FM Index: Efficient matching with BWT

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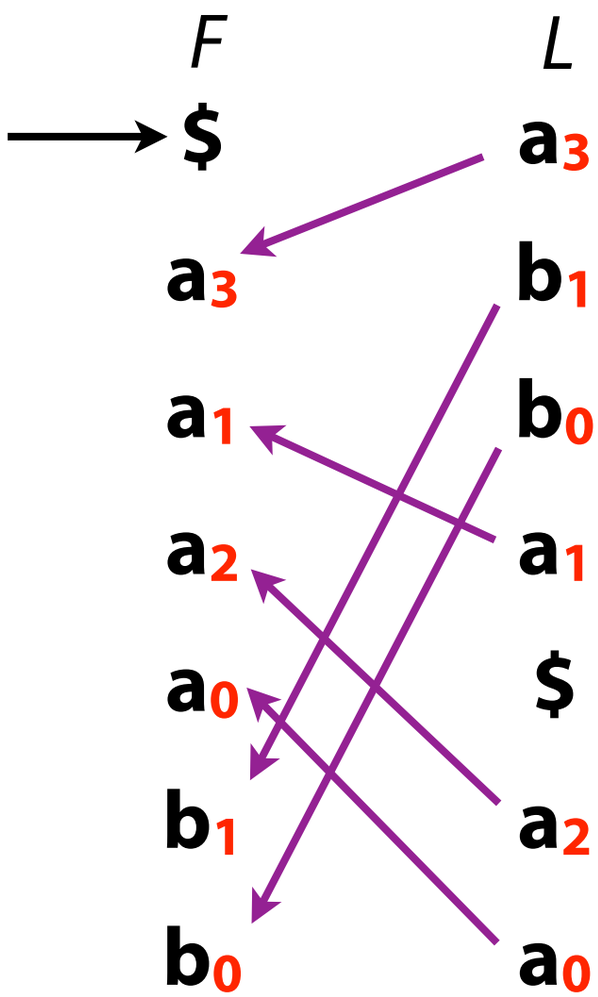
Wavelet trees

Armed with Wavelet Trees, let's return to the Burrows-Wheeler Transform

We can reverse it efficiently now!

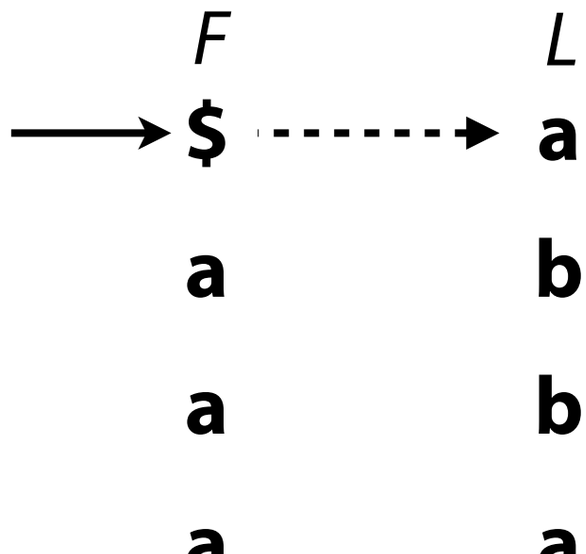
Burrows-Wheeler Transform

T: a_0 b_0 a_1 a_2 b_1 a_3 $

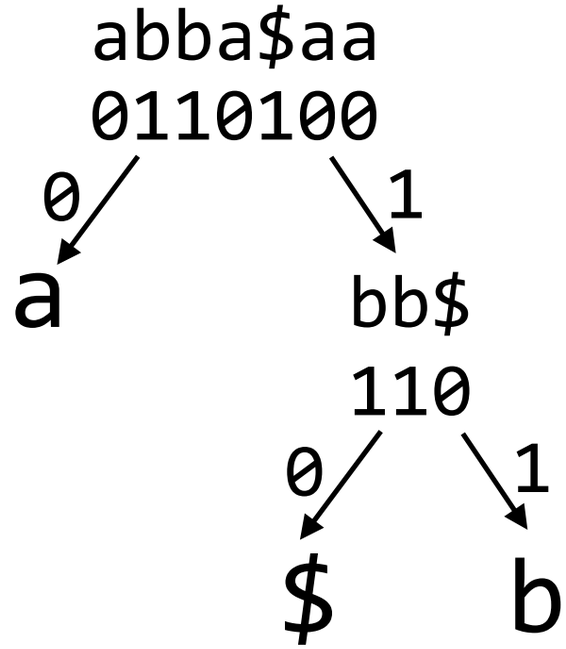


LF Mapping: The i^{th} occurrence of a character *c* in *L* and the i^{th} occurrence of *c* in *F* correspond to the *same* occurrence in *T* (i.e. have same rank)

