Zero-Knoweldge Proofs

Abhishek Jain

Zero Knowledge: History

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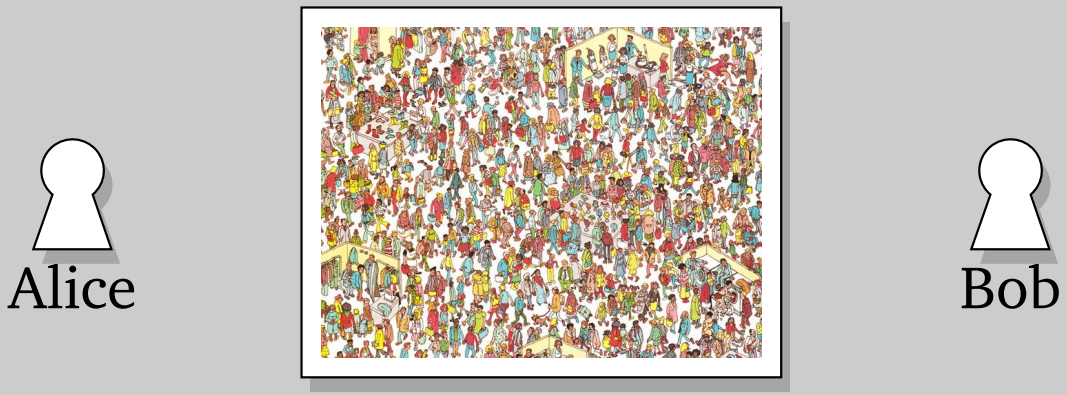
- Shafi and Silvio won the 2012 Turing Award for work on encryption and proof systems.

Scenario: Where's Waldo?



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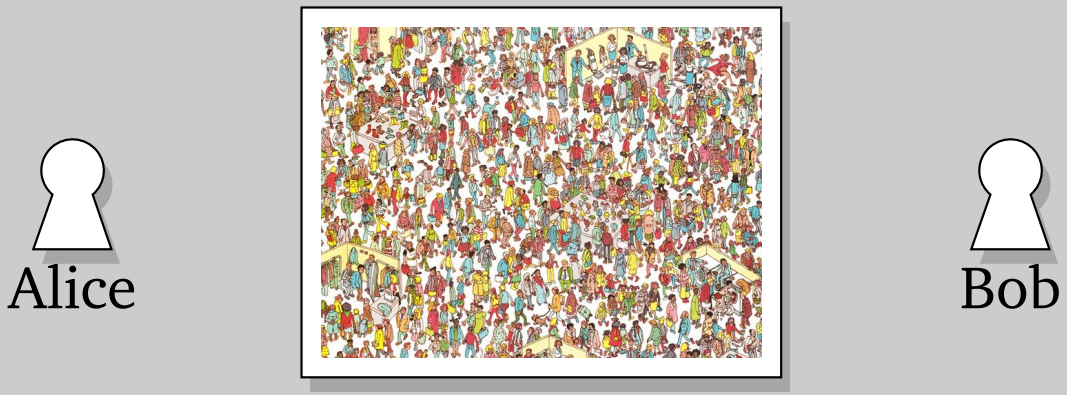
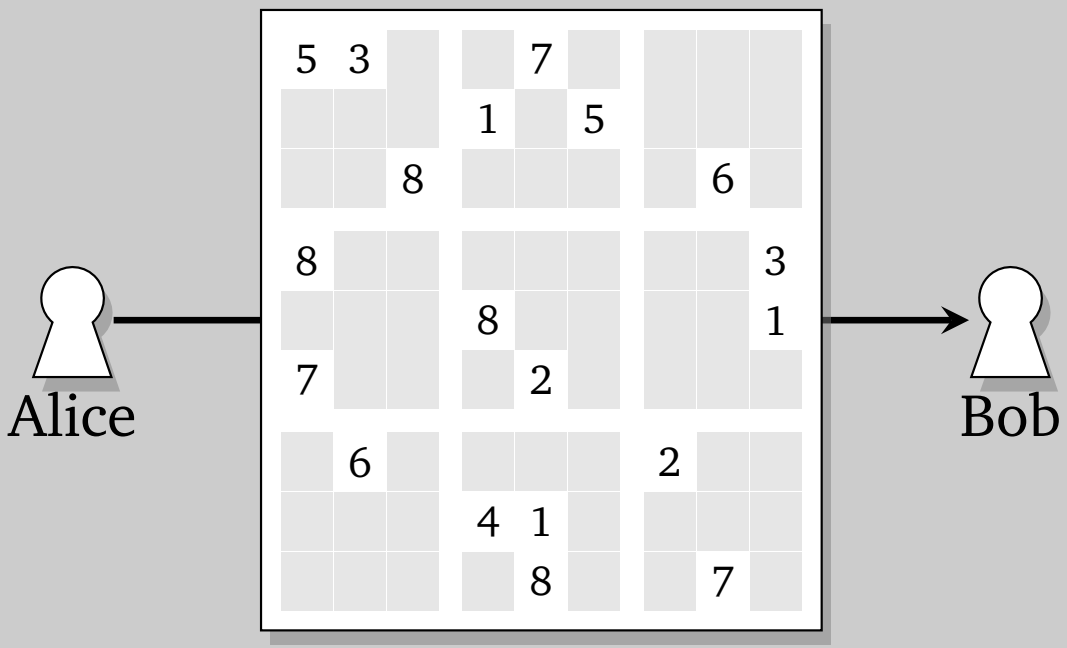


A "Hey Bob, I found Waldo!"

B "That was way too fast, I don't believe you."

