Bloom filters

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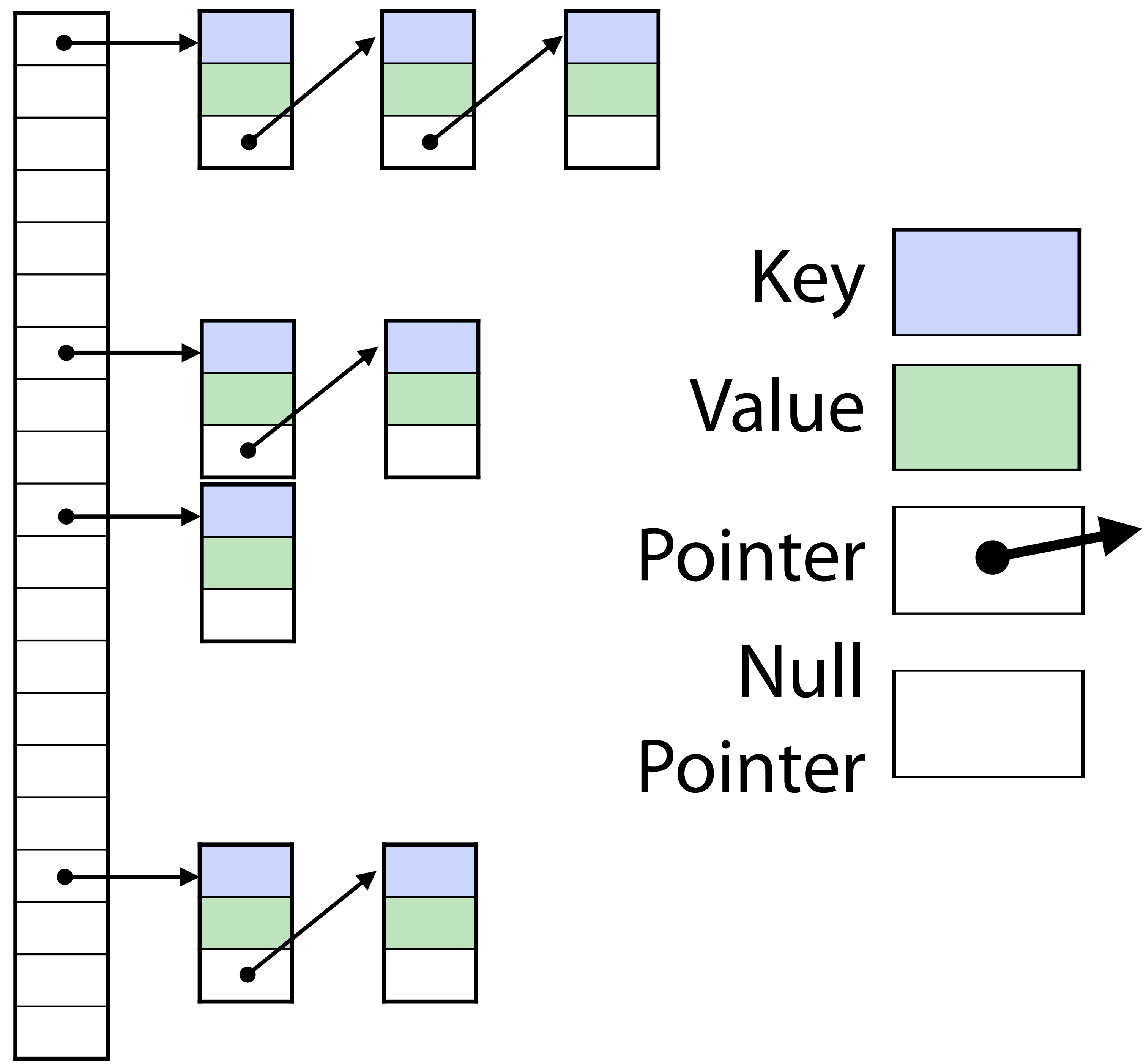
Bloom filter

Imagine we start with a hash table ...

... and start taking things away



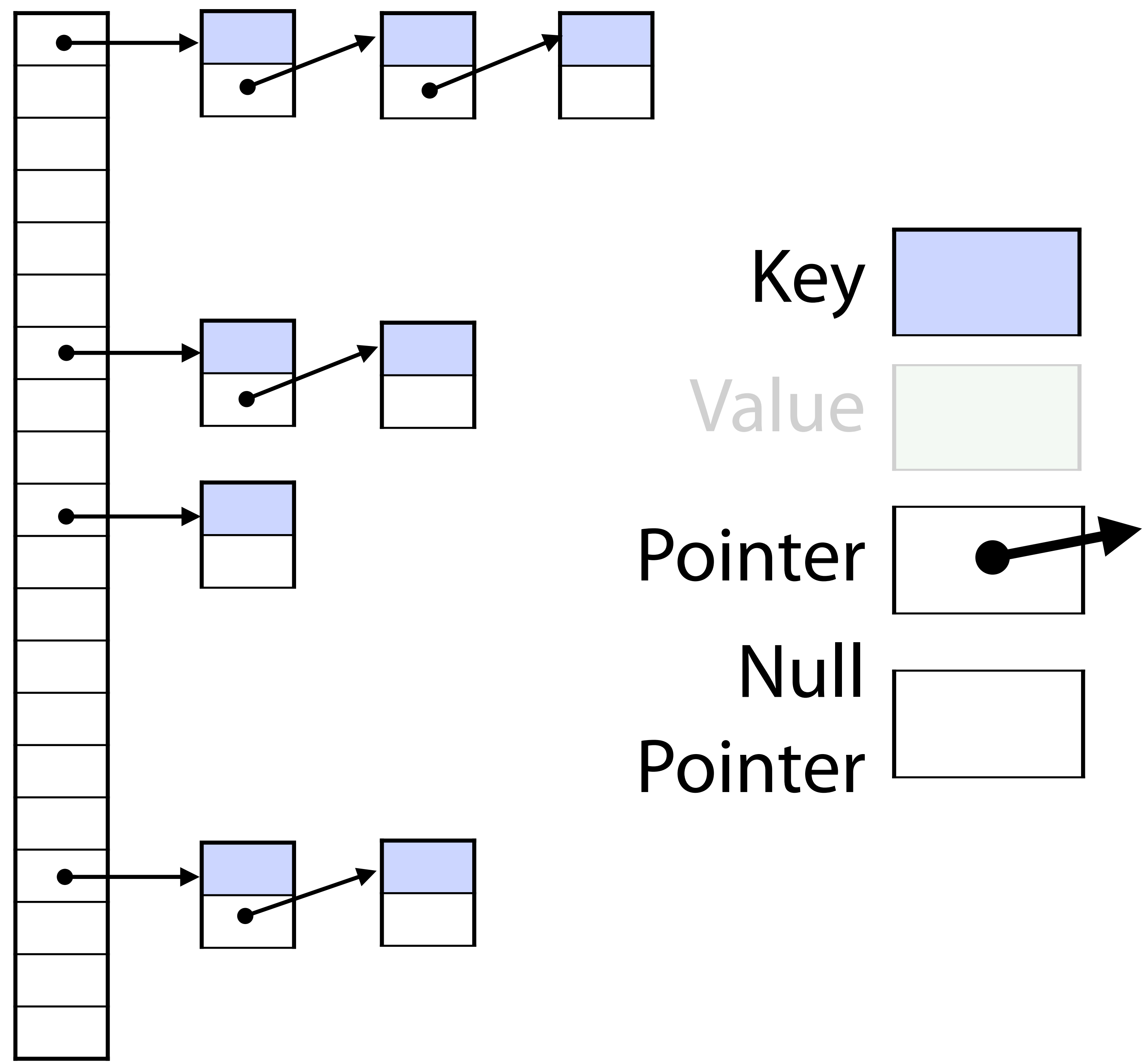
Bloom filter

Imagine we start with a hash table ...

... and start taking things away

Take away values

Now we have a **hash set**



Bloom filter

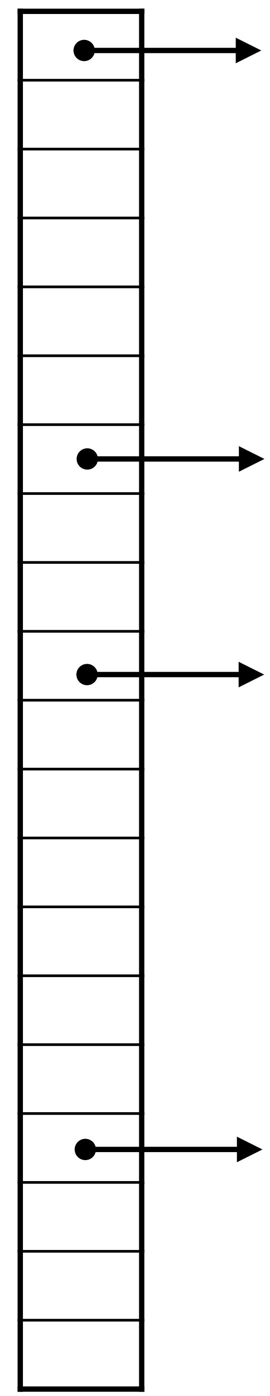
Imagine we start with a hash table ...