Burrows-Wheeler Transform



$\mathrm{WT}(L) =$

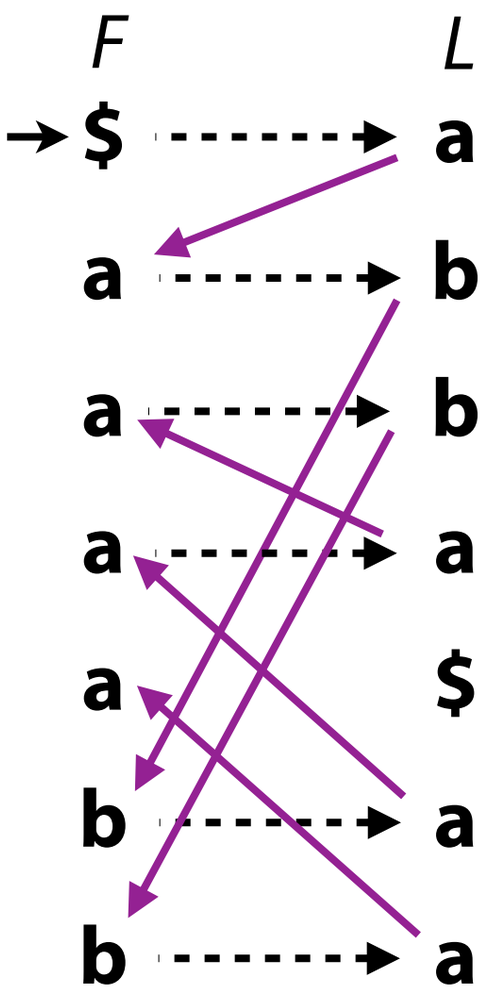


Recall: 1st row has \$ in F, so start there

In L, we see an **a**. What's its rank in L?

$$L\,.\,\mathrm{rank}_{\mathrm{a}}(0) = 0$$

Burrows-Wheeler Transform



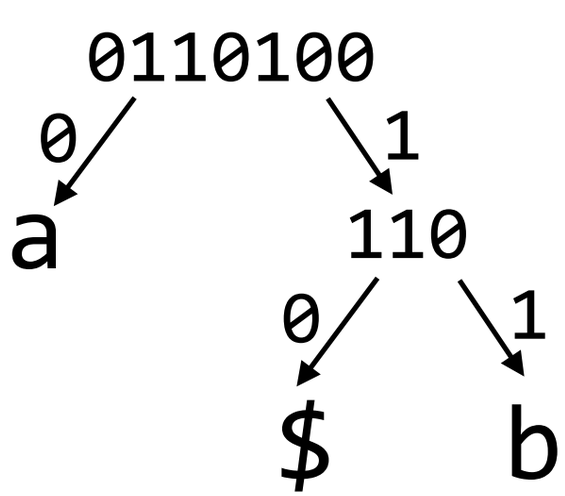
Rank	Skip	Next row	Next char
			a
$L.\text{rank}_{\text{a}}(0) = 0$	1 x \$ = 1	0 + 1 = 1	**b**
$L.\text{rank}_{\text{b}}(1) = 0$	1 x \$ + 4 x **a** = 5	0 + 5 = 5	**a**
$L.\text{rank}_{\text{a}}(5) = 2$	1 x \$ = 1	2 + 1 = 3	**a**
$L.\text{rank}_{\text{a}}(3) = 1$	1 x \$ = 1	1 + 1 = 2	**b**
$L.\text{rank}_{\text{b}}(2) = 1$	1 x \$ + 4 x **a** = 5	1 + 5 = 6	**a**
$L.\text{rank}_{\text{a}}(6) = 3$	1 x \$ = 1	3 + 1 = 4	**\$**

Skip amount can be looked up; pre-calculate C where $C[c]$ (c is a character) equals the number of characters alphabetically smaller than c in T

