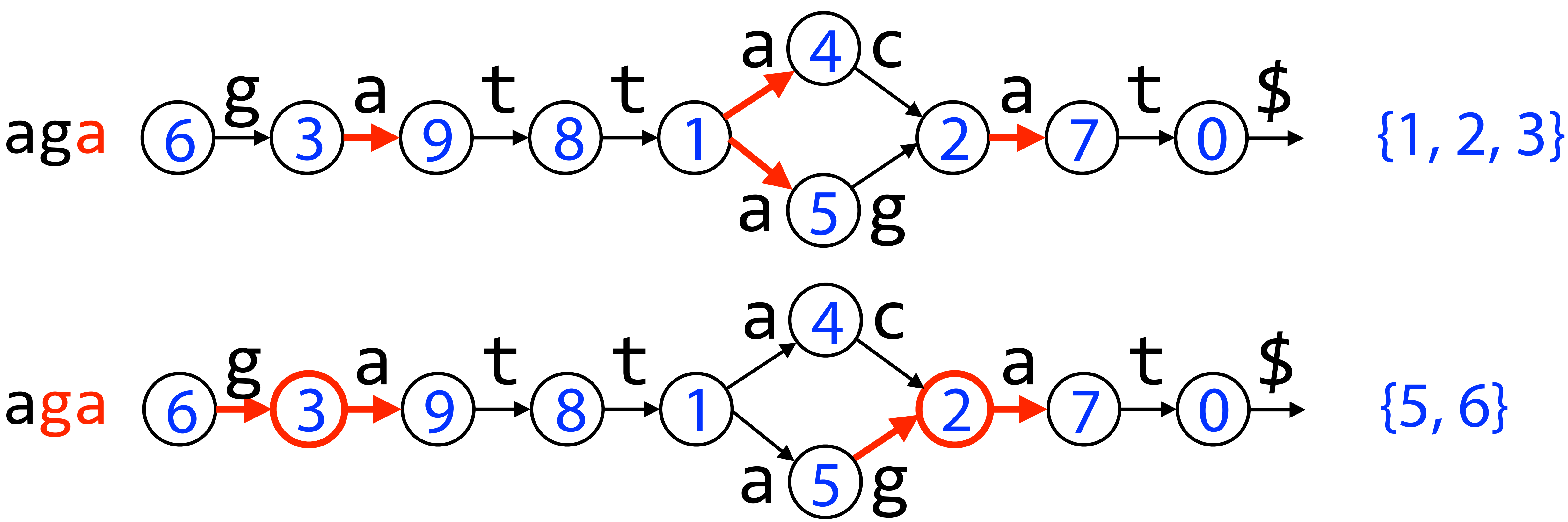
Can we show Wheeler Graphs are “special,” giving us analogs to FM index queries?

We start with “path coherence,” the graph version of ***consecutivity***

Wheeler graphs

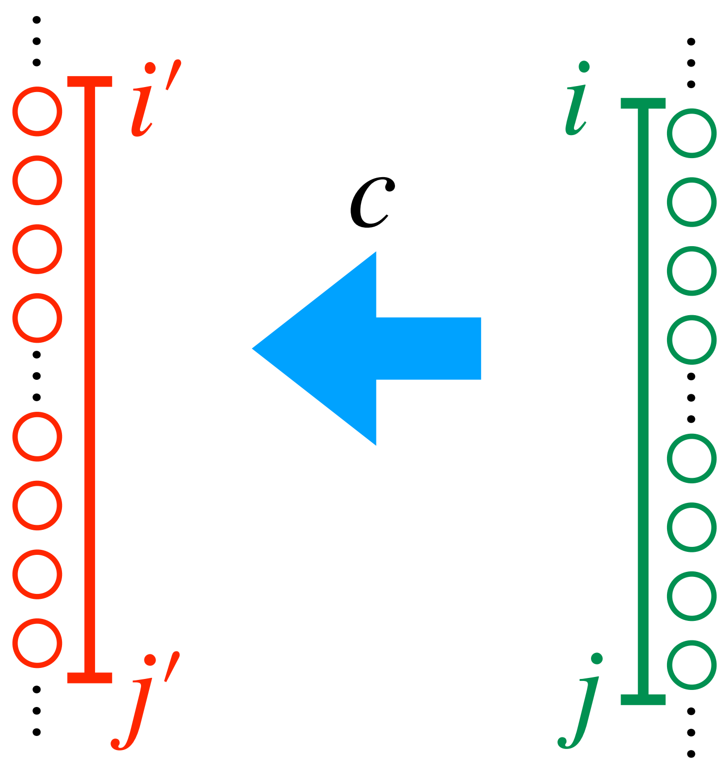
A graph is ***path coherent*** if there is a total order of the nodes such that:

For any consecutive range $[i, j]$ of nodes and string α, the nodes reached by following edges matching α also form a consecutive range



Wheeler graphs

Consider a single step where our initial set of nodes are in consecutive range $[i, j]$ and, after advancing on a single character $c \in \Sigma$, $[i', j']$ is the smallest range containing our next set of nodes

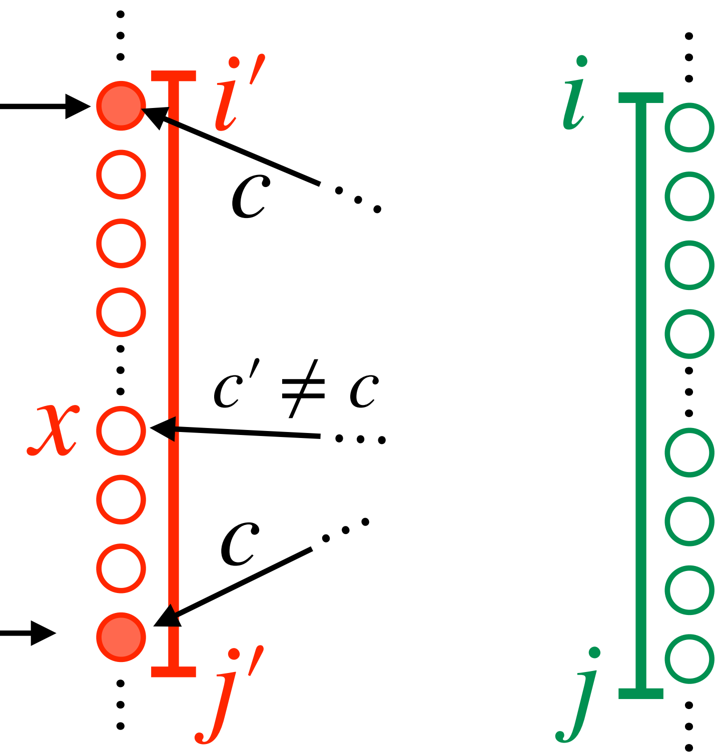


We want to show that the nodes in $[i', j']$ consist *only* of the c-successors of nodes $[i, j]$

Wheeler graphs

As defined, i' is reachable via an edge labeled c from a node in $[i, j]$

Same for j'



Consider node x, where $i' < x < j'$ with incoming edge labeled c'. Suppose $c' \neq c$.