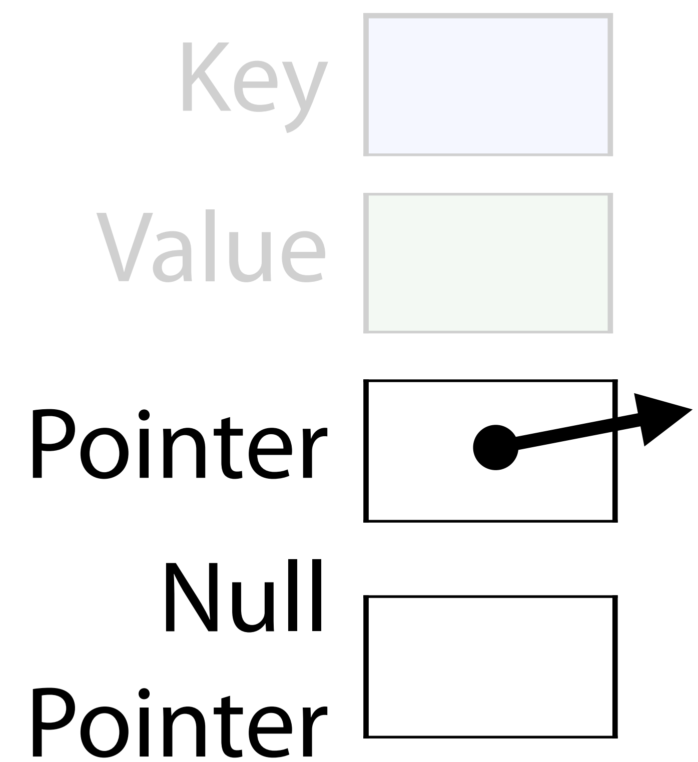
... and start taking things away

Take away values

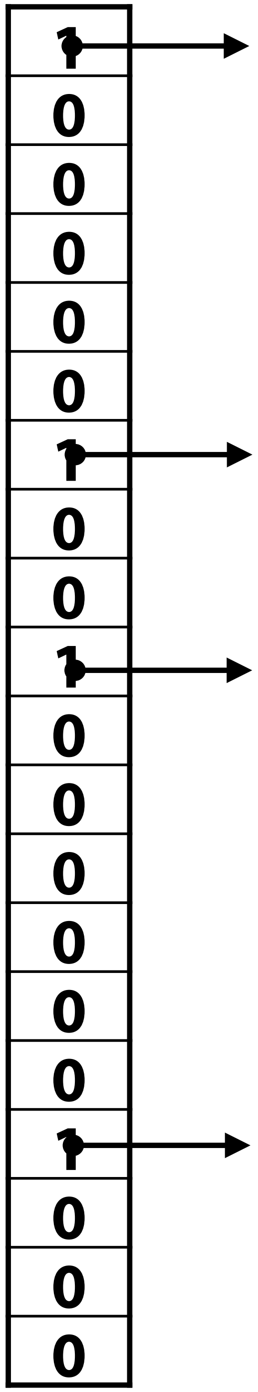
Now we have a hash set

Now take away keys





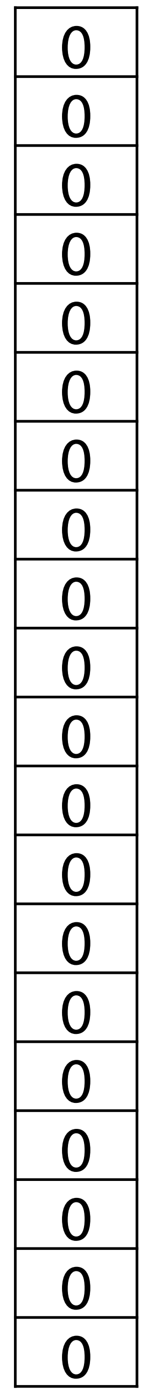
Bloom filter



Bloom filter

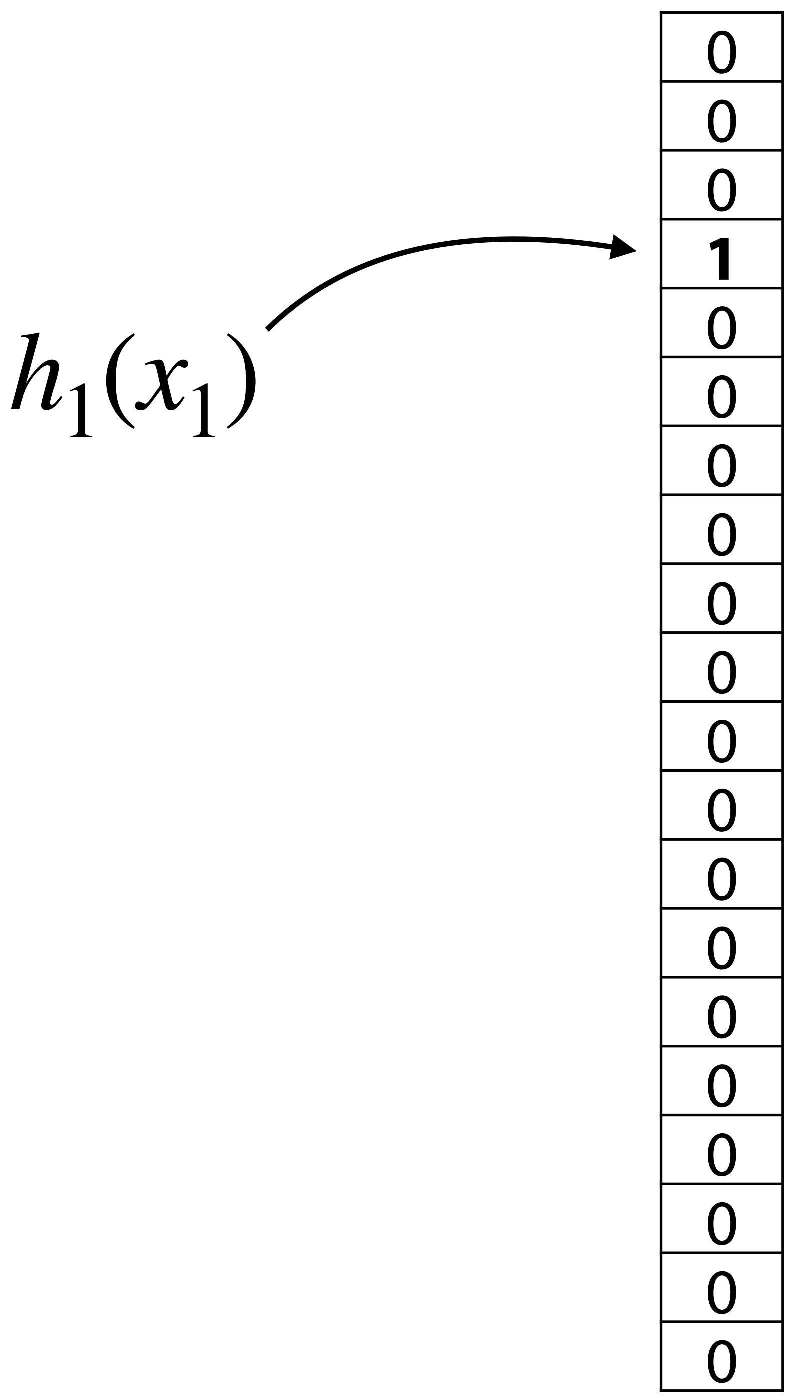