Here, $C[\$] = 0,\ C[\text{a}] = 1,\ C[\text{b}] = 5$

Burrows-Wheeler Transform

WT(BWT(T))



Rank	Skip	Next row	Next char
			a
$L.\text{rank}_a(0) = 0$	1 x \$ = 1	0 + 1 = 1	**b**
$L.\text{rank}_b(1) = 0$	1 x \$ + 4 x **a** = 5	0 + 5 = 5	**a**
$L.\text{rank}_a(5) = 2$	1 x \$ = 1	2 + 1 = 3	**a**
$L.\text{rank}_a(3) = 1$	1 x \$ = 1	1 + 1 = 2	**b**
$L.\text{rank}_b(2) = 1$	1 x \$ + 4 x **a** = 5	1 + 5 = 6	**a**
$L.\text{rank}_a(6) = 3$	1 x \$ = 1	3 + 1 = 4	**\$**

Reversing is $O\left(n \log_2 \sigma \right)$

steps

rank query

Rank + skip = LF mapping

Burrows-Wheeler Transform

Principles of navigation

Use WT(BWT(T)) to reverse: BWT(T) $\rightarrow T$

How do we do *indexing*?

Indexing

A ***full-text index*** for text $T \in \Sigma^n$ is a structure giving efficient answers to queries:

Locate(P), where $P \in \Sigma^m$, returns all offsets where P matches a substring of T

Count(P) returns # of offsets where P matches a substring of T

Extract(i, m) returns $T[i : i + m - 1]$ (length-m substring starting at i)

FM Index: querying

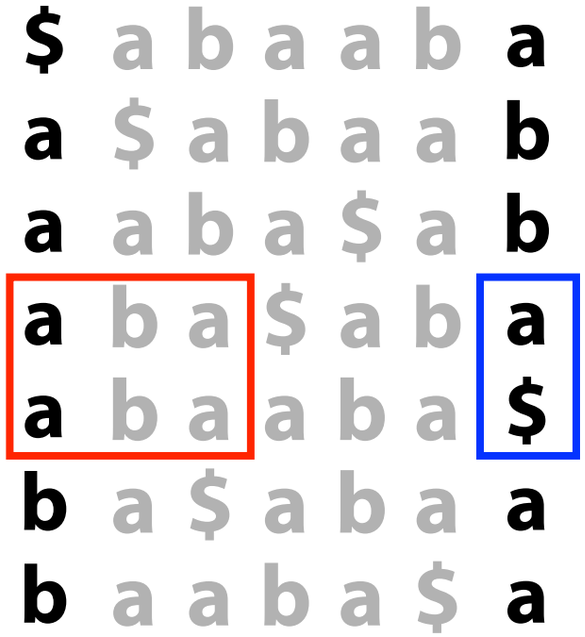
How to ***find***, ***count*** and ***locate*** substrings matching a query?

$ a b a a b **a**
a $ a b a a **b**
a a b a $ a **b**
a b a $ a b **a**
a b a a b a **$**
b a $ a b a **a**
b a a b a $ **a**

Ferragina, Paolo, and Giovanni Manzini. "Opportunistic data structures with applications." *Proceedings 41st Annual Symposium on Foundations of Computer Science*. IEEE, 2000.

FM Index: querying

Observation 1: Rows with same prefix are consecutive



Observation 2: Characters in last column are those *preceding* the prefixes (to their *left* in T)

FM Index: querying

Given pattern P, $|P| = m$, start with shortest suffix of and match successively longer suffixes

$P =$ **aba**

Easy to find all the rows beginning with **a**

$[C[\text{a}], C[\text{b}]) = [1,5)$

F	L
$	a b a a b a$_0$
a$_0$	$ a b a a b$_0$
a$_1$	a b a $ a b$_1$
a$_2$	b a $ a b a$_1$
a$_3$	b a a b a $
b$_0$	a $ a b a a$_2$
b$_1$	a a b a $ a$_3$

Subscripts are ranks in L

FM Index: querying

We have rows beginning with **a**, now we want rows beginning with **ba**

$P =$ **aba**

F						*L*
$	a	b	a	a	b	$\mathbf{a_0}$
$\mathbf{a_0}$	$	a	b	a	a	$\mathbf{b_0}$
$\mathbf{a_1}$	a	b	a	$	a	$\mathbf{b_1}$
$\mathbf{a_2}$	b	a	$	a	b	$\mathbf{a_1}$
$\mathbf{a_3}$	b	a	a	b	a	**$**
$\mathbf{b_0}$	a	$	a	b	a	$\mathbf{a_2}$
$\mathbf{b_1}$	a	a	b	a	$	$\mathbf{a_3}$

← Look at those rows in *L*.
$\mathbf{b_0}$, $\mathbf{b_1}$ are **b**s occuring just to left.