Scenario: Sudoku



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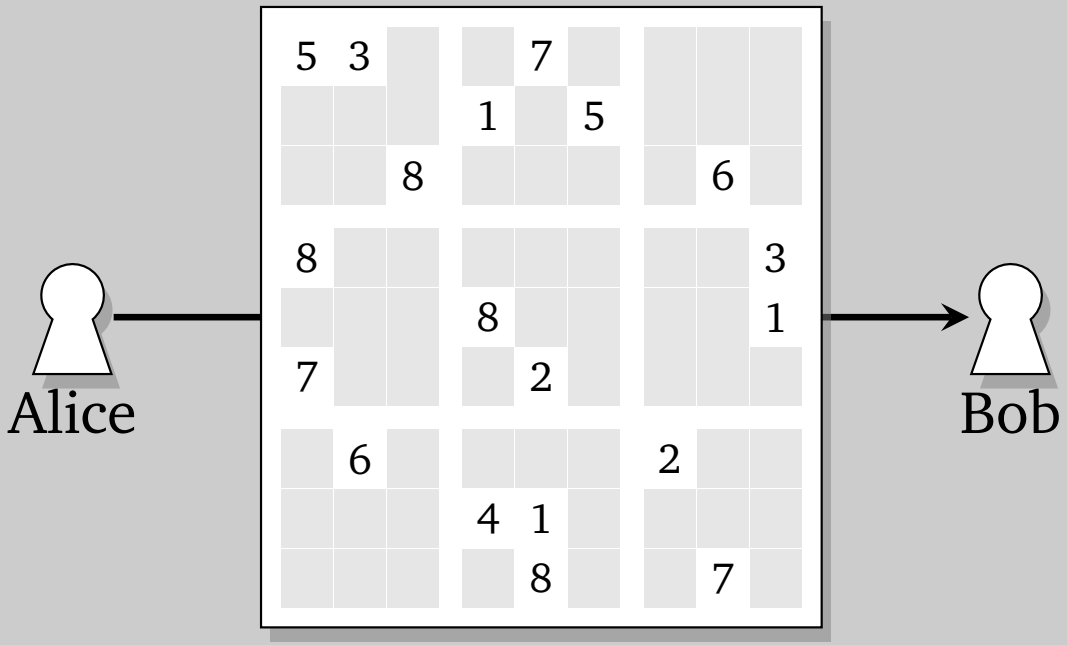

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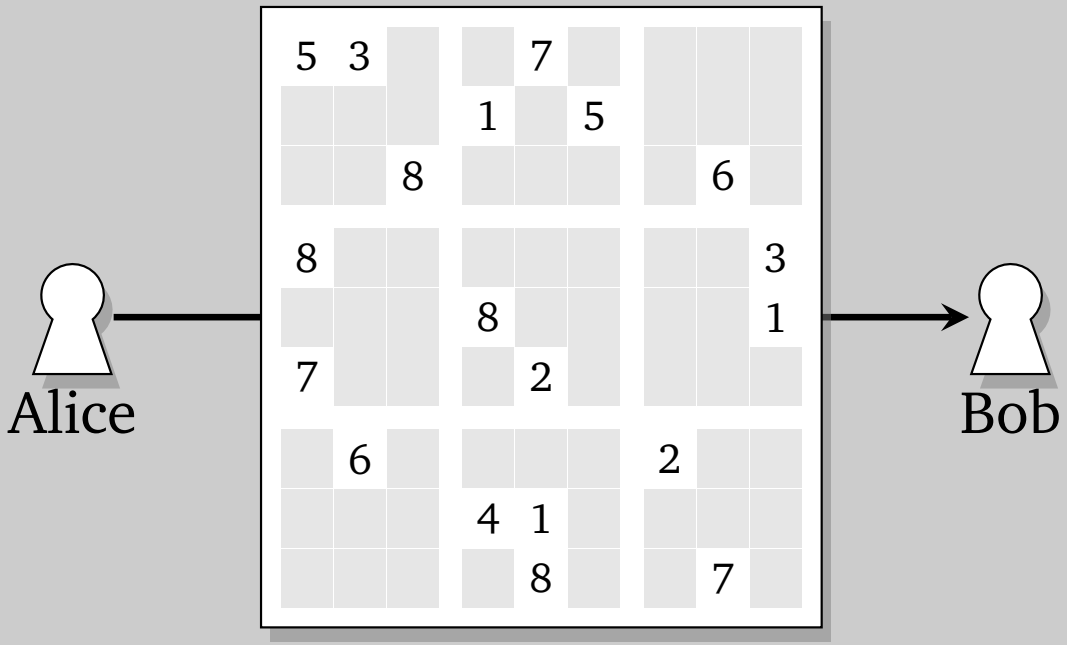

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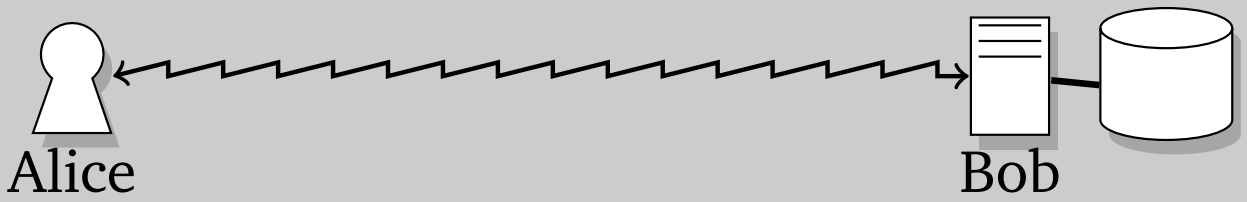
A "This one has a solution, trust me."

Scenario: Authentication



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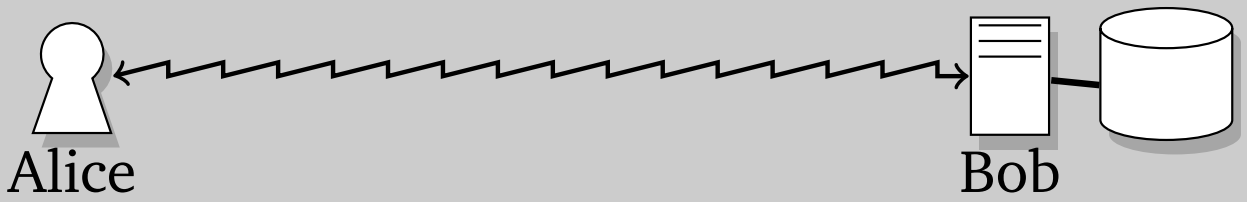

A "Can I have access to the database? It's me, Alice."

B "OK, send me your password so I know it's you."

A Problem of Trust and Information

Alice wants to convince Bob of something

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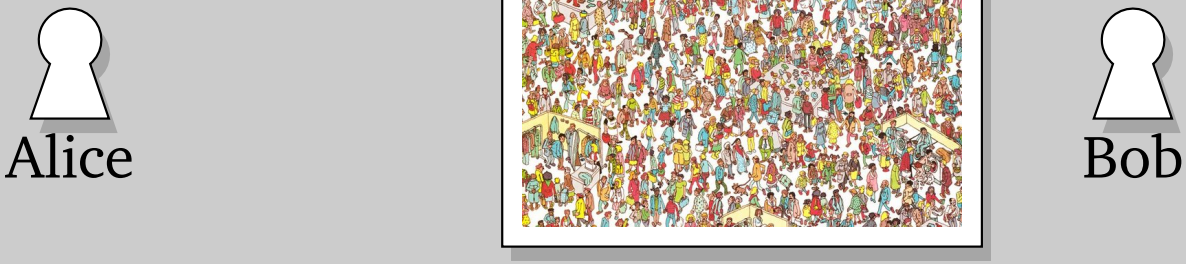
What might a possible solution look like?