1
0
0
0
0
0
1
0
0
1
0
0
0
0
0
0
1
0
0
0

Bloom filter: inserting



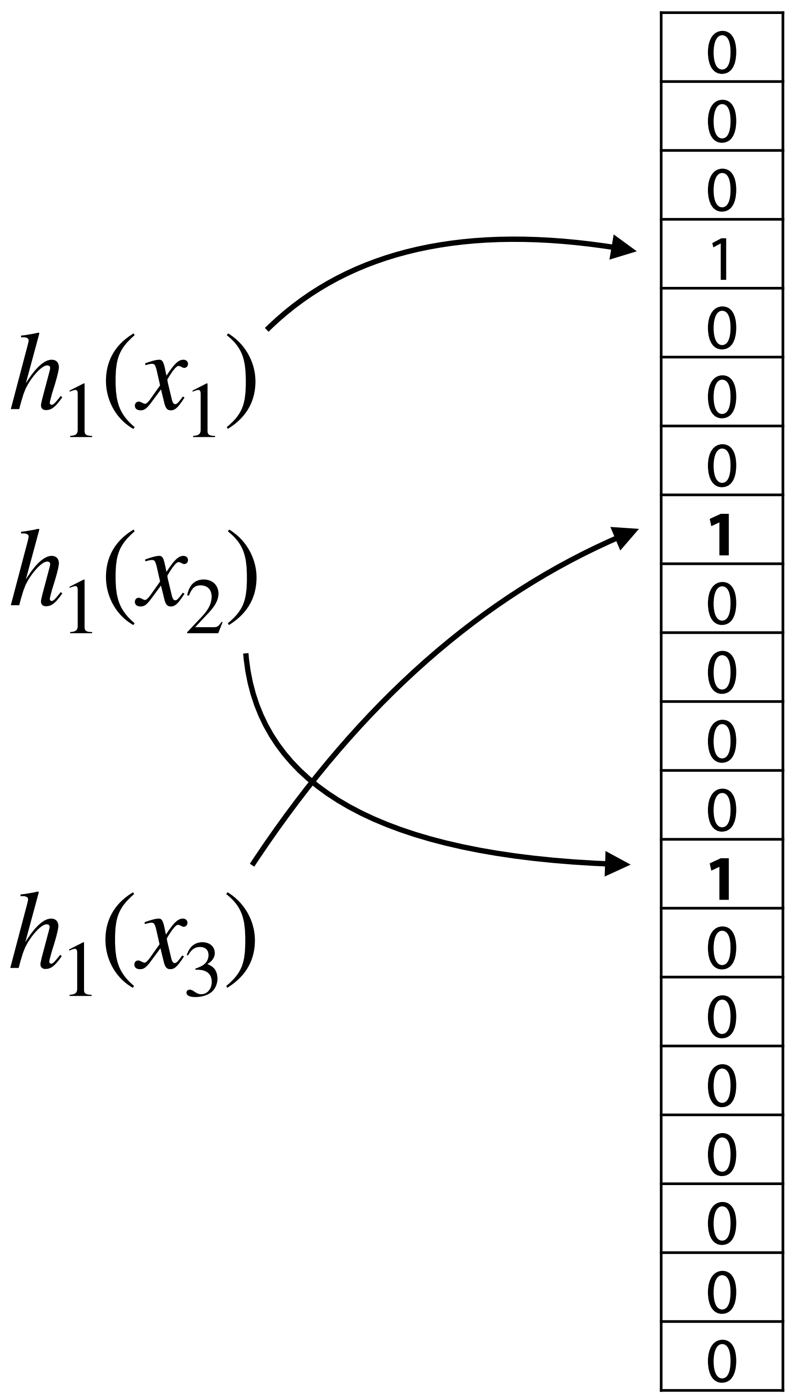
Start with all 0s

Bloom filter: inserting

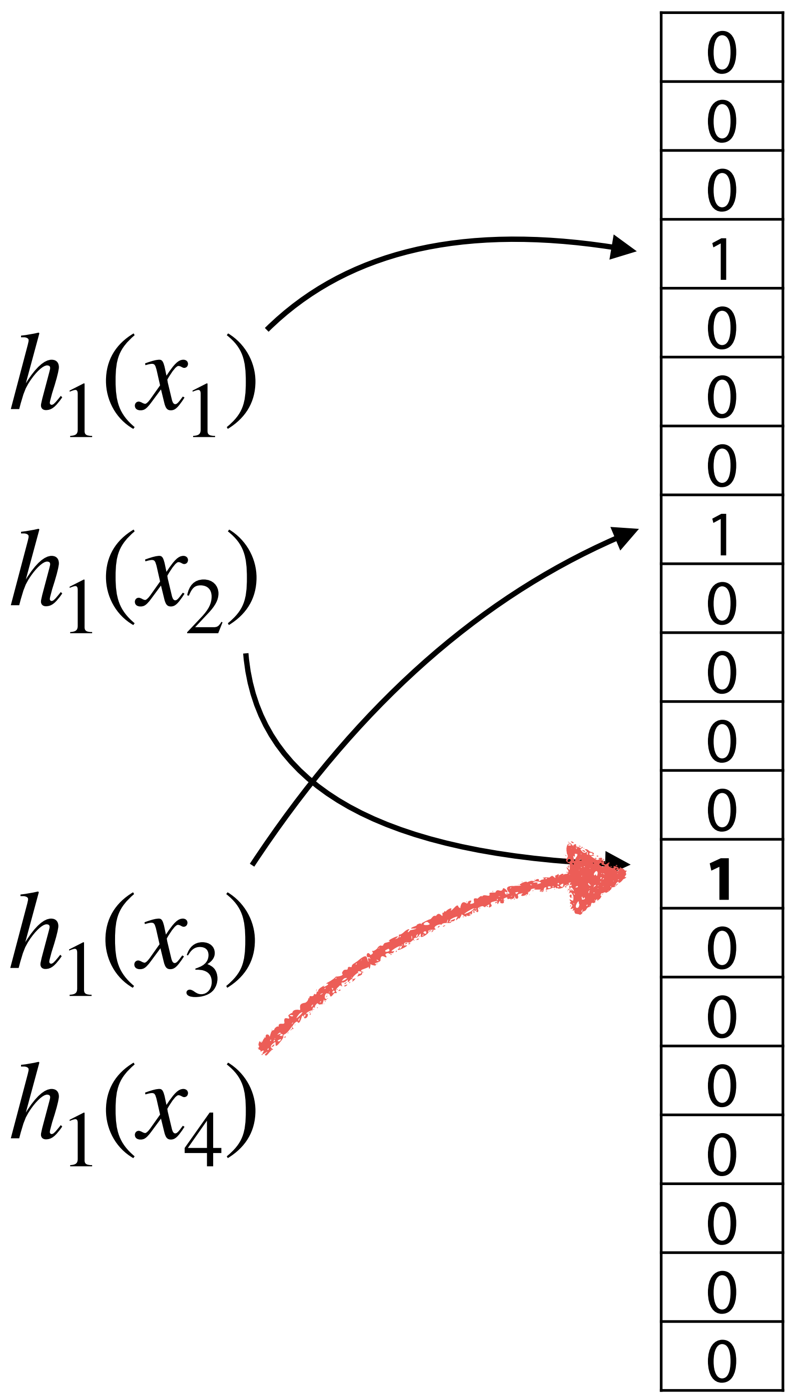


To insert item, hash it to a bucket and set bit to 1 (if not already)