Use LF Mapping. Let new range delimit those **b**s →

$P =$ **aba**

F						*L*
$	a	b	a	a	b	$\mathbf{a_0}$
$\mathbf{a_0}$	$	a	b	a	a	$\mathbf{b_0}$
$\mathbf{a_1}$	a	b	a	$	a	$\mathbf{b_1}$
$\mathbf{a_2}$	b	a	$	a	b	$\mathbf{a_1}$
$\mathbf{a_3}$	b	a	a	b	a	**$**
$\mathbf{b_0}$	a	$	a	b	a	$\mathbf{a_2}$
$\mathbf{b_1}$	a	a	b	a	$	$\mathbf{a_3}$

Now we have the rows with prefix **ba**

FM Index: querying

We have rows beginning with **ba**, now we seek rows beginning with **aba**

$P =$ **aba**

F						*L*
$	a	b	a	a	b	**a**$_0$
a $_0$	**$**	a	b	a	a	**b**$_0$
a $_1$	a	b	a	**$**	a	**b**$_1$
a $_2$	b	a	**$**	a	b	**a**$_1$
a $_3$	b	a	a	b	a	**$**
b $_0$	a	**$**	a	b	a	**a**$_2$
b $_1$	a	a	b	a	**$**	**a**$_3$

← **a**$_2$, **a**$_3$ occur just to left.

Use LF Mapping →

$P =$ **aba**

F						*L*
$	a	b	a	a	b	**a**$_0$
a $_0$	**$**	a	b	a	a	**b**$_0$
a $_1$	a	b	a	**$**	a	**b**$_1$
a $_2$	b	a	**$**	a	b	**a**$_1$
a $_3$	b	a	a	b	a	**$**
b $_0$	a	**$**	a	b	a	**a**$_2$
b $_1$	a	a	b	a	**$**	**a**$_3$

Now we have the rows with prefix **aba**

$$T.\text{count}(\text{aba}) = 2$$

FM Index: querying

When *P* does not occur in *T*, we eventually fail to find next character in *L*:

$$P = \mathbf{bba}$$

F						*L*
$	a	b	a	a	b	$\mathbf{a_0}$
$\mathbf{a_0}$	$	a	b	a	a	$\mathbf{b_0}$
$\mathbf{a_1}$	a	b	a	$	a	$\mathbf{b_1}$
$\mathbf{a_2}$	b	a	$	a	b	$\mathbf{a_1}$
$\mathbf{a_3}$	b	a	a	b	a	**$**
$\mathbf{b_0}$	a	$	a	b	a	$\mathbf{a_2}$
$\mathbf{b_1}$	a	a	b	a	$	$\mathbf{a_3}$

Rows with **ba** prefix

← No **b**s!

FM Index: querying

$P = \mathbf{aba}$

F						*L*
$	a	b	a	a	b	$\mathbf{a_0}$
$\mathbf{a_0}$	$	a	b	a	a	$\mathbf{b_0}$
$\mathbf{a_1}$	a	b	a	$	a	$\mathbf{b_1}$
$\mathbf{a_2}$	b	a	$	a	b	$\mathbf{a_1}$
$\mathbf{a_3}$	b	a	a	b	a	**$**
$\mathbf{b_0}$	a	$	a	b	a	$\mathbf{a_2}$
$\mathbf{b_1}$	a	a	b	a	$	$\mathbf{a_3}$

Next char	Rank	Skip	Next range
a		1 x $ = 1 1 x $ + 5 x **a** = 5	1 5

FM Index: querying

$P = \mathbf{aba}$

F						L	
$	a	b	a	a	b	$\mathbf{a}_0$	
$\mathbf{a}_0$	$	a	b	a	a	$\mathbf{b}_0$	←
$\mathbf{a}_1$	a	b	a	$	a	$\mathbf{b}_1$	
$\mathbf{a}_2$	b	a	$	a	b	$\mathbf{a}_1$	
$\mathbf{a}_3$	b	a	a	b	a	**$**	
$\mathbf{b}_0$	a	$	a	b	a	$\mathbf{a}_2$	←
$\mathbf{b}_1$	a	a	b	a	$	$\mathbf{a}_3$	