Where's Waldo? Solution



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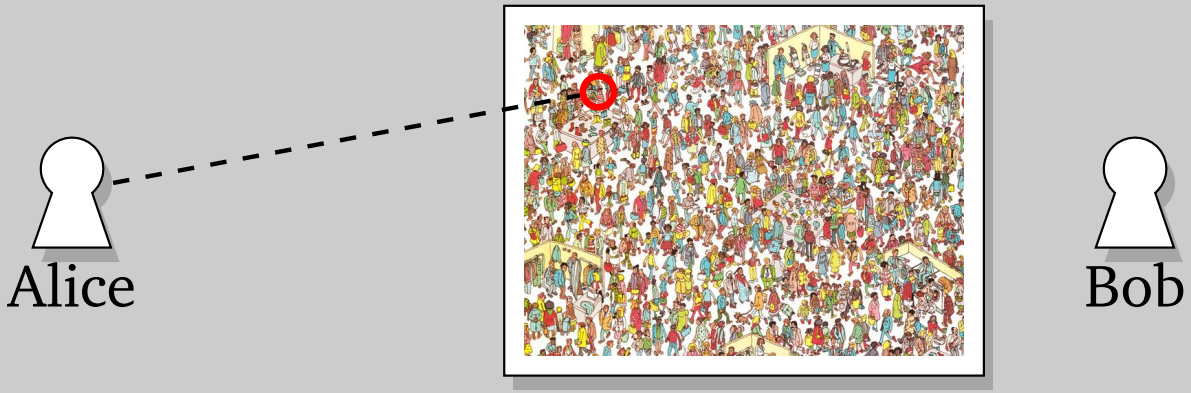

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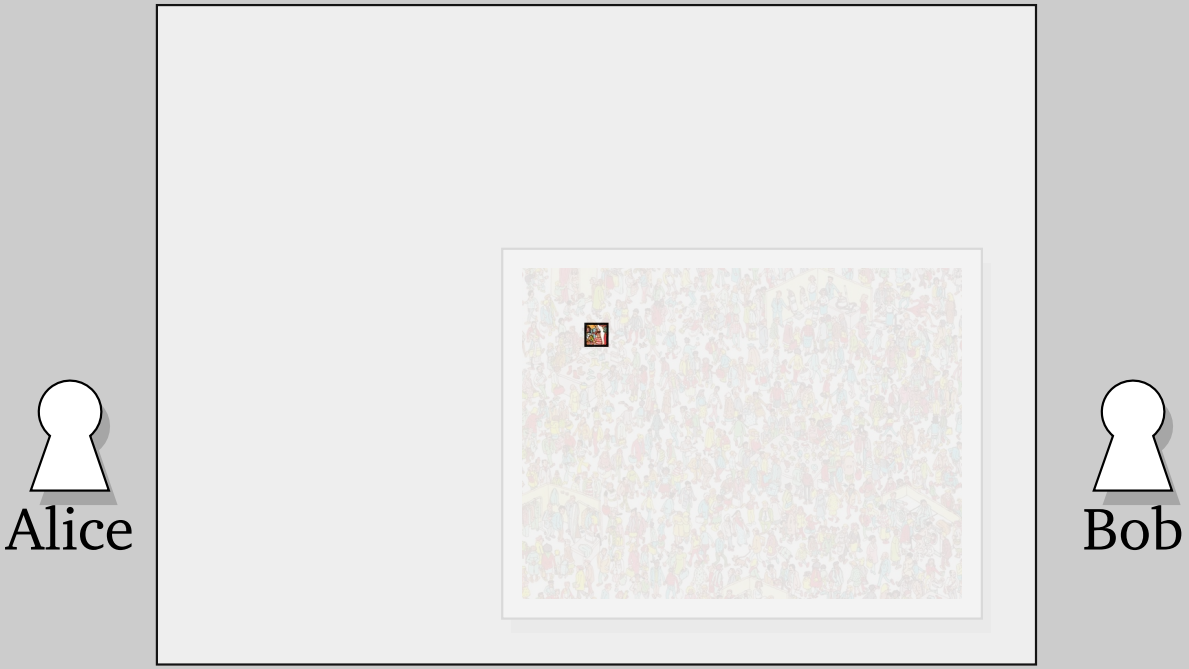

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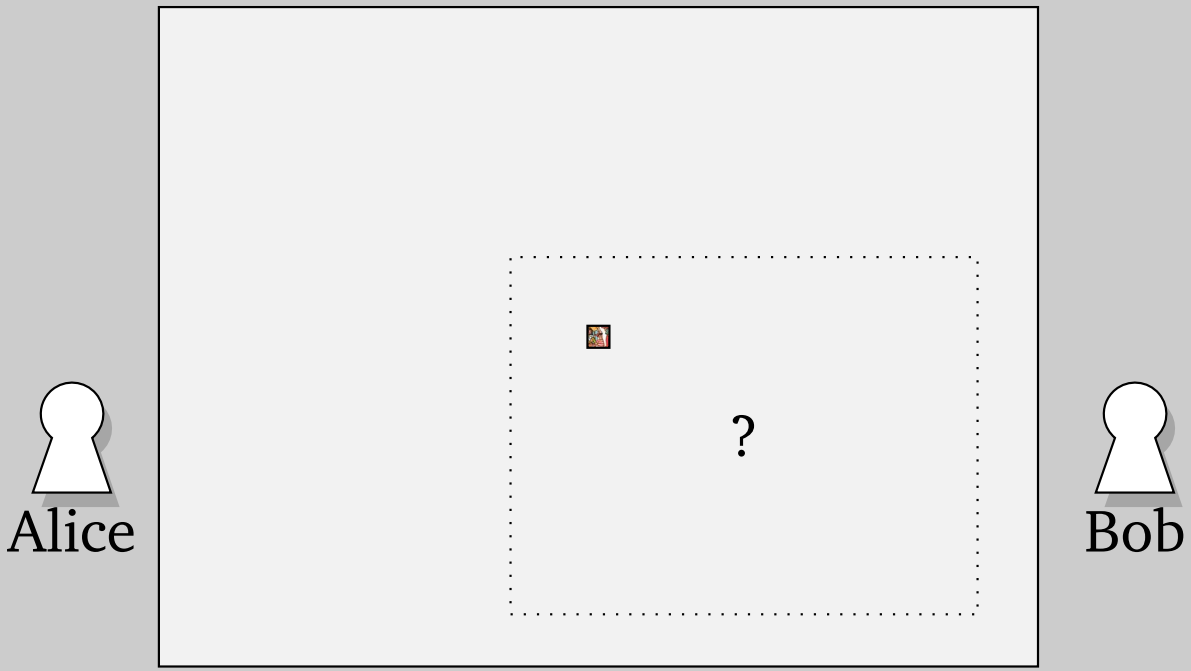