Bloom filter: inserting



Bloom filter: inserting



If already 1, this is a *collision*

Life goes on

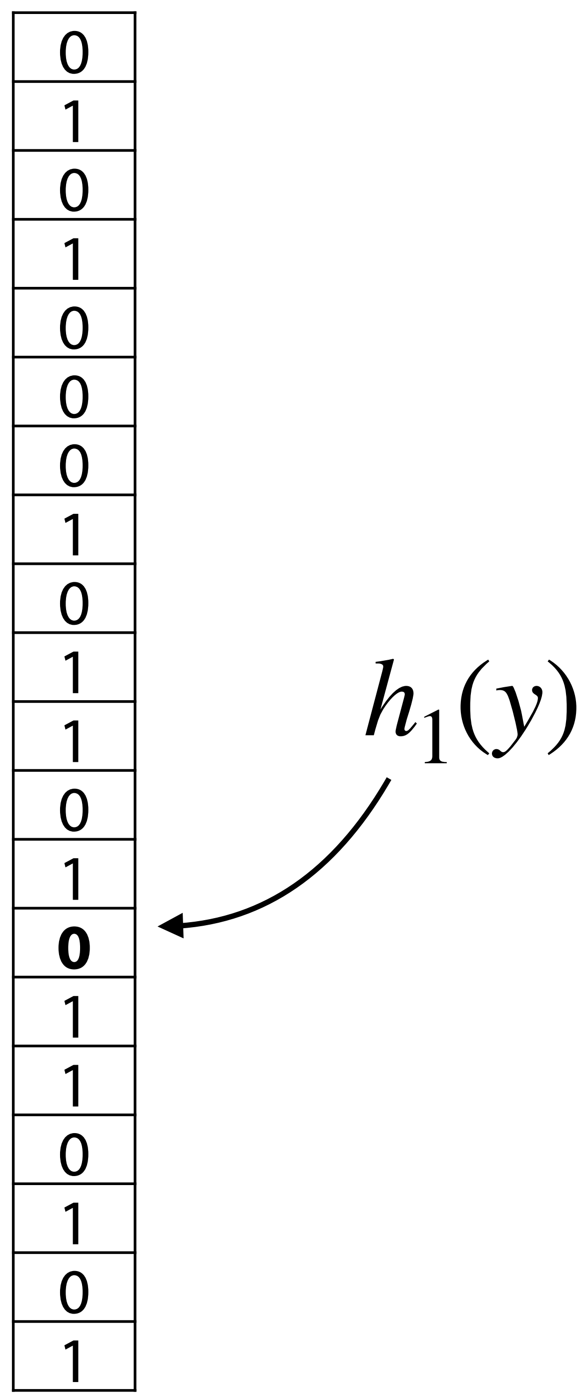
Bloom filter: querying

0
1
0
1
0
0
0
1
0
1
1
0
1
0
1
1
0
1
0
1

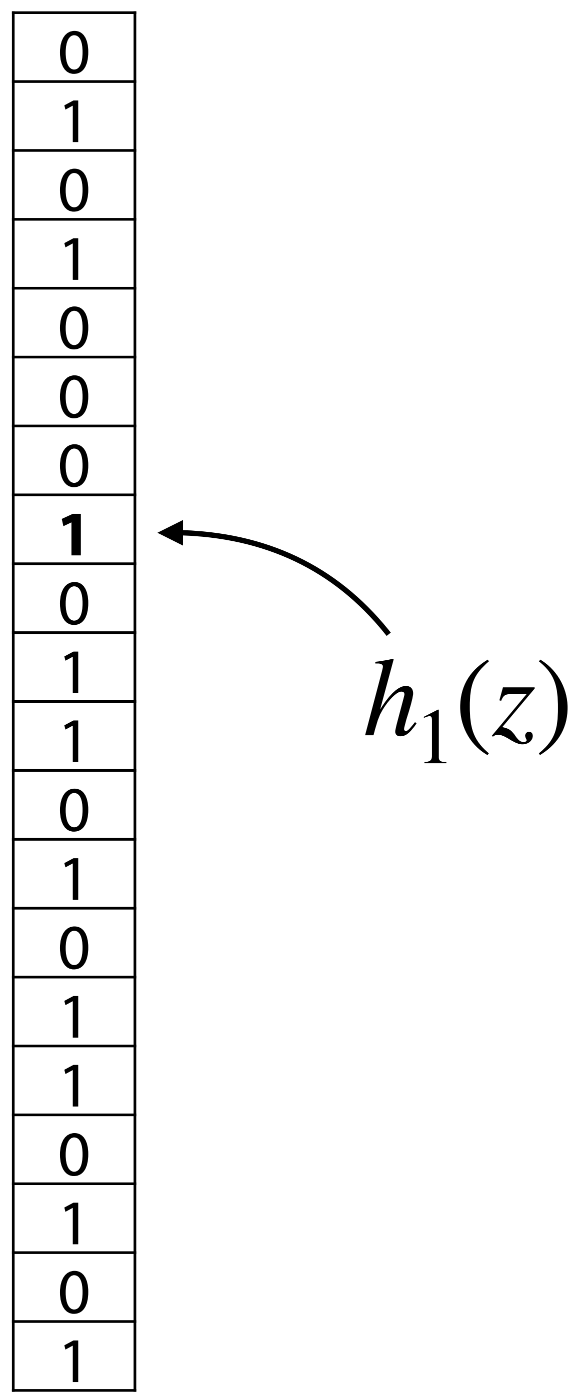
Use same hash function h_1 to hash query item; check if bucket is 0 or 1

Bloom filter: querying

If filter says "0" -- item is definitely not present



Bloom filter: querying



If filter says "0" -- item is definitely not present

If filter says "1" -- item *may be* present

...or may be a *collision*

One-sided error

Randomized algorithms can exploit one-sided error via repeated trials

Bloom filter

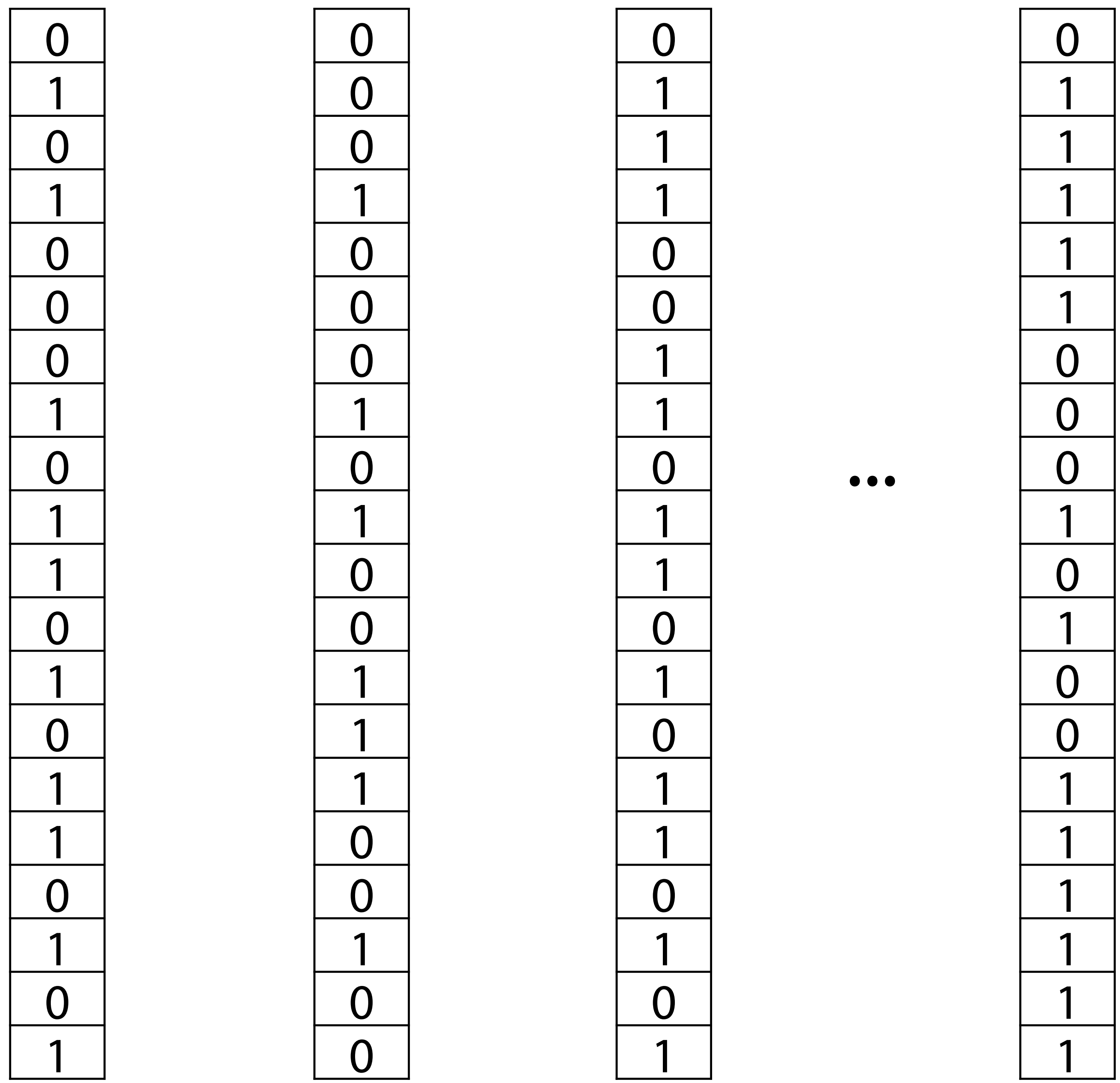
What if we used many hashes/filters; adding each item to each?

h_1	h_2	h_3	...	h_k
0	0	0		0
1	0	1		1
0	0	1		1
1	1	1		1
0	0	0		1
0	0	0		1
0	0	1		0
1	1	1		0
0	0	0		0
1	1	1		1
1	0	1		0
0	0	0		1
1	1	1		0
0	1	0		0
1	1	1		1
1	0	1		1
0	0	0		1
1	1	1		1
0	0	0		1
1	0	1		1

Will all filters have the same number of set (=1) bits?

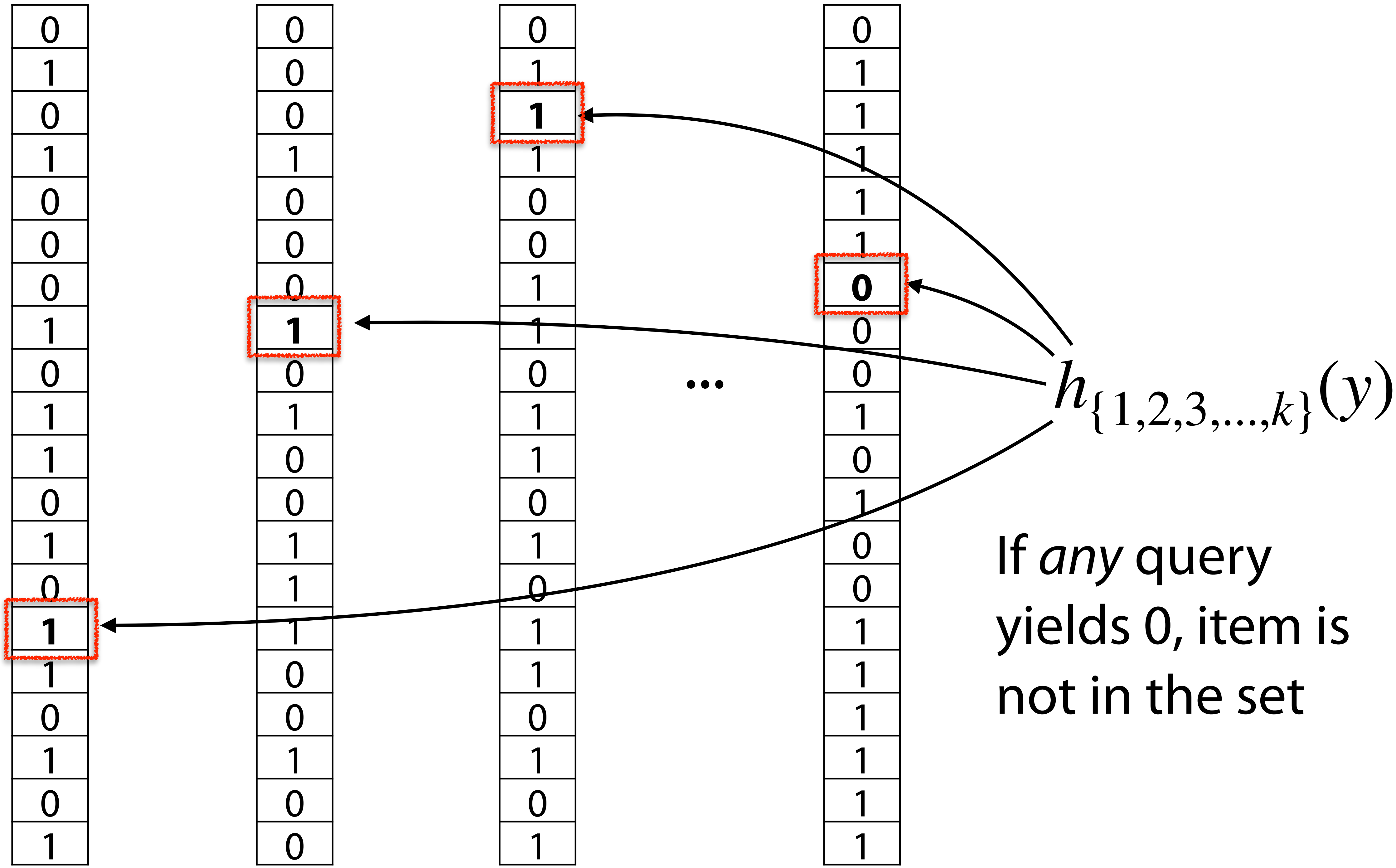
Not necessarily; one hash/filter might have more/fewer collisions than another