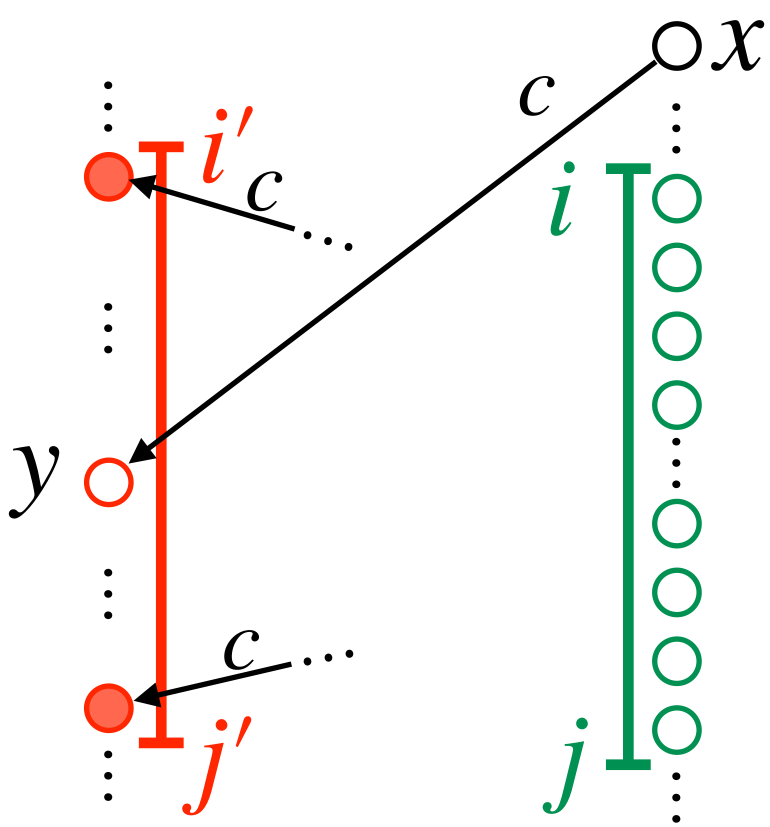
Recall: $a \prec a' \implies v < v'$

Since $x \not< i'$, we have $c' \nprec c$

Since $j' \not< x$, we have $c \nprec c'$

We have $c' \succcurlyeq c$, $c \succcurlyeq c'$, and $c' \neq c$, giving a contradiction

Wheeler graphs



Could node $x \notin [i, j]$ be a $c-$predecessor of a node y, $i' < y < j'$?

No. Proof is by contradiction, similar to previous argument and using:

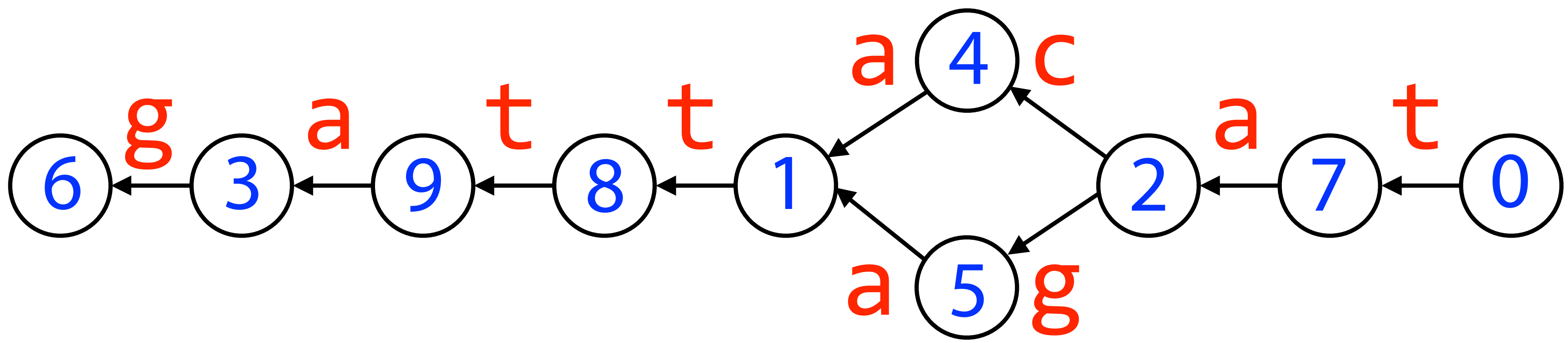
$$(a = a') \wedge (u < u') \implies v \leq v'$$

Wheeler graphs

For any consecutive range $[i, j]$ of nodes and string α, the nodes reached by following edges matching α also form a consecutive range

Proof takes previous arguments and extends them to string α inductively

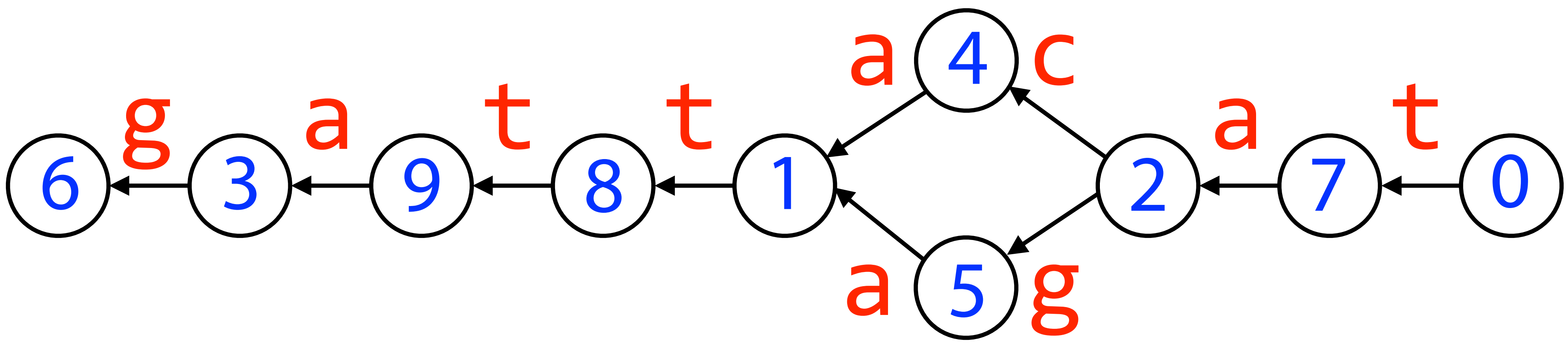
Wheeler graphs



How would we represent a Wheeler graph with bitvectors?

