600.464/664 Rand. Algos.
Key to Midsem Exam (2007 Sp)

I. $f(i) = \sum_{j=1}^{n^2} X_{ij}$ in which $X_{ij} = 1$ if jth ball falls into bin i. $E(X_{ij}) = \frac{1}{n}$. Hence $E(f(i)) = n^2 \frac{1}{n} = n$

Similarly $E(f(j)) = n$.

Failure happens if $\exists\, i,j$ s.t. $f(i) - f(j) > c\sqrt{n \ln n}$.

Fix i & j

$$P\left(f(i) - f(j) > c\sqrt{n\ln n}\right) \le P\left(f(i) \ge n + \frac{c}{2}\sqrt{n \ln n}\right) + P\left(f(j) \le n - \frac{c}{2}\sqrt{n\ln n}\right)$$

[there is nothing special about the offset; it could be any value. But when it is n, we can apply the chernoff bounds].

$$P\left(f(i) \ge n + \frac{c}{2}\sqrt{n\ln n}\right) = P\Big(f(i) \ge n\big(1 + \underbrace{\tfrac{c}{2}\sqrt{\tfrac{\ln n}{n}}}_{\delta}\big)\Big)$$
$$\le e^{-n \frac{c^2}{4} \frac{\ln n}{n} \frac{1}{3}} = \frac{1}{n^{c^2/12}}$$

$$P\left(f(j) \le n - \frac{c}{2}\sqrt{n\ln n}\right) = P\left(f(j) \le n\left(1 - \frac{c}{2}\sqrt{\frac{\ln n}{n}}\right)\right)$$
$$\le e^{-n \frac{c^2}{4} \frac{\ln n}{n} \frac{1}{2}} = \frac{1}{n^{c^2/8}}$$

Make $\frac{c^2}{12} \ge \cancel{2}\,3$; i.e. $c^2 \ge \cancel{24}\,36$.

$c = \cancel{5}\,6$ is fine.

Then $P\left(f(i) - f(j) > c\sqrt{n\ln n}\right) \le \frac{1}{n^3} + \frac{1}{n^3} = \frac{2}{n^3}$

There are $n(n-1)$ ways of choosing (i,j)'s.

Hence $P(\text{there exists } i,j \text{ s.t. } f(i) - f(j) > c\sqrt{n\ln n})$

$$\le \frac{2}{n^3} n(n-1) < \frac{2}{n}.$$

II $$P[\underbrace{AB+B+C+D \le 0}_{E_1}] + P[\underbrace{AC+D > 0}_{E_2}] < 1.$$

$$P(E_1)+P(E_2) = \tfrac{1}{2} P(E_1 | A=-1) + \tfrac{1}{2} P(E_1 | A=+1) + \tfrac{1}{2} P(E_2 | A=-1) + \tfrac{1}{2} P(E_2 | A=+1)$$

$$= \tfrac{1}{2}\Big(\underbrace{[P(E_1|A=-1) + P(E_2|A=-1)]}_{S_1} + \underbrace{[P(E_1|A=+1) + P(E_2|A=+1)]}_{S_2}\Big) < 1$$

since $\frac{S_1+S_2}{2} < 1$, ~~S_1 or S_2 is less than~~ $S_1 < 1$ or $S_2 < 1$.

now compute S_1 & S_2, then choose the value of A that minimizes the sum

$P(E_1 | A=-1) = P(C+D \le 0) = \frac{3}{4}$

$P(E_2 | A=-1) = P(-C+D > 0) = P(C+D > 0) = \frac{1}{4}$

Hence $S_1 = \frac{3}{4} + \frac{1}{4} = 1$

$P(E_1 | A=+1) = P(2B+C+D \le 0) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$

$P(E_2 | A=+1) = P(C+D > 0) = \frac{1}{4}$

Hence $S_2 = \frac{5}{8} + \frac{1}{4} = \frac{7}{8}$.

since $S_2 < S_1$, choose $A = +1$.

III Fix $i,j \in \{1,2,3\}$, $i \ne j$. Let k be the third index; i.e. $\{k\} = \{1,2,3\} \setminus \{i,j\}$. Fix $S \subseteq A_i$ $|S| = dn$, and $S' \subseteq A_j$, $|S'| = cdn-1$. Let $|S| = s$, $|S'| = s'$

$$P(\text{all edges from } S \text{ fall in } S' \cup A_i \cup A_k) < \left[\frac{s'+2n}{3n} \cdot \frac{s'+2n-1}{3n-1} \cdots \frac{s'+2n-s+1}{3n-s+1}\right]^d$$

$$< \left(\frac{s'+2n}{3n}\right)^{sd}$$

Applying Boole's inequality for all choices of i, j, S & S'

$$P(\text{failure}) < 6 \binom{n}{s}\binom{n}{s'}\left(\frac{s'+2n}{3n}\right)^{sd}.$$

If we make $s' = \frac{n}{2}$ & d large, $P(\text{failure})$ becomes less than 1.

Hence $P(\text{the given properties are satisfied}) > 0$.

Hence a graph with the given properties exists.