Bloom filter



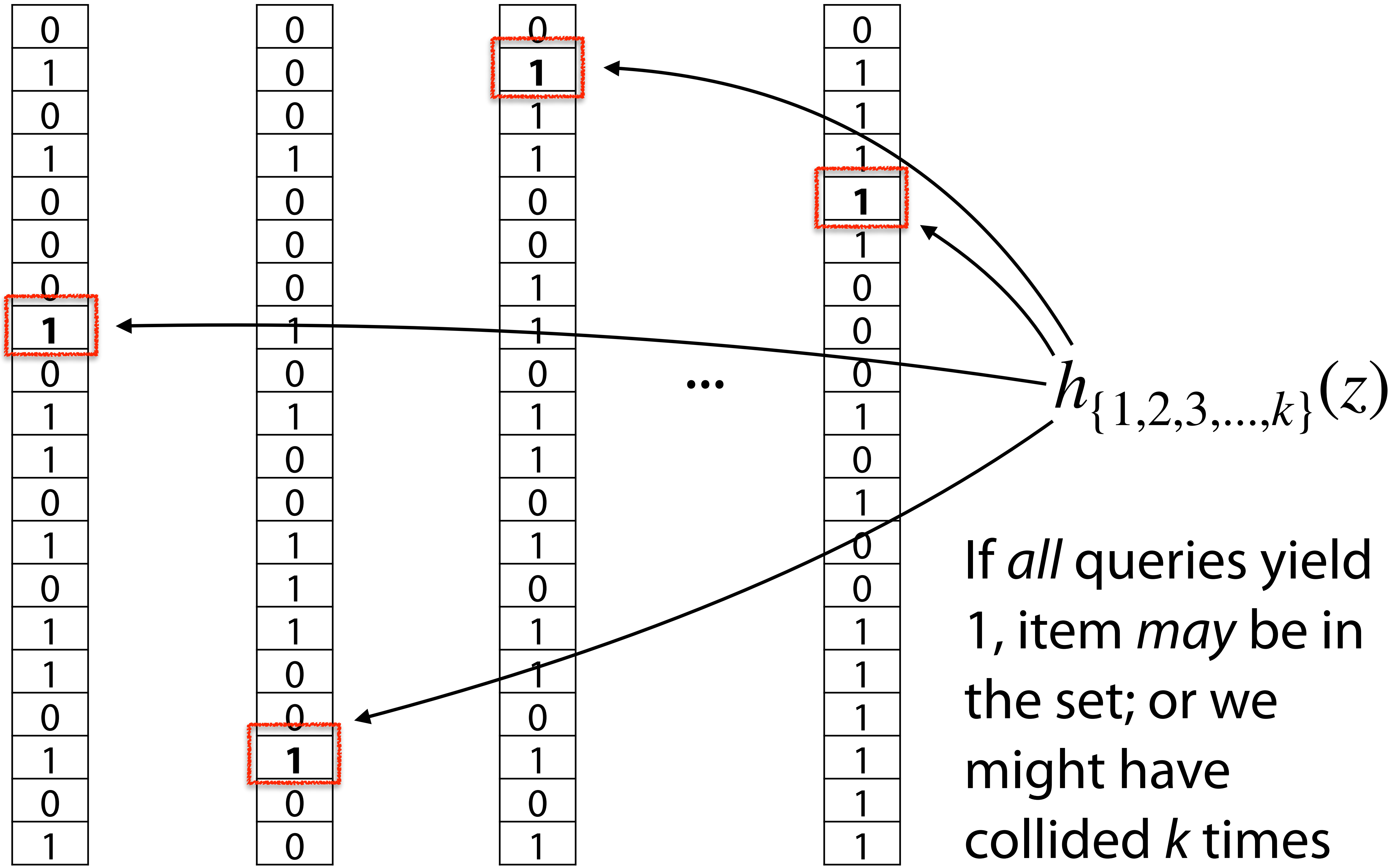
$$h_{\{1,2,3,\ldots,k\}}(y)$$

Bloom filter



If *any* query yields 0, item is not in the set

Bloom filter



If *all* queries yield 1, item *may* be in the set; or we might have collided *k* times

Bloom filter

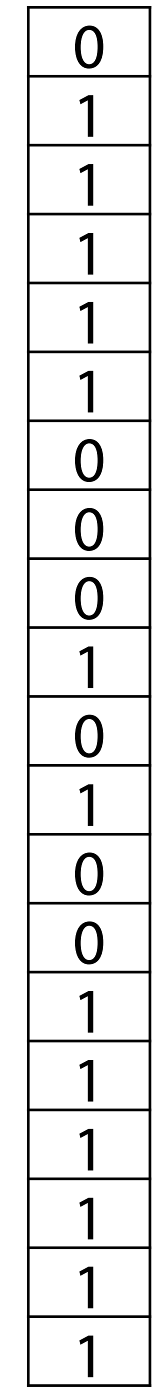
Say item is not present; chance that k filters all return 1 is ... p_1^k

...increasing k reduces error *exponentially*

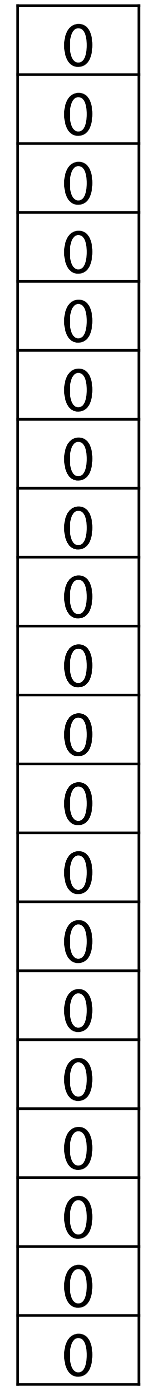
Say $p_1 = 50\,\%$ for all filters; 10 filters give collective error rate of $\left(\frac{1}{2}\right)^{10} < 0.1\,\%$