Need to represent *structure* as well as *node and edge labels*

Wheeler graphs

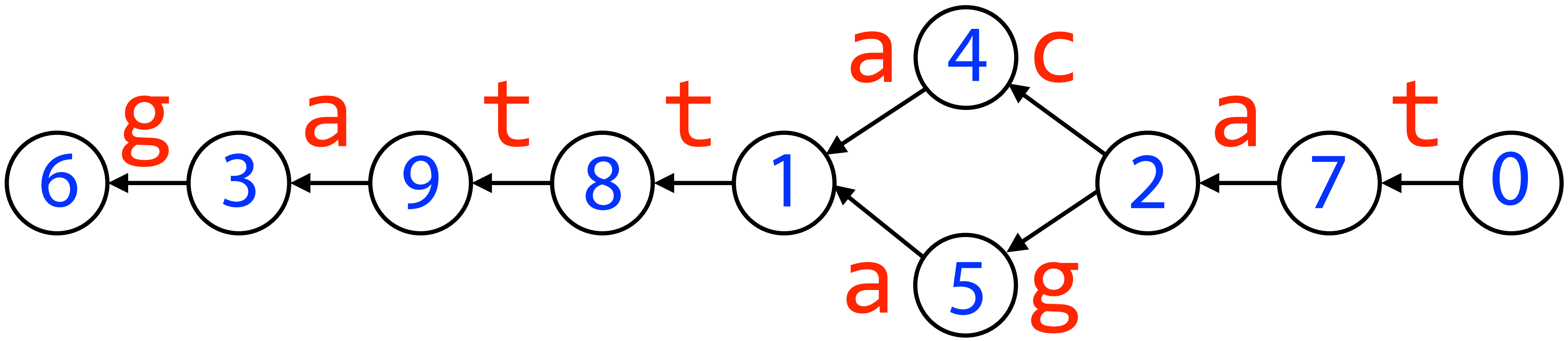


Idea 1: Encode in- and outdegree of each node in unary

Idea 2: Concatenate in order by node

Here, O = 01010010101011010101
0 1 2 3 4 56 7 8 9

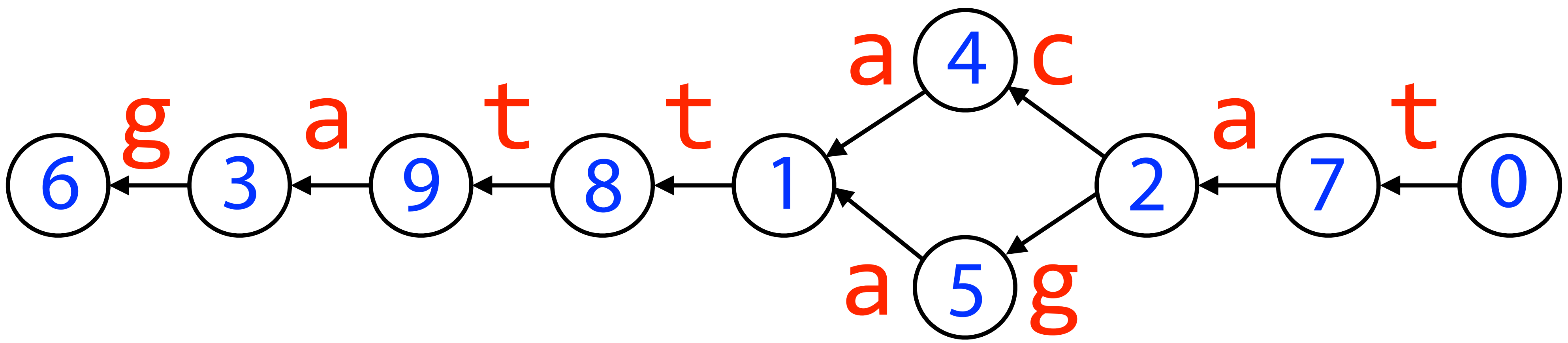
Wheeler graphs



$I =$ 10010101010101010101

0 1 2 3 4 5 6 7 8 9

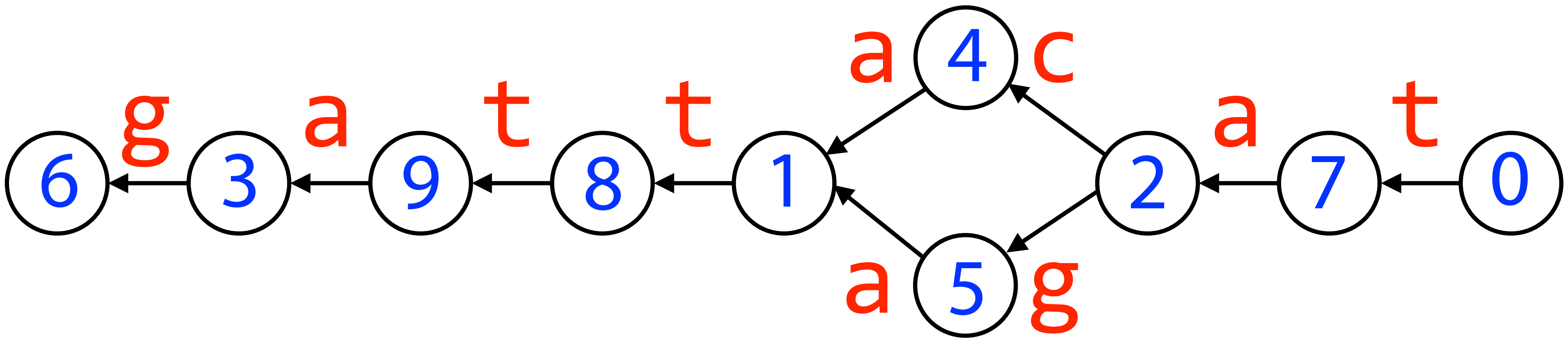
Wheeler graphs



Idea 3: Encode edge labels corresponding to 0s in O

$O =$ 01010010101011010101
$L =$ t t cg g a a a t a

Wheeler graphs



$I =$ 10010101010101010101

$O =$ 01010010101010110101

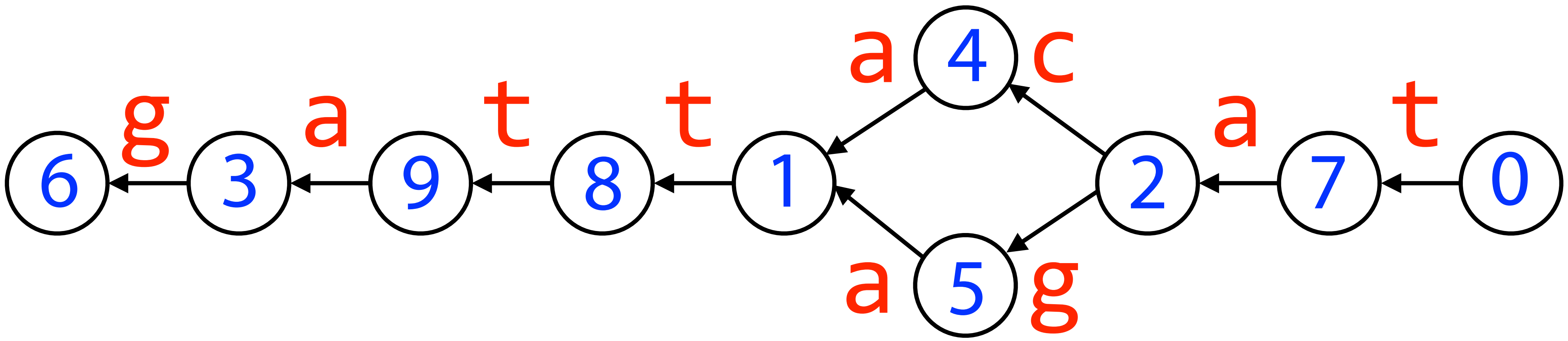
