600.464/664 Rand. Algos.

Key to HW 6

I. ~~(a)~~ Let the canonical path between vertices i and j, $i<j$, be $i, i+1, i+2, \ldots, j$.

(This is not the best choice since no path uses the edge $(1,n)$. Shortest path is a better choice. But the order of value will not change).

$\pi = (\frac{1}{n} \; \frac{1}{n} \cdots \frac{1}{n})$.

For any edge $(i, i+1)$, $Q_e = \frac{1}{4} \cdot \frac{1}{n} = \frac{1}{4n}$.

For a path $\overbrace{\text{to use}}^{u \to \cdots \to v}$ this edge, $u \in \{1, 2, \ldots, i\}$ & $v \in \{i+1, \ldots, n\}$. Hence the number of paths using that edge $= i(n-i)$.

Hence $\rho_e = 4n \; i(n-i)\left(\frac{1}{n} \cdot \frac{1}{n}\right)$

This is maximized when $i = \frac{n}{2}$. Then

$$\rho = 4n \frac{n}{2} \cdot \frac{n}{2} \frac{1}{n^2} = n$$

Hence $\Phi \geq \frac{1}{2n}$

Hence $1 - \lambda_2 \geq \frac{\Phi^2}{2} \geq \frac{1}{8n^2}$.

II. For G, $\pi = (\frac{1}{n} \; \frac{1}{n} \cdots \frac{1}{n})$.

For any edge e, $Q_e = \frac{1}{d} \frac{1}{n} = \frac{1}{dn}$ (Sorry: forgot to state uniform deg d).

Consider any spanning tree for G. ~~For~~ any 2 vertices u & v, let the canonical path be the shortest path in the tree.

~~Any~~ ~~For~~ Any edge e, in the tree breaks up the tree into 2 subtrees. Let their sizes be

k and $n-k$.

Total # of canonical paths that go through the edge are $k(n-k)$

Hence $\rho_e = dn\, k(n-k) \frac{1}{n^2}$

This is maximized when $k = n/2$.

Hence $\rho = dn \frac{n}{2} \frac{n}{2} \frac{1}{n^2} = \frac{dn}{4}$.

Hence $\phi \geq \frac{2}{dn}$

Hence $1-\lambda_2 \geq \frac{2}{d^2 n^2}$.

III The canonical paths between any u & v is the Shortest path in the given spanning tree.

$$q_e = \frac{1}{dn} \quad (\text{as before})$$

$$\hat{\rho}_e = \frac{1}{q_e} \sum (\text{length of } i \text{ to } j \text{ path}) \pi_i \pi_j$$

$$\hat{\rho}_e \leq dn \left(2 \log n \frac{1}{n^2}\right) k(n-k)$$

Once again setting $k = n/2$

$$\hat{\rho} \leq dn\, 2 \log n \frac{1}{n^2} \frac{n^2}{4} = \frac{dn \log n}{2}.$$

Hence $1-\lambda_2 \geq \frac{2}{dn \log n}$

(Note: this is a larger value than the value in II)

IV I could not derive any clean bound for any $\epsilon < \frac{1}{2}$ & $\lambda_2 < 1$.

Some Students have nice analyses under some ~~mild~~ mild assumptions.