p_1 : fraction of bits set

k : number of hash functions

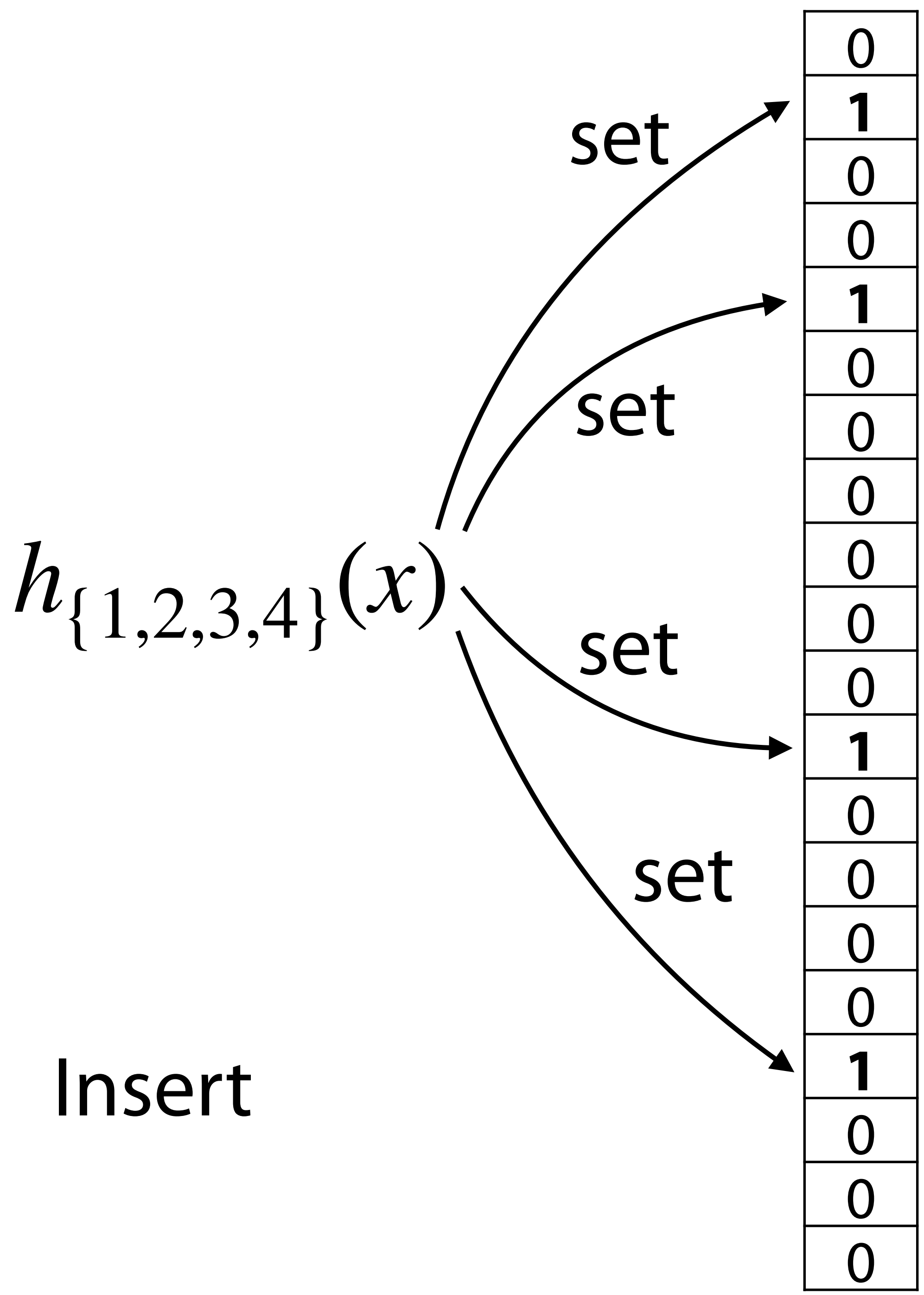


Bloom filter



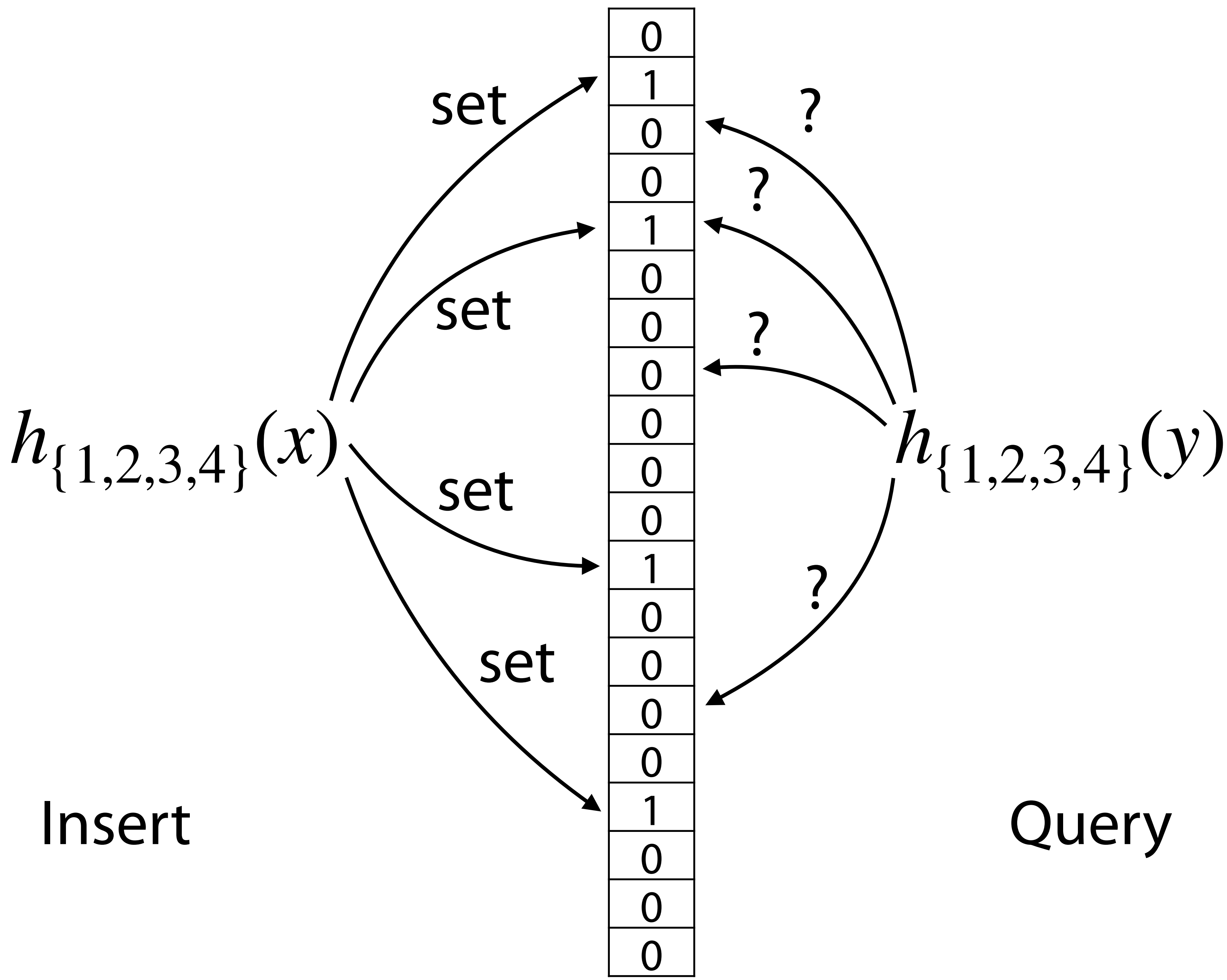
Rather than use a new filter for each hash, one filter can use k hashes

Bloom filter



Rather than use a new filter for each hash, one filter can use k hashes ($k = 4$ here)

Bloom filter



Rather than use a new filter for each hash, one filter can use k hashes ($k = 4$ here)

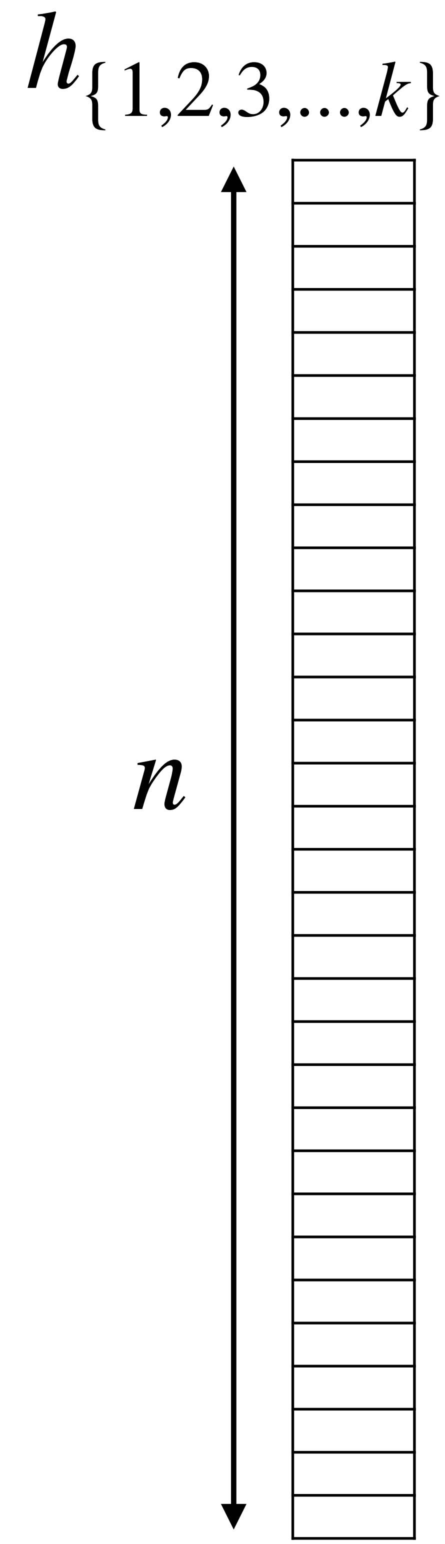
Bloom filter

Say filter has n bits and uses k hash functions

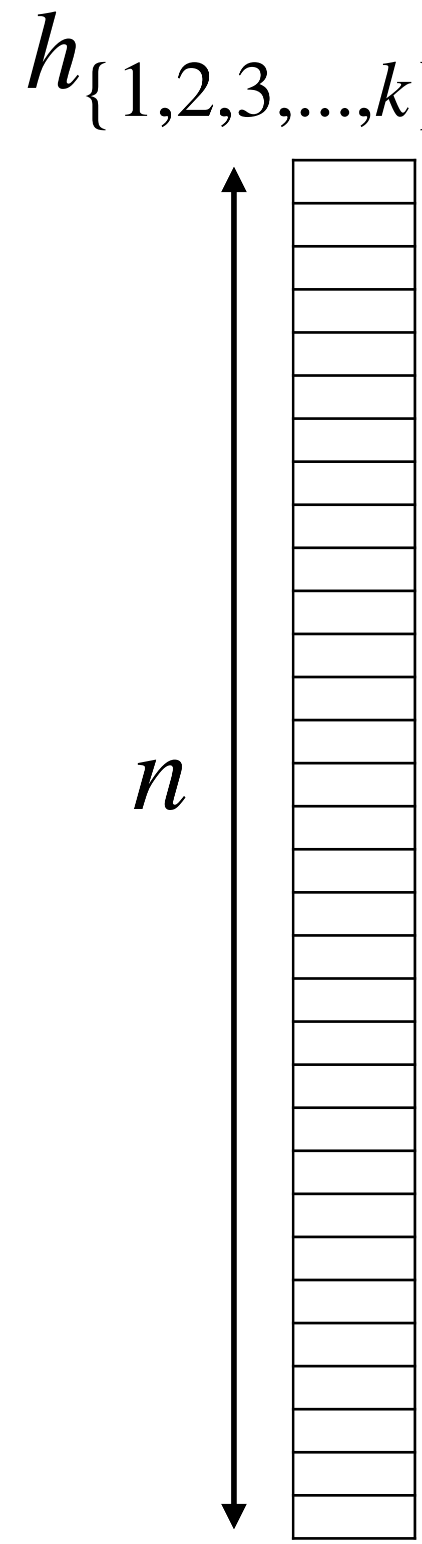
Assume hashes are well behaved, i.e. uniform and independent