$L =$ ttcggaaata

How long is I? (# edges) + (# nodes) bits

How long is O? (# edges) + (# nodes) bits

How long is L? (# edges) chars

Wheeler graphs



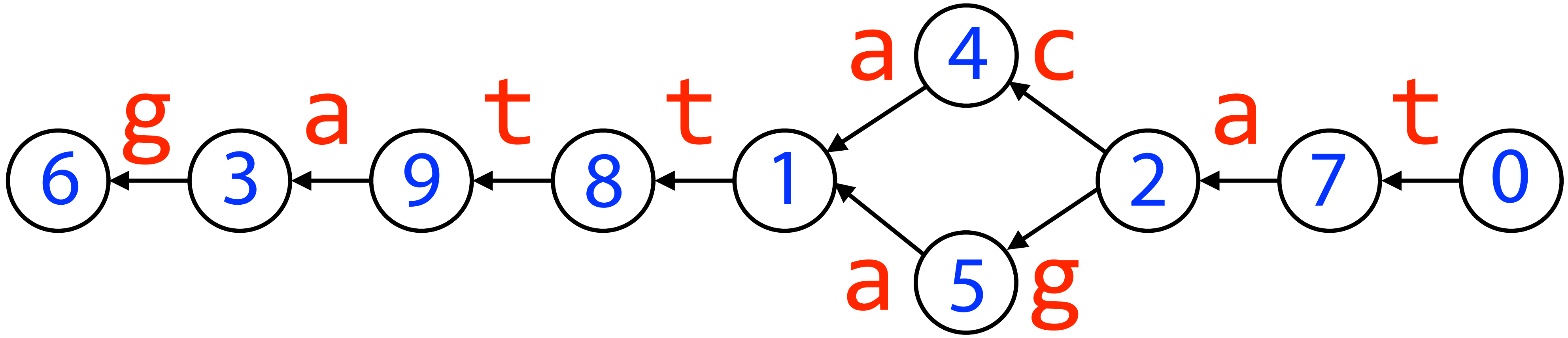
$$I = 10010101010101010101$$

$$O = 01010010101011010101$$

$$L = \texttt{ttcggaaata}$$

How to find in- or outdegree of a node?

Wheeler graphs



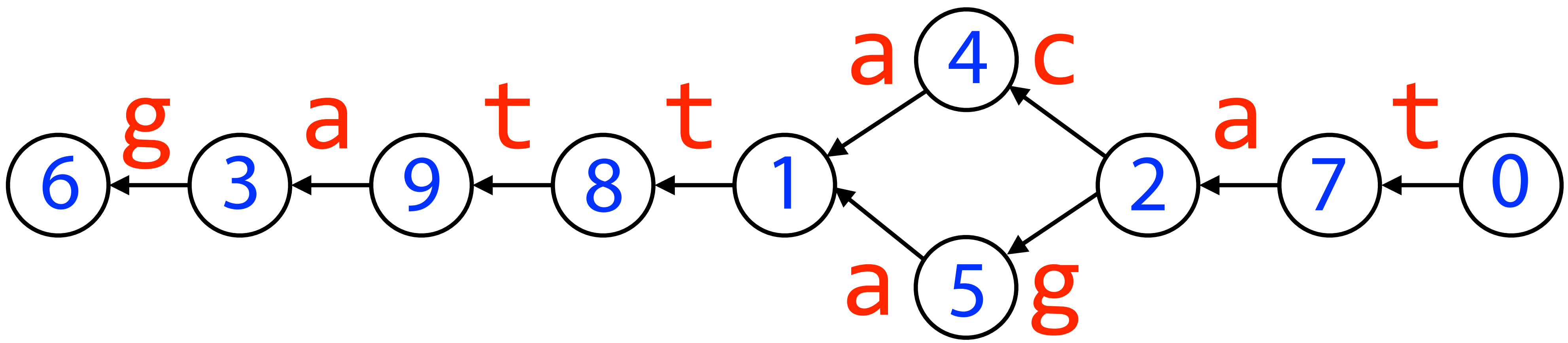
$$I = \texttt{10010101010101010101}$$

$$G.\text{indegree}(i) = I.\text{select}_1(i) - I.\text{select}_1(i-1) - 1$$

(let I. select(-1) = 0)

$$G.\text{indegree}(1) = I.\text{select}_1(1) - I.\text{select}_1(0) - 1$$
$$= 3 - 0 - 1 = 2$$