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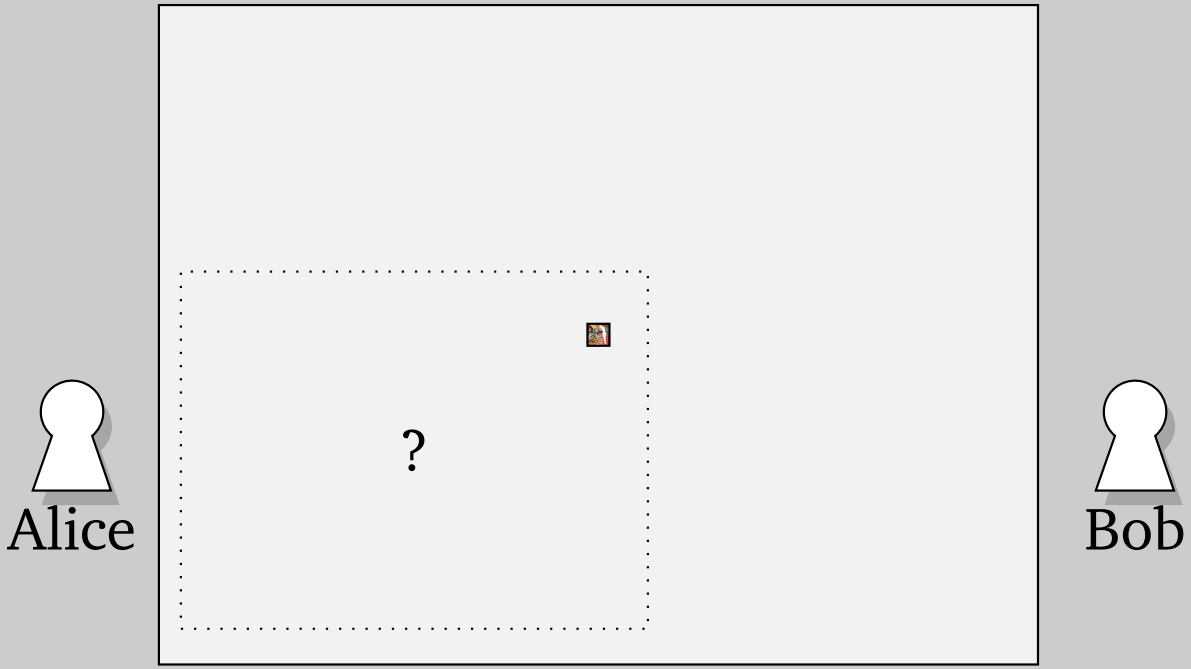

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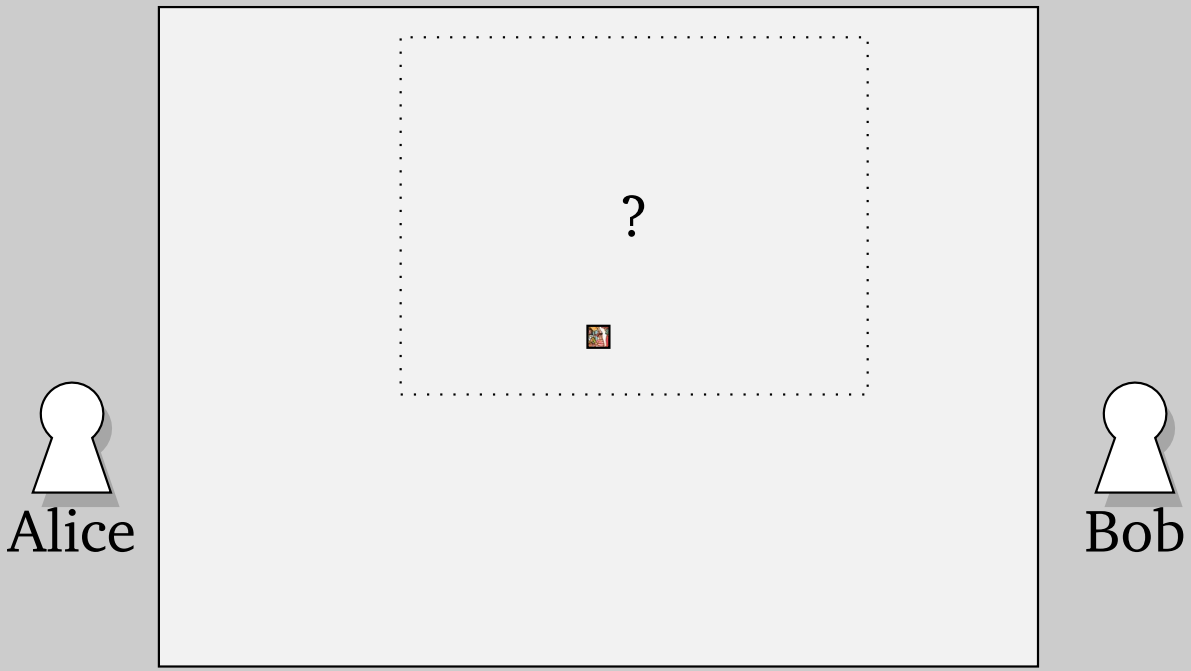

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Philosophy

Fuzzy Definition

A zero-knowledge proof is a way to convince someone of a fact without giving out "any additional information"

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What does it mean to

- prove something?
- give out information?

What is a proof?