Next char	Rank	Skip	Next range
a		1 x $ = 1 1 x $ + 5 x **a** = 5	1 5
b	$L.\text{rank}_{\text{b}}(1) = 0$ $L.\text{rank}_{\text{b}}(5) = 2$	1 x $ + 5 x **a** = 5	0 + 5 = 5 2 + 5 = 7

FM Index: querying

$P = \mathbf{aba}$

F						L
$	a	b	a	a	b	$\mathbf{a}_0$
$\mathbf{a}_0$	$	a	b	a	a	$\mathbf{b}_0$
$\mathbf{a}_1$	a	b	a	$	a	$\mathbf{b}_1$
$\mathbf{a}_2$	b	a	$	a	b	$\mathbf{a}_1$
$\mathbf{a}_3$	b	a	a	b	a	**$**
$\mathbf{b}_0$	a	$	a	b	a	$\mathbf{a}_2$ ←
$\mathbf{b}_1$	a	a	b	a	$	$\mathbf{a}_3$
						←

Next char	Rank	Skip	Next range
a		1 x $ = 1 1 x $ + 5 x **a** = 5	1 5
b	$L.\text{rank}_b(1) = 0$ $L.\text{rank}_b(5) = 2$	1 x $ + 5 x **a** = 5	0 + 5 = 5 2 + 5 = 7
a	$L.\text{rank}_a(5) = 2$ $L.\text{rank}_a(7) = 4$	1 x $ = 1	0 + 1 = 3 2 + 1 = 5

FM Index: querying

$P = \mathbf{aba}$

F						L
$	a	b	a	a	b	$\mathbf{a_0}$
$\mathbf{a_0}$	$	a	b	a	a	$\mathbf{b_0}$
$\mathbf{a_1}$	a	b	a	$	a	$\mathbf{b_1}$
$\mathbf{a_2}$	b	a	$	a	b	$\mathbf{a_1}$
$\mathbf{a_3}$	b	a	a	b	a	**$**
$\mathbf{b_0}$	a	$	a	b	a	$\mathbf{a_2}$
$\mathbf{b_1}$	a	a	b	a	$	$\mathbf{a_3}$

$T.\text{count}(\text{aba}) = 2$

Next char	Rank	Skip	Next range
a		1 x $ = 1 1 x $ + 5 x **a** = 5	1 5
b	$L.\text{rank}_b(1) = 0$ $L.\text{rank}_b(5) = 2$	1 x $ + 5 x **a** = 5	0 + 5 = 5 2 + 5 = 7
a	$L.\text{rank}_a(5) = 2$ $L.\text{rank}_a(7) = 4$	1 x $ = 1	0 + 1 = 3 2 + 1 = 5

FM Index: querying

FM index match(P):

Given query string P

top $\leftarrow 0$

bot $\leftarrow |T|$

$i \leftarrow |P| - 1$

while $i \geq 0$ and bot $>$ top

  $c \leftarrow P[i]$

  top $\leftarrow$ BWT . $C[c]$ + BWT . rank$_c$(top)

  bot $\leftarrow$ BWT . $C[c]$ + BWT . rank$_c$(bot)

  $i \leftarrow i - 1$

return (top, bot)

Skip

Rank

Rank

(For simplicity, version starts with the all-inclusive range rather than using 2 initial BWT . C[. . .] lookups to get the range for the length-1 suffix)

FM Index: querying

A ***full-text index*** for text $T \in \Sigma^n$ is a structure giving efficient answers to queries:

Locate(P), where $P \in \Sigma^m$, returns all offsets where P matches a substring of T

Count(P) returns # of offsets where P matches a substring of T

Extract(i, m) returns $T[i : i + m - 1]$ (length-m substring starting at i)

FM Index: querying

F						*L*
$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

Where are these occurrences in *T*?

FM Index: querying

Where are these occurrences in *T*?