Wheeler graphs



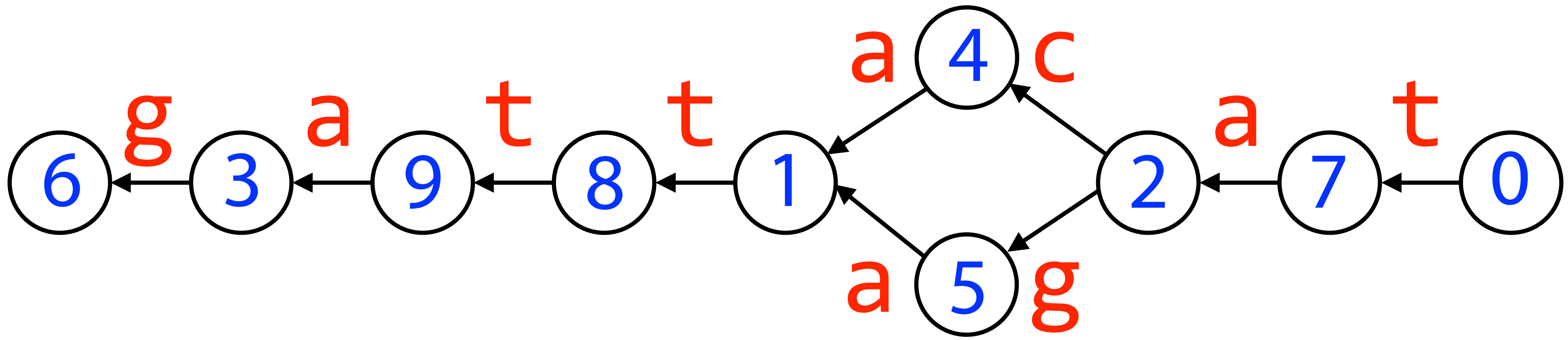
$$O = 0101001010101011010101$$

$$G.\text{outdegree}(i) = O.\text{select}_1(i) - O.\text{select}_1(i-1) - 1$$

(let O . select(-1) = 0)

$$G.\text{outdegree}(2) = O.\text{select}_1(2) - O.\text{select}_1(1) - 1$$
$$= 6 - 3 - 1 = 2$$

Wheeler graphs



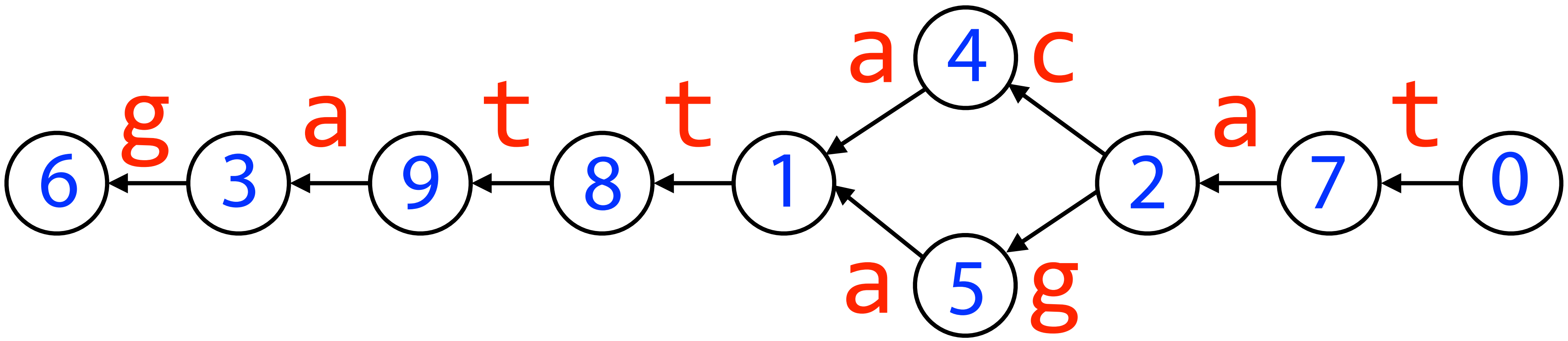
$$I = 1001010101010101010101$$

$$O = 0101001010101011010101$$

$$L = \texttt{ttcggaaata}$$

How to access the labels of the edges outgoing from a node?