What's the probability a given bit is 0 after m items have been inserted?

$$\left(1 - \frac{1}{n}\right)^{mk}$$



Bloom filter



Probability a given bit is 0 after m items are inserted

$$p_0 = \left(1 - \frac{1}{n}\right)^{mk}$$

$$p_1 = 1 - p_0$$

What's the probability of a *false positive*? Where all k hash functions find a 1?

$$p_1^k = (1 - p_0)^k$$

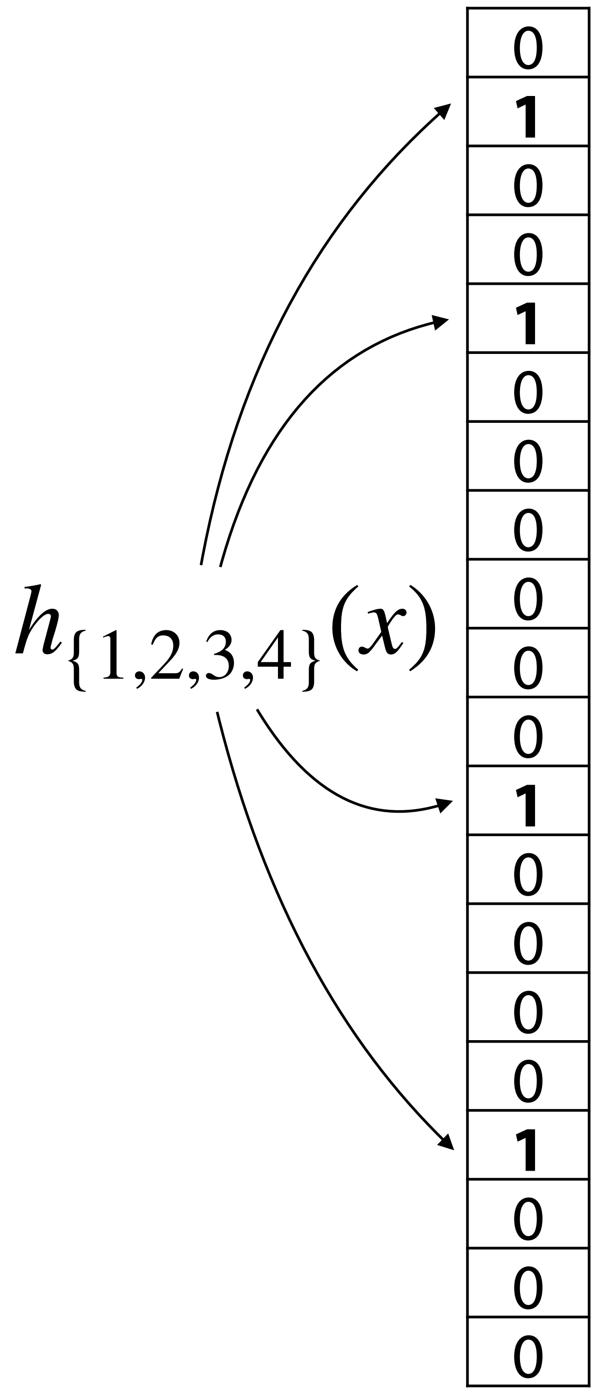
$$= \left(1 - \left(1 - \frac{1}{n}\right)^{mk}\right)^k$$

Call this f

Bloom analysis: road ahead

$$p_0 = \left(1 - \frac{1}{n}\right)^{mk}$$

$$f = \left(1 - \left(1 - \frac{1}{n}\right)^{mk}\right)^{k} = \left(1 - p_0\right)^{k} = p_1^{k}$$



1. Approximations in terms of e

2. Choosing k (# hashes) for fixed m, n

3. Choosing m/n

4. Analyzing fullness with Balls & Bins

Bloom filter

Rewrite $\left(1-\frac{1}{n}\right)^{mk}$ in terms of e:

$$\left(1-\frac{1}{n}\right)^{mk} = e^{\ln\left[\left(1-\frac{1}{n}\right)^{mk}\right]}$$

$$= e^{\ln\left(1-\frac{1}{n}\right)mk}$$

Can we approximate $\ln\left(1-\frac{1}{n}\right)$?

Bloom filter

Taylor expansion of $\ln(1+x)$: $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

"Mercator series"

If x is small (e.g. 0.001), terms beyond first are *really* small (<1 millionth)

$$\ln\left(1 - \frac{1}{n}\right) \approx -\frac{1}{n} \qquad e^{\ln\left(1 - \frac{1}{n}\right)mk} \approx e^{-\frac{mk}{n}}$$

Bloom filter

$$p_0 = \left(1 - \frac{1}{n}\right)^{mk} \approx e^{\frac{-mk}{n}} = \tilde{p}_0$$

$$f = (1 - p_0)^k \approx \left(1 - e^{\frac{-mk}{n}}\right)^k = (1 - \tilde{p}_0)^k$$

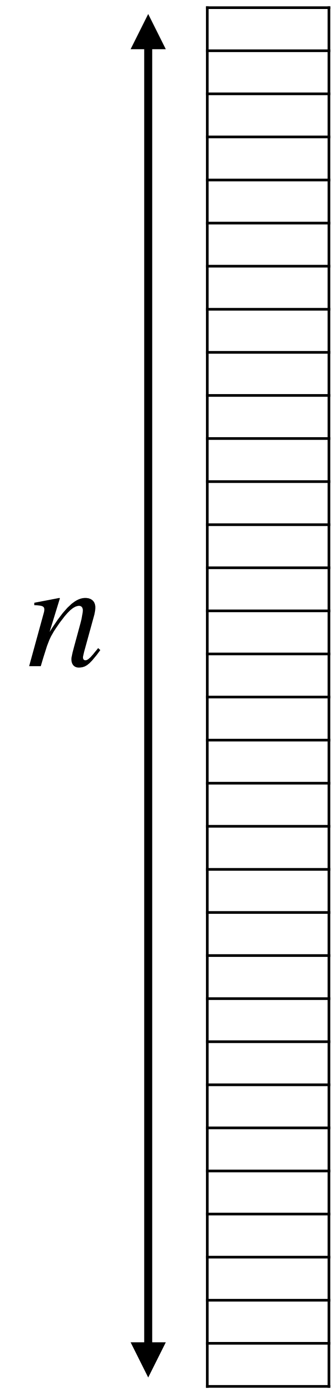
Approximation using Taylor expansion of $\ln(1 + x)$

$$e^{-\frac{m}{n} \cdot k}$$

Where $\frac{m}{n}$ = # data items per slot

Bloom filter

m items



How do we choose # of hash functions k?

Can we have too many?

Yes, quickly clogging the filter with 1s

$$p_1^k$$

p is too close to 1

Too few?