Classical Definition

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Today: Interactive Proofs

A Lady Testing Tea

Muriel

Ronald

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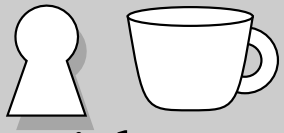

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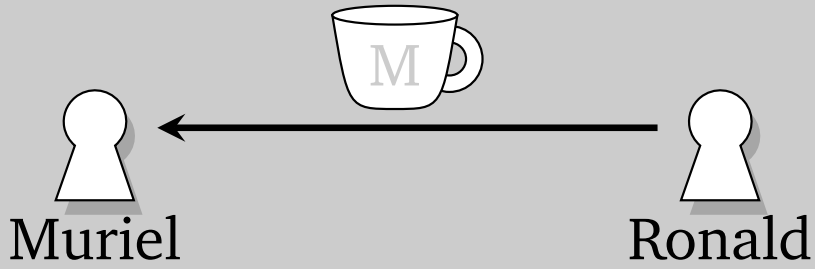

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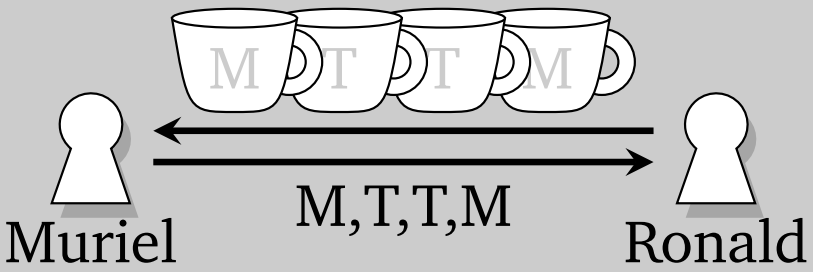

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Repeat Repeat n times.

- If no difference in two kinds of teas, she has $(1/2)^n$ chance of guessing *all* correctly.

Epistemology: What is Knowledge?

Bad situation

x is true

Alice

Bob

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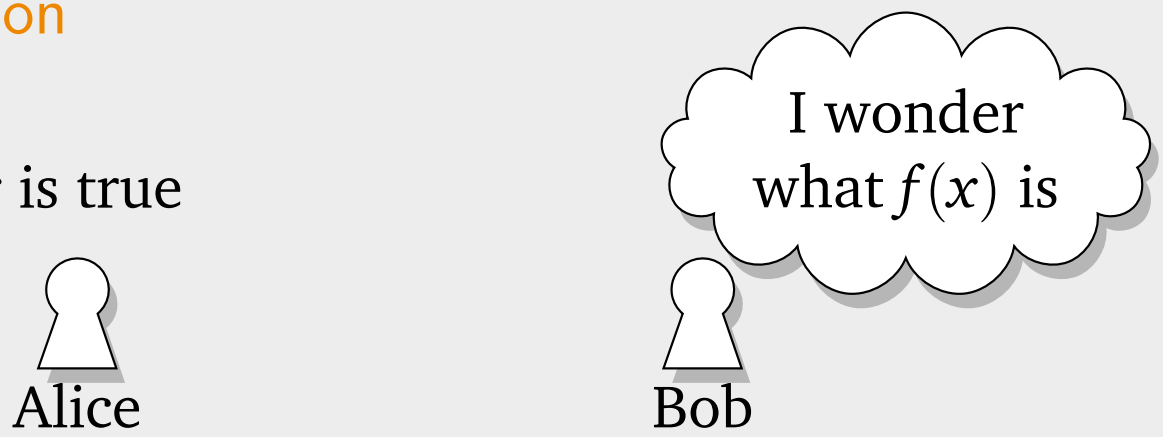

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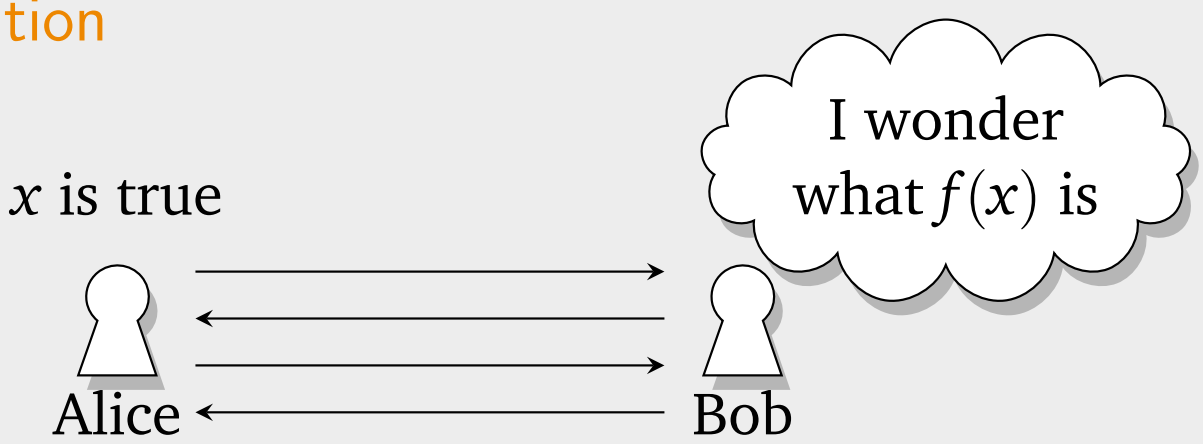

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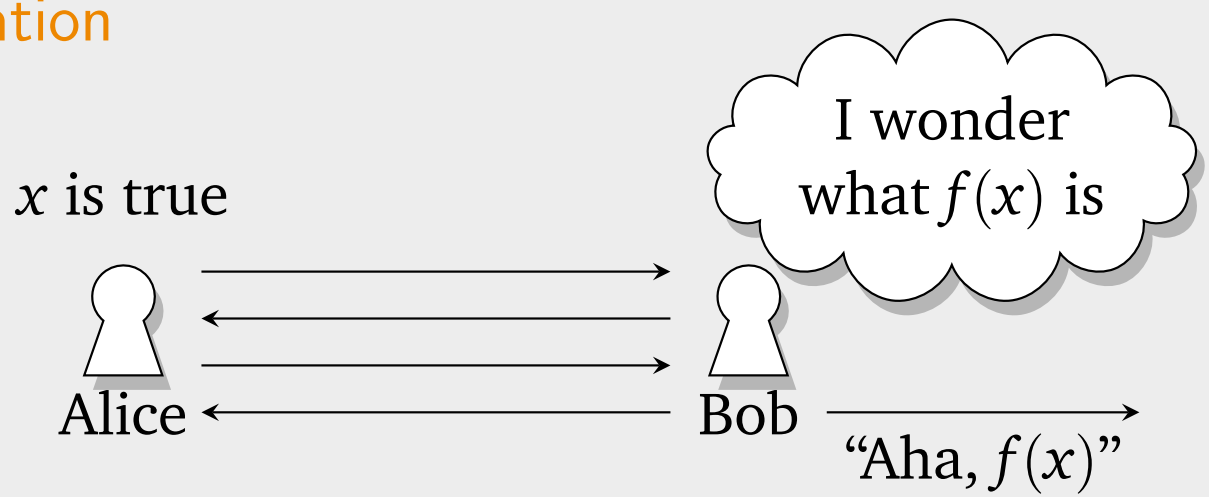

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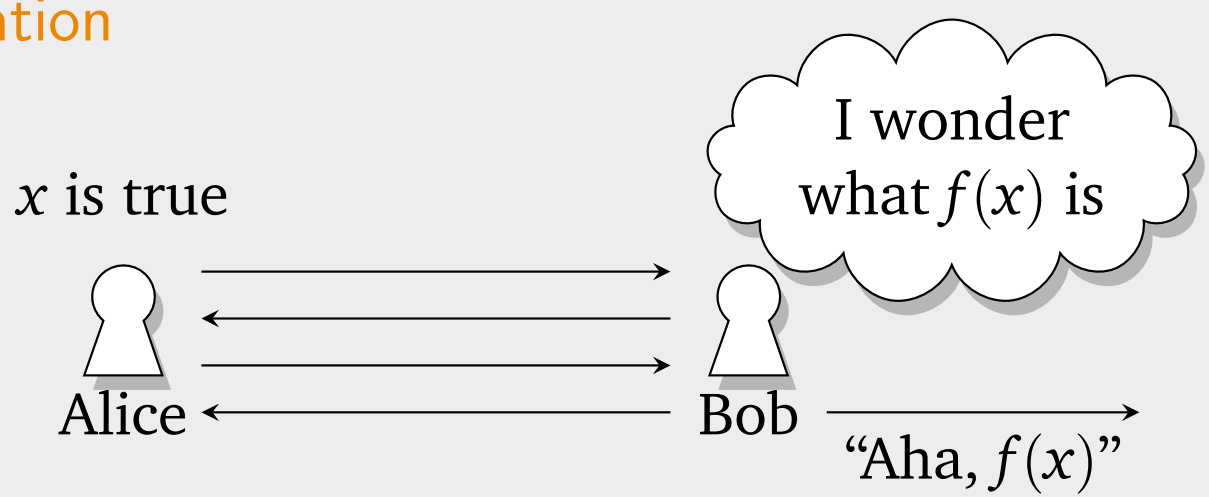

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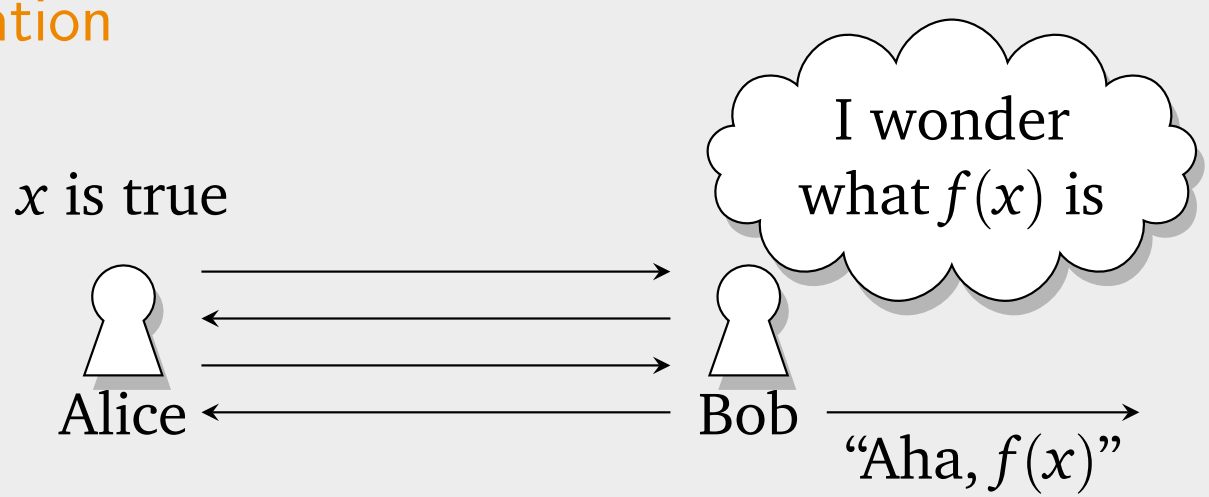

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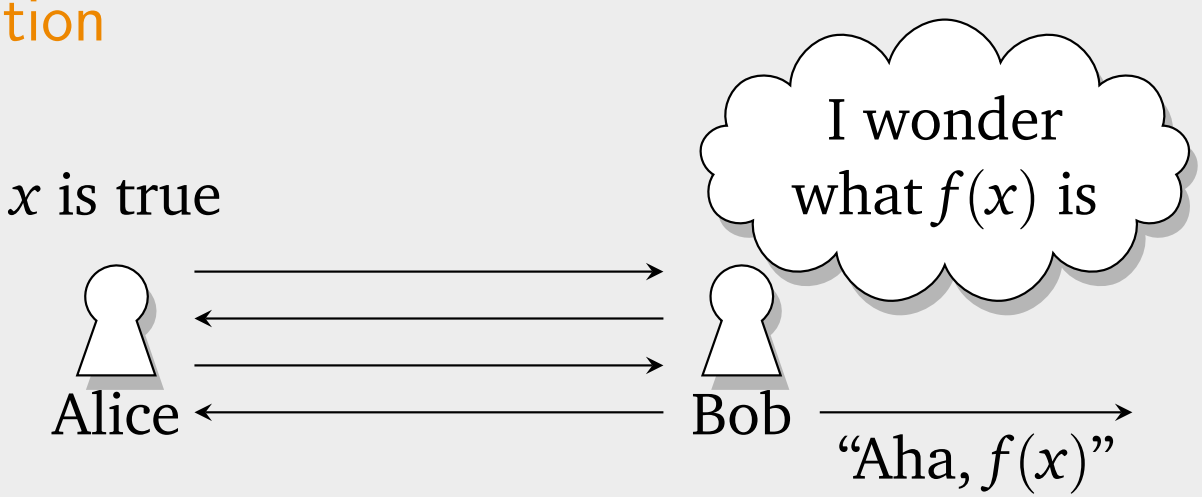

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Zero Knowledge (Want to say):

Everything Bob can compute *after* seeing the transcript, he could have computed *before* seeing the transcript.

Transcript Simulation

Clever Definition

Interaction is zero-knowledge if Bob could generate transcripts without interacting with Alice:

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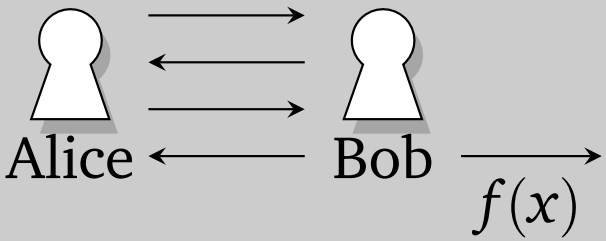

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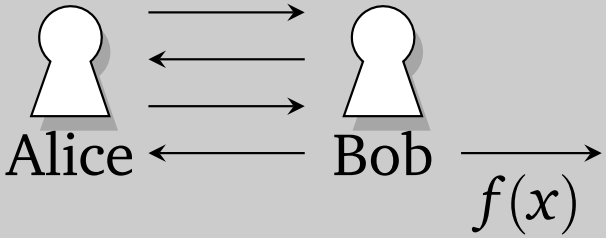

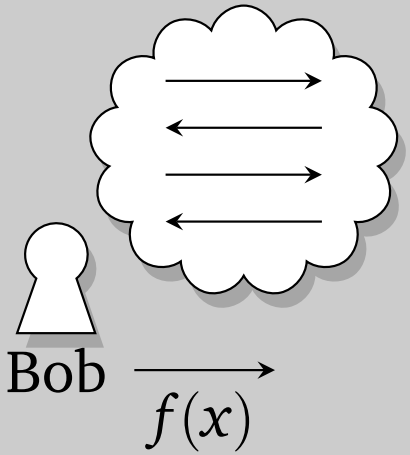

Whatever Bob could compute *after* seeing the transcript ...

... there is a way to compute without interaction!

Apparent Paradox

Paradox?

- Transcript should *convince* Bob of something new
- Bob could have generated transcript himself

Apparent Paradox


Bob


Charlie

- “Alice can tell whether tea is poured into milk or vice-versa!”

Apparent Paradox


Bob


Charlie

B "Alice can tell whether tea is poured into milk or vice-versa!"

C "Oh really?"

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Bob already knew the correct responses to challenges

- Convinced by *how* the transcript was generated (in response to his challenges)

Formal Definition

Definition [GMR 1985]

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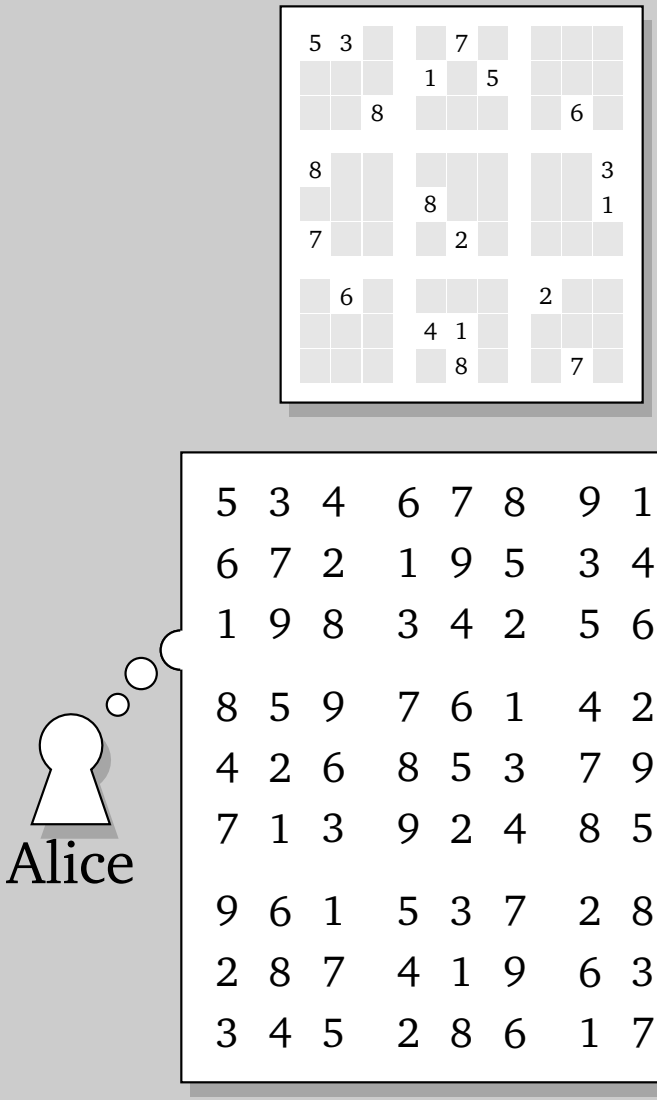
Definition [GMR 1985]

A zero-knowledge proof is an interactive protocol satisfying:

- The prover can always convince the verifier of any true statement
- The verifier can't be convinced of a false statement (even by a cheating prover), except with very low probability
- There is an efficient procedure to output "same-looking" protocol transcripts

Sudoku Zero-Knowledge Proof

Zero-knowledge protocol:



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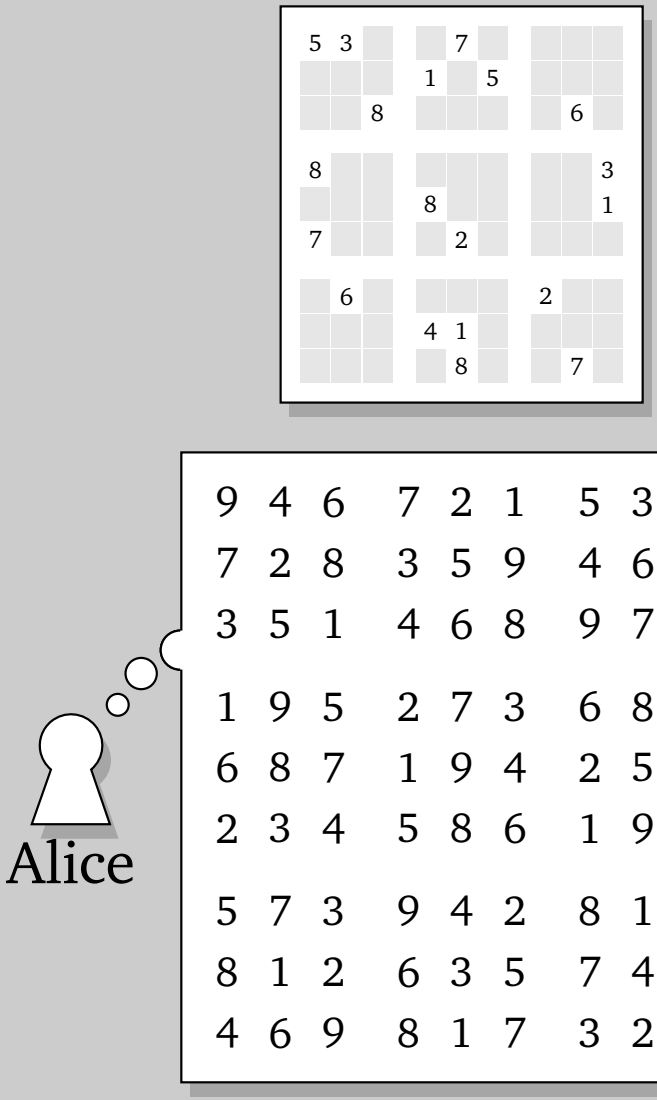
1. Alice randomly relabels $\{1, \ldots, 9\}$



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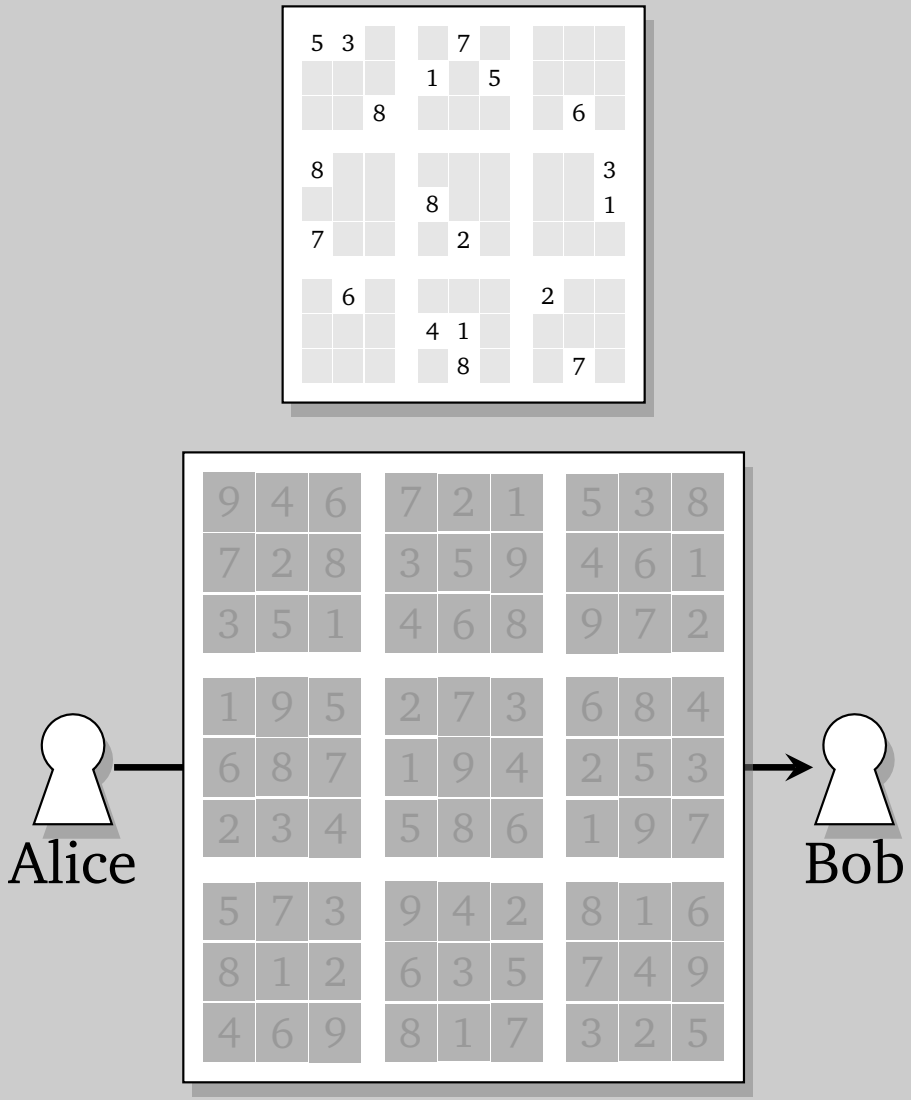

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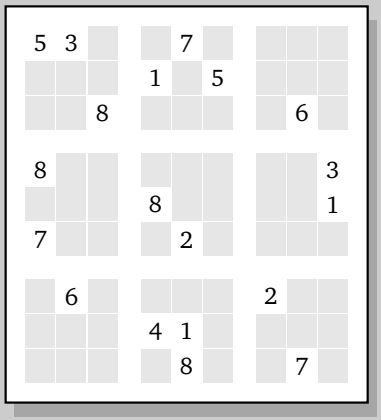
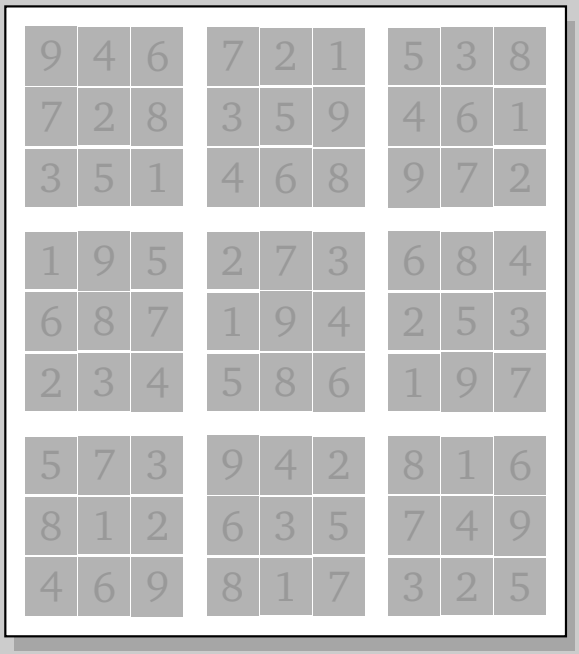
3. Bob asks Alice to scratch off either:

and checks consistency





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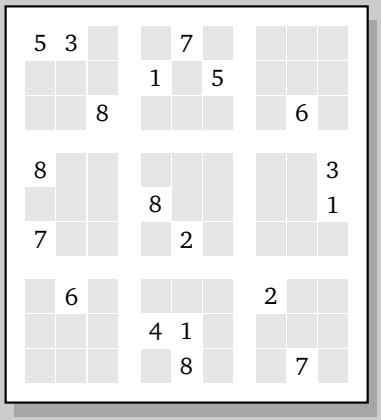