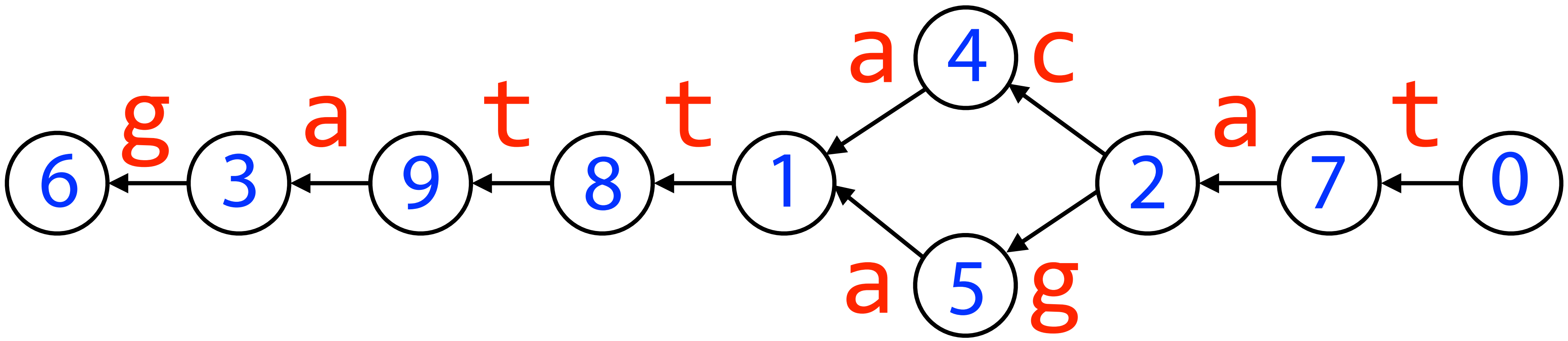
Wheeler graphs



$$O = \texttt{01010010101011010101}$$
$$L = \texttt{ttcggaaata}$$

$$\text{off} = O\,.\,\text{rank}_0(O\,.\,\text{select}_1(i-1))$$

Wheeler graphs



$O =$ 01010010101011010101

$L =$ ttcggaaata

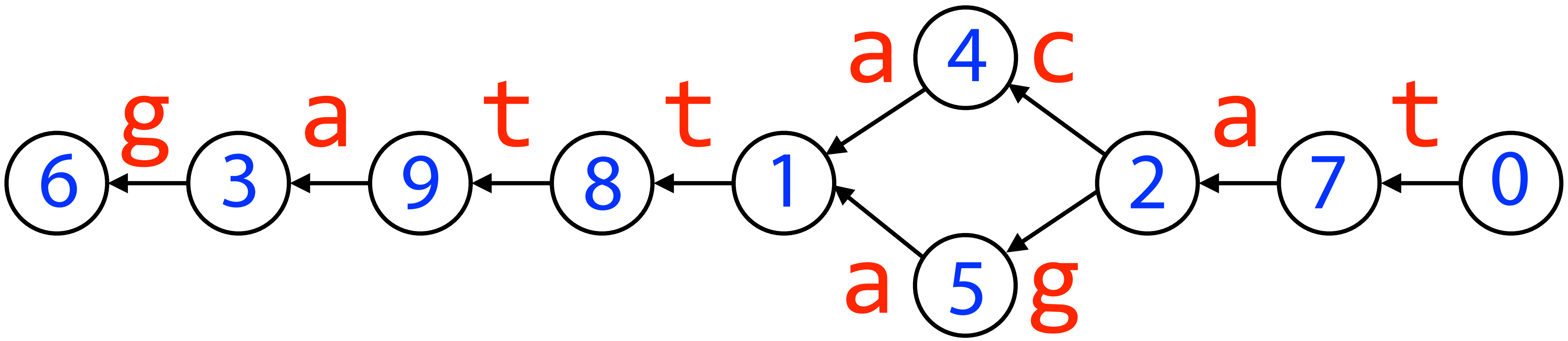
num $= G$. outdegree(i)

off $= O$. rank$_0(O$. select$_1(i-1))$

for j in {0, 1, …, num-1}:

  L . access(off $+ j)$

Wheeler graphs



$O =$ `010100101010110101`

$L =$ `ttcggaaata`

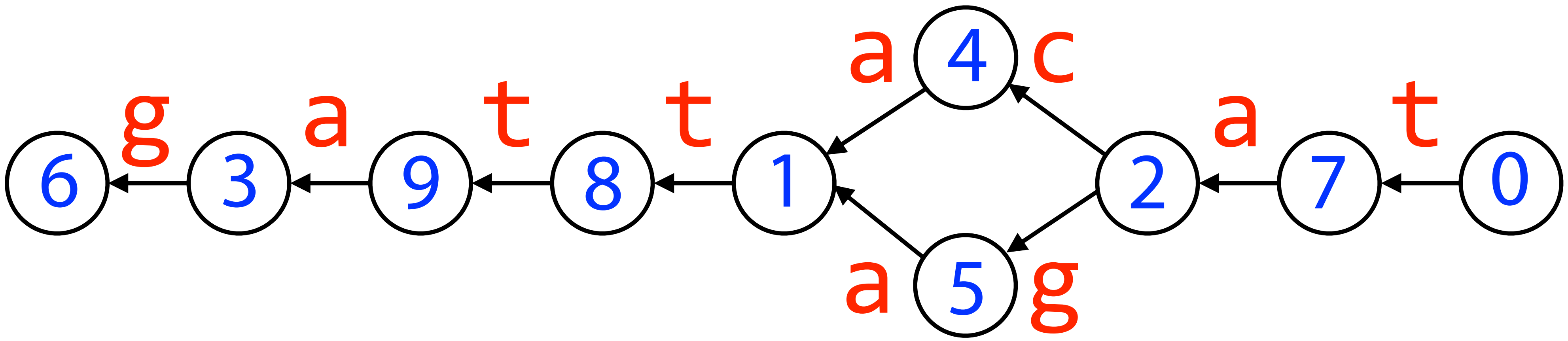
→ num $= G\,.\,\text{outdegree}(2) =$ **2**

off $= O\,.\,\text{rank}_0(O\,.\,\text{select}_1(2-1))$

for j in {0, 1, …, num-1}:

  $L\,.\,\text{access}(\text{off} + j)$

Wheeler graphs



$O =$ `0101001010101011010101`

$L =$ `ttcggaaata`

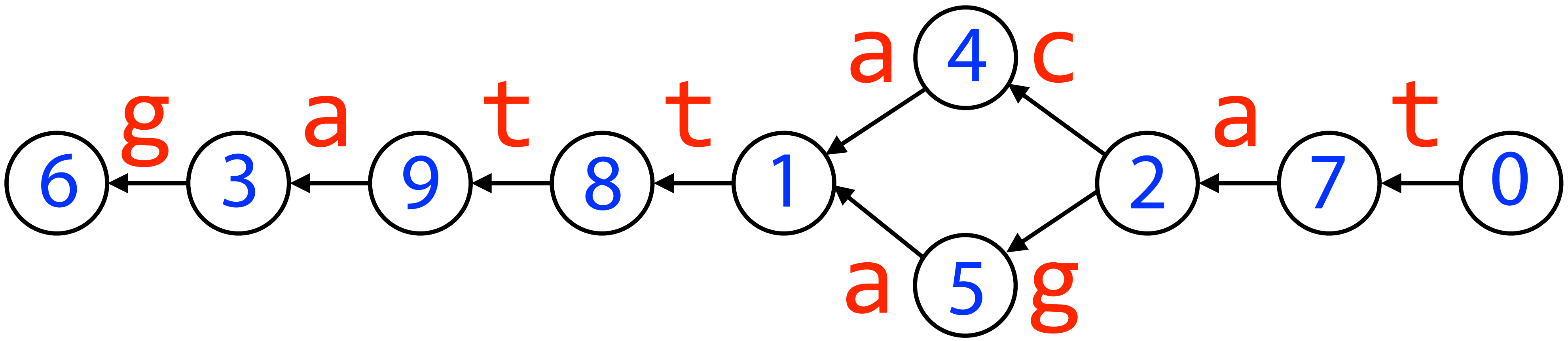
num $= 2$

off $= O\,.\,\text{rank}_0(O\,.\,\text{select}_1(2-1) = 3)$

for j in {0, 1, …, num-1}:

$L\,.\,\text{access}(\text{off} + j)$

Wheeler graphs



$O = $ 01010010101011010101

$L = $ ttcggaaata

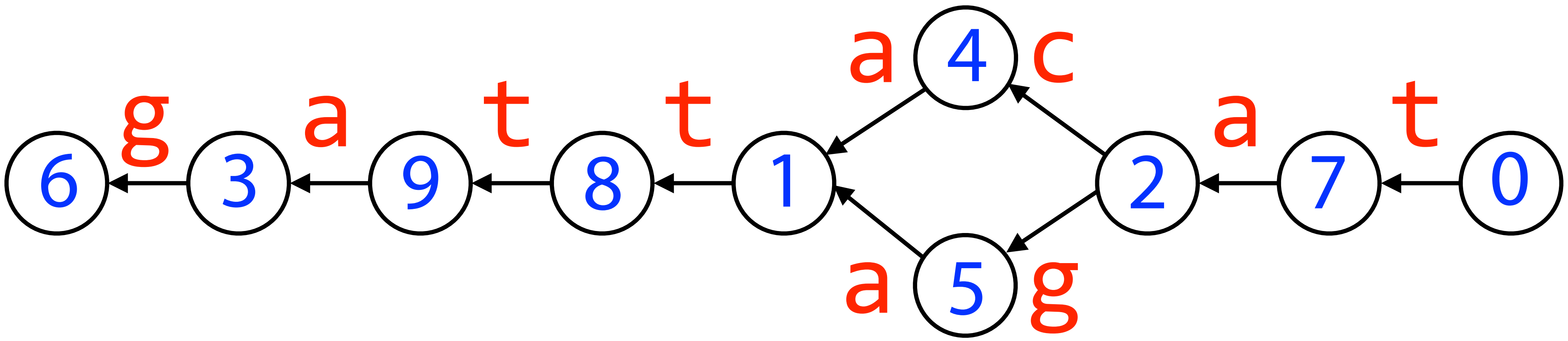
num $= 2$

off $= O\,.\,\text{rank}_0(3) = 2$

for j in {0, 1, …, num-1}:

$L\,.\,\text{access}(\text{off} + j)$

Wheeler graphs



$$O = \texttt{010100101010110101 01}$$

$$L = \texttt{ttcggaaata}$$

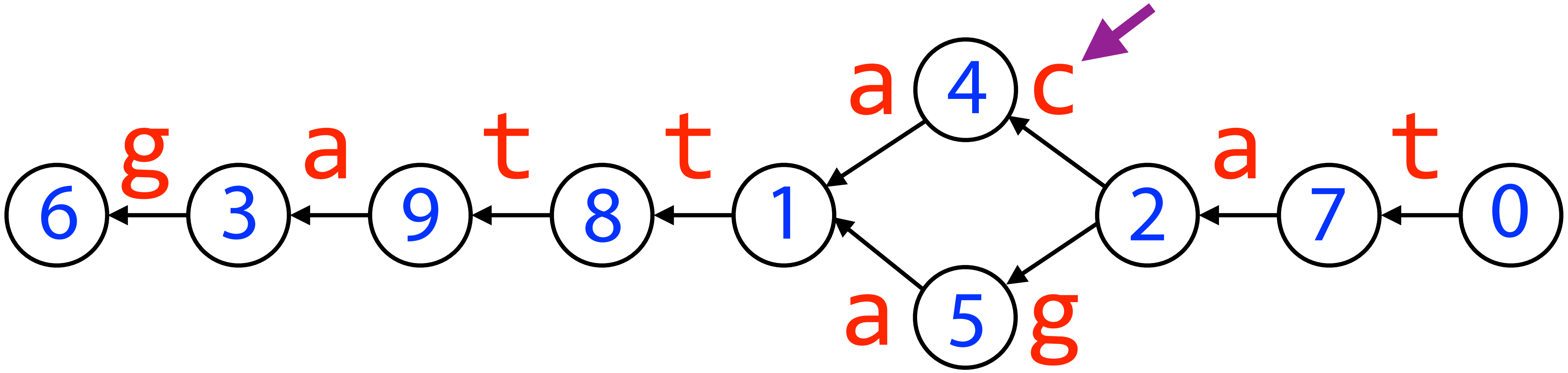
num $= 2$

off $= 2$

→ for j in $\{0, 1\}$:

  L . access(off $+ j$)

Wheeler graphs



$O =$ 01010010101011010101

$L =$ ttcggaaata

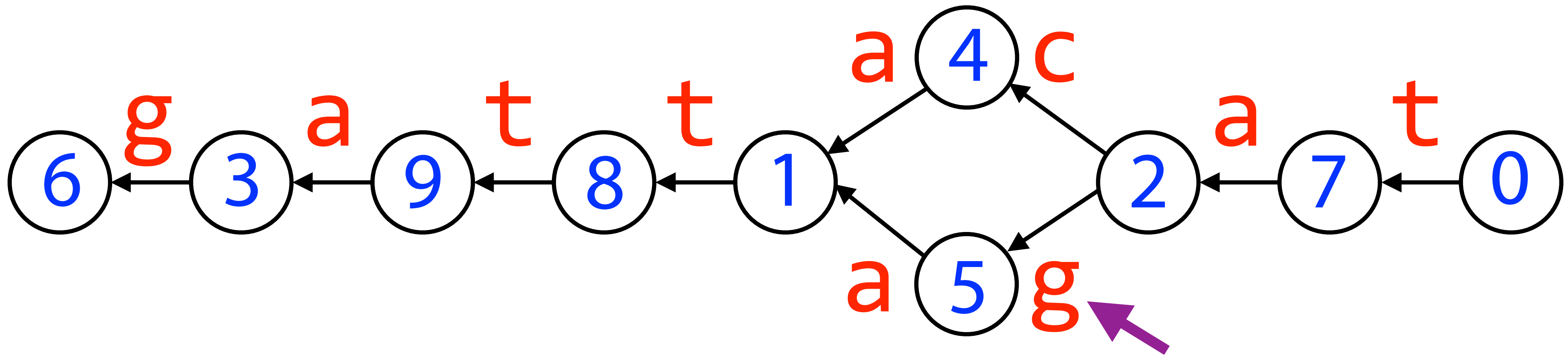
num $= 2$

off $= 2$

for j in {0, 1}:

L.access$(2 + 0) =$ c

Wheeler graphs



$$O = \texttt{01010010101011010101}$$

$$L = \texttt{ttcggaaata}$$

num $= 2$

off $= 2$

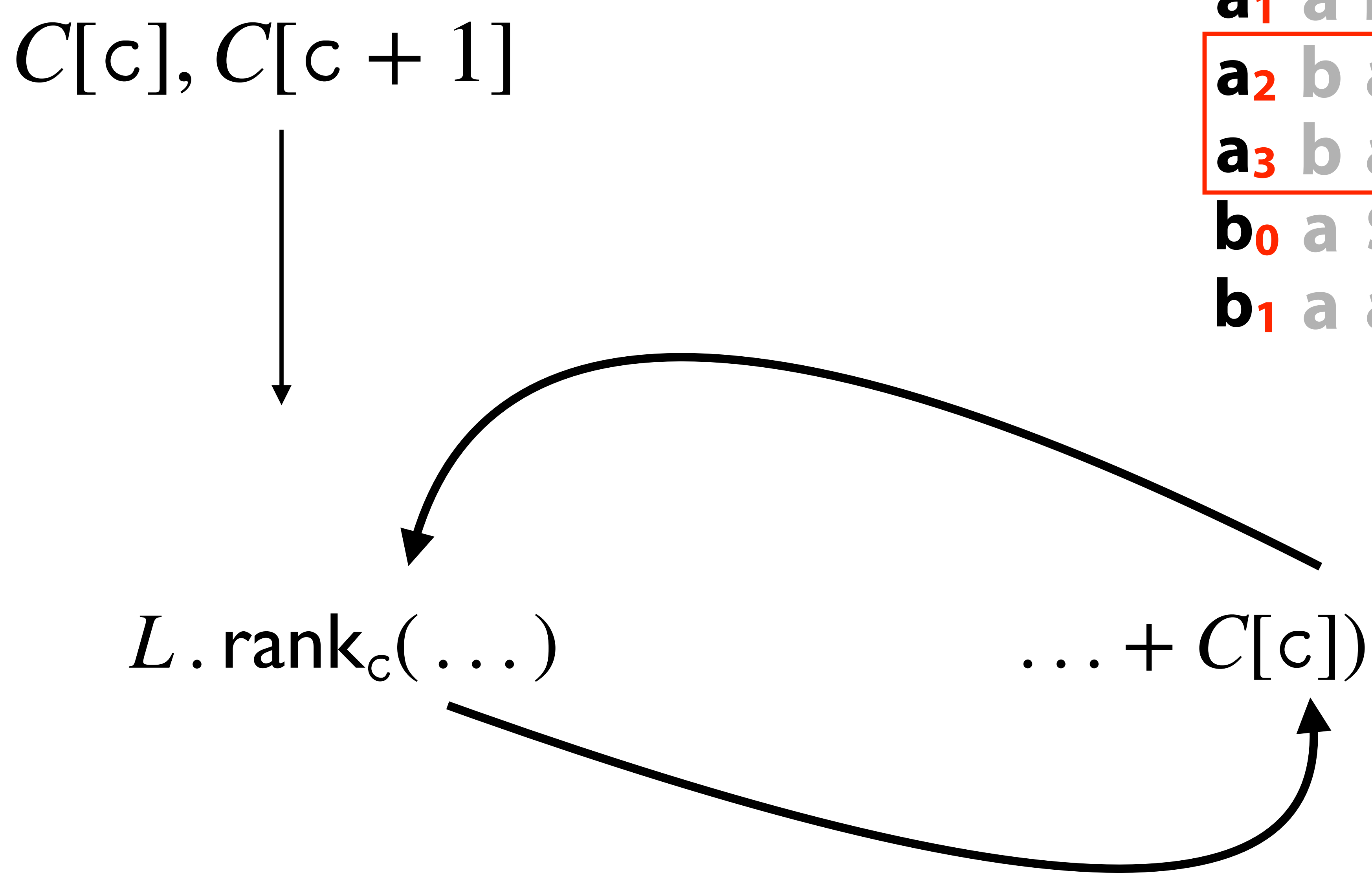
for j in {0, 1}:

$L.\text{access}(2 + 1) = \text{g}$

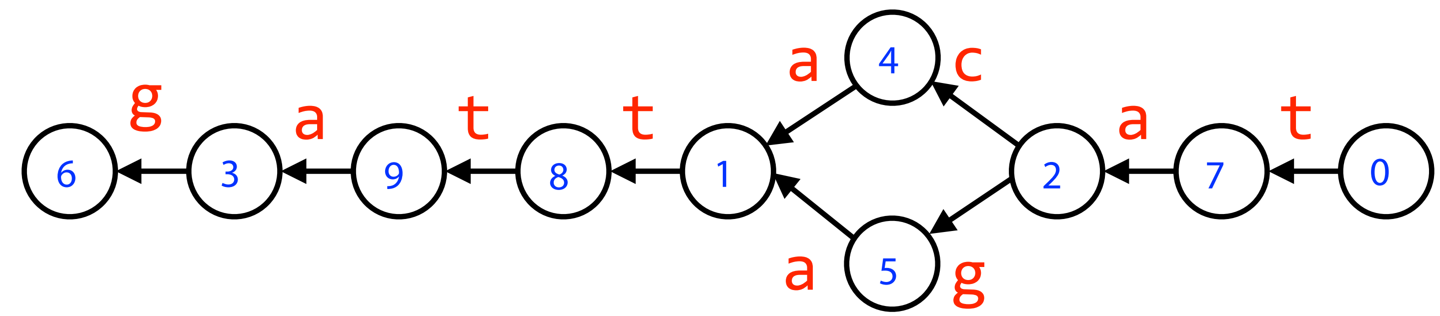
Wheeler graphs

FM Index match query loop:

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3



Wheeler graphs



Wheeler graph match query:

I : 1001010101010101010101

O : 0101001010101011010101

L : ttcggaaata

$C[\mathtt{c}], C[\mathtt{c}+1]$

$O.\text{rank}_0(O.\text{select}_1(\ldots))$

$I.\text{rank}_1(I.\text{select}_0(\ldots + C[\mathtt{c}]))$

$L.\text{rank}_\mathtt{c}(\ldots)$