$	a	b	a	a	b	**$a₀**
a₀	$	a	b	a	a	**ba₀**
a₁	a	b	a	$	a	**ba₁**
a₂	b	a	$	a	b	**aa₁**
a₃	b	a	a	b	a	**$**
b₀	a	$	a	b	a	**aa₂**
b₁	a	a	b	a	$	**aa₃**

If we had suffix array, we could look up offsets...

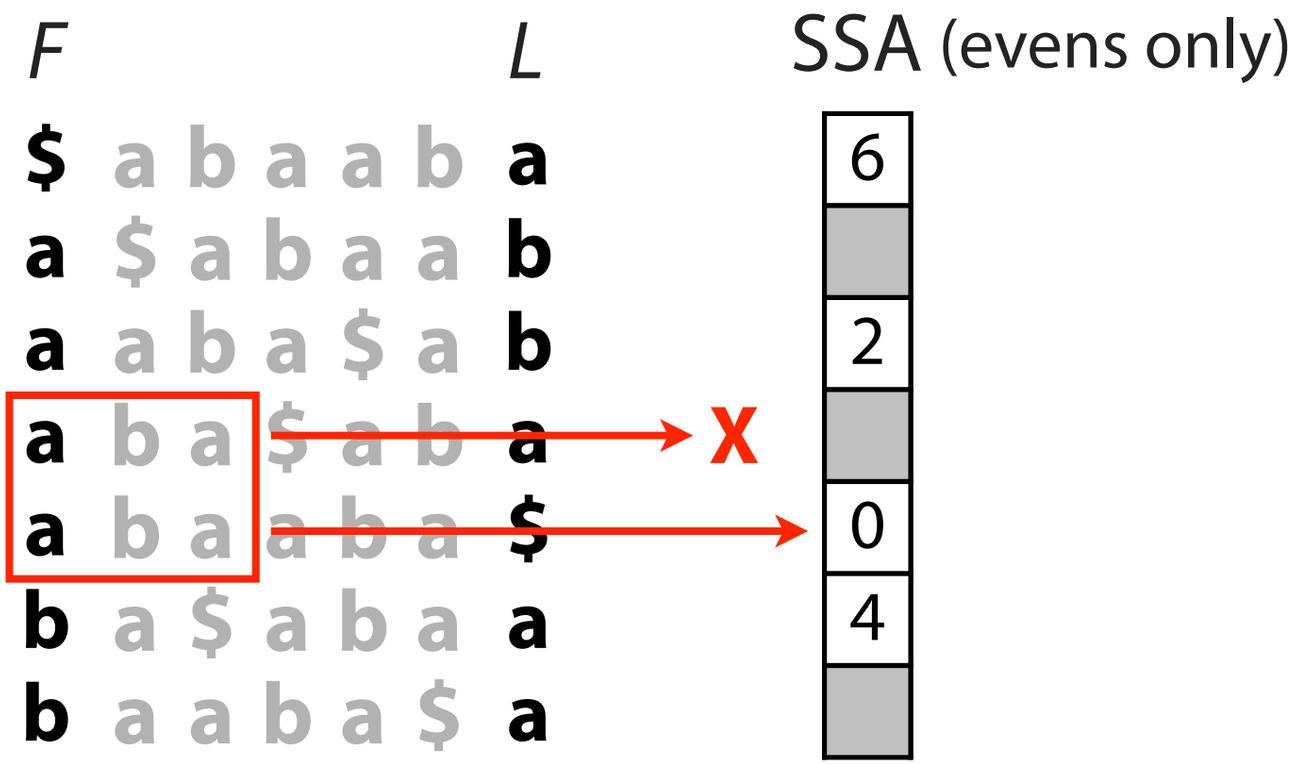
F						*L*
$	a	b	a	a	b	**a**
a	$	a	b	a	a	**b**
a	a	b	a	$	a	**b**
a	b	a	$	a	b	**a**
a	b	a	a	b	a	**$**
b	a	$	a	b	a	**a**
b	a	a	b	a	$	**a**

Offsets: 0, 3

SA	
6	**$**
5	**a $**
2	**a a b a $**
3	**a b a $**
0	**a b a a b a $**
4	**b a $**
1	**b a a b a $**

FM Index: resolving offsets

Sampled Suffix Array (SSA): store some suffix array elements, not all



Lookup for row 4 succeeds

Lookup for row 3 fails - SA entry was discarded