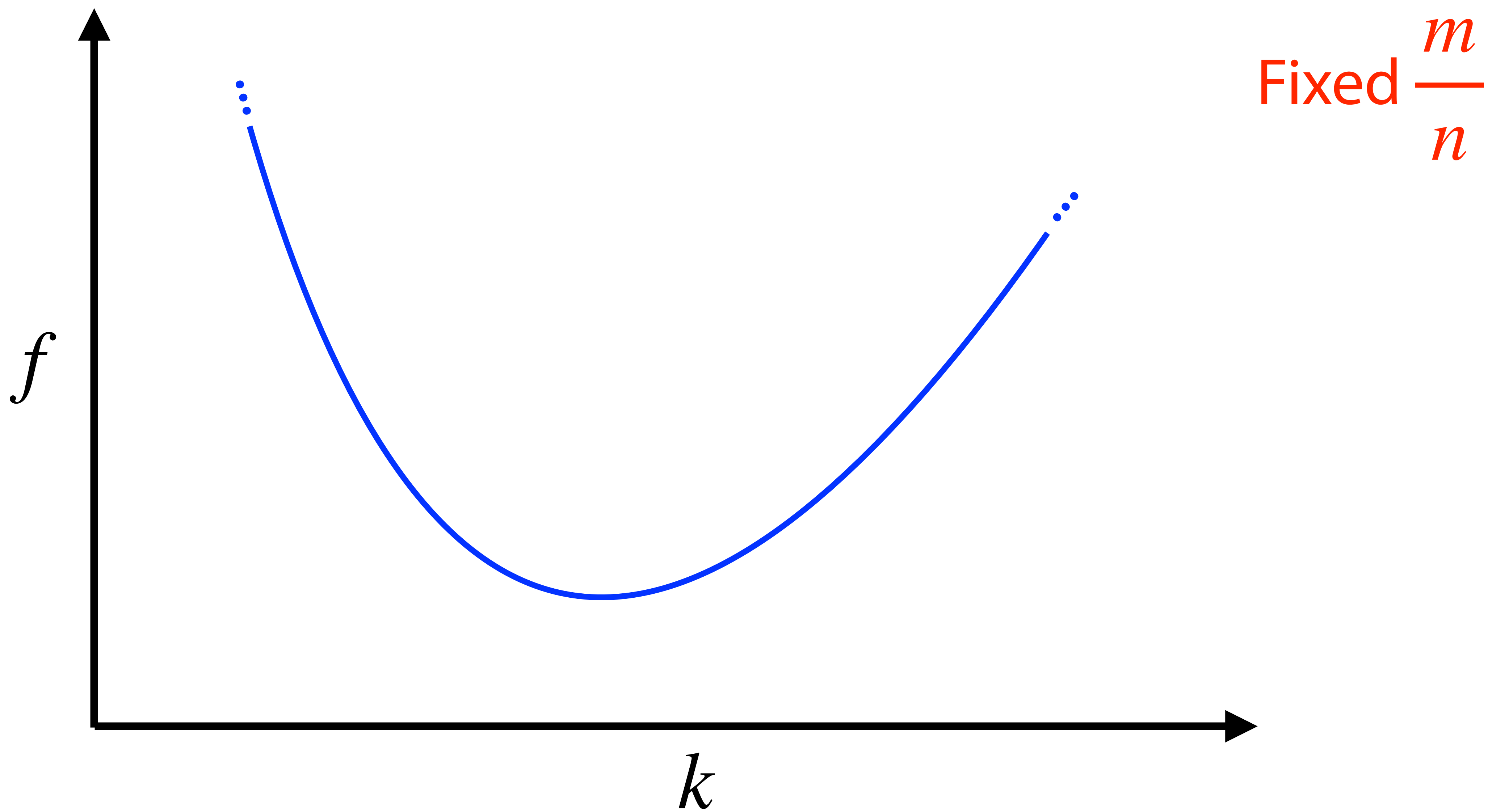
Yes, failing to get much exponential decrease in error

$$p_1^k$$

small k, not much compounding

Bloom filter

We know that f.p. rate $\left(1 - e^{\frac{m}{n} \cdot -k}\right)^k$ gets large for small and large k. To find minimum, find where derivative is 0



Bloom filter

$$\frac{d}{dk}\left(1 - e^{\frac{-mk}{n}}\right)^{k}$$ is tricky, so log the function first

$$\frac{d}{dk}\left[\, k \; \ln\left(1 - e^{\frac{-mk}{n}}\right) \,\right]$$

At top level, use product rule

$$\frac{d}{dk}[k] \cdot \ln\left(1 - e^{\frac{-mk}{n}}\right) = \ln\left(1 - e^{\frac{-mk}{n}}\right) + k \cdot \frac{d}{dk}\left[\ln\left(1 - e^{\frac{-mk}{n}}\right)\right]$$

Need chain rule in here

Bloom filter

$$\frac{d}{dk}\left[k\ln\left(1-e^{\frac{-mk}{n}}\right)\right]=\ln\left(1-e^{-km/n}\right)+\frac{km}{n}\cdot\frac{e^{-km/n}}{1-e^{-km/n}}$$

Derivative is zero when $k=\ln 2\cdot n/m$

$$e^{-km/n}=e^{\ln 2\cdot n/m\cdot -m/n}$$
$$=e^{-\ln 2}=\frac{1}{2}$$

$$\frac{km}{n}=\ln 2\cdot n/m\cdot m/n$$
$$=\ln 2$$

Bloom filter

$$\frac{d}{dk}\left[k\ln\left(1-e^{\frac{-mk}{n}}\right)\right]=\ln\left(1-e^{-km/n}\right)+\frac{km}{n}\cdot\frac{e^{-km/n}}{1-e^{-km/n}}$$

$$e^{-km/n}=\frac{1}{2}$$

$$\frac{km}{n}=\ln 2$$

$$=\ln(1-1/2)+\ln 2\cdot\frac{1/2}{1-1/2}$$

$$=\ln 1/2+\ln 2$$

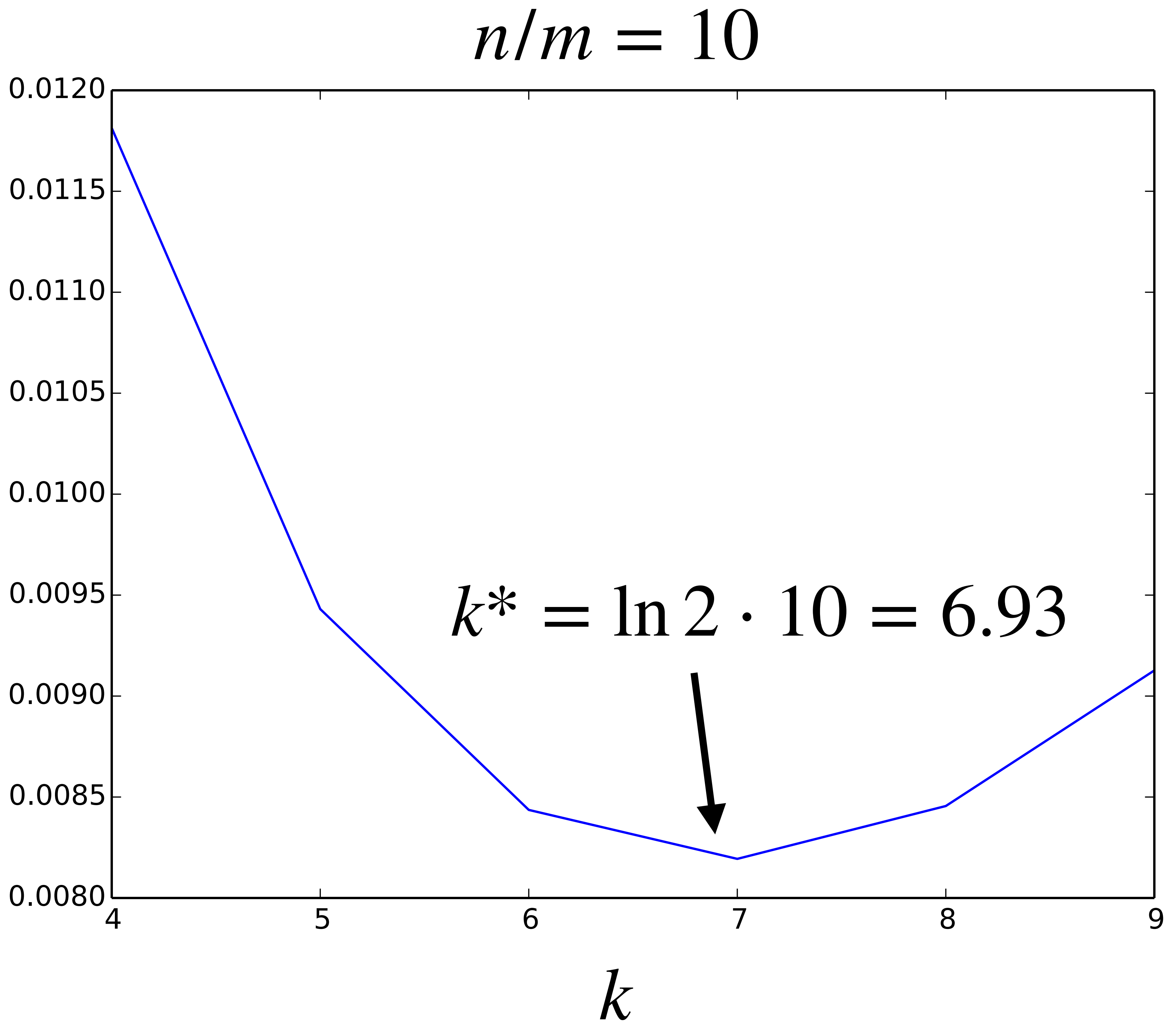
$$=-\ln 2+\ln 2=0$$

Derivative is zero when $k=\ln 2\cdot n/m$

Best choice of k: $k^*=\ln 2\cdot n/m$

Bloom filter

$$\left(1 - e^{\frac{-mk}{n}}\right)^k$$



Bloom filter

If we pick ideal k (# hashes) for fixed m, n, what fraction of the filter do we expect to be set bits?

0
1
0
1
0
1
0
1
0
1
1
0
1
0
1
1
0
1
0
1

Bloom filter

$$e^{-mk/n} = \tilde{p}_0$$

$$\frac{-mk}{n} = \ln \tilde{p}_0$$

$$k = -\ln \tilde{p}_0 \cdot \frac{n}{m}$$

Best choice of k:

$$k^* = \ln 2 \cdot \frac{n}{m}$$

$$-\ln \tilde{p}_0 \cdot \frac{n}{m} = \ln 2 \cdot \frac{n}{m}$$

$$\ln \tilde{p}_0 = -\ln 2$$

$$\tilde{p}_0 = \frac{1}{2}$$

With k chosen optimally, filter is 50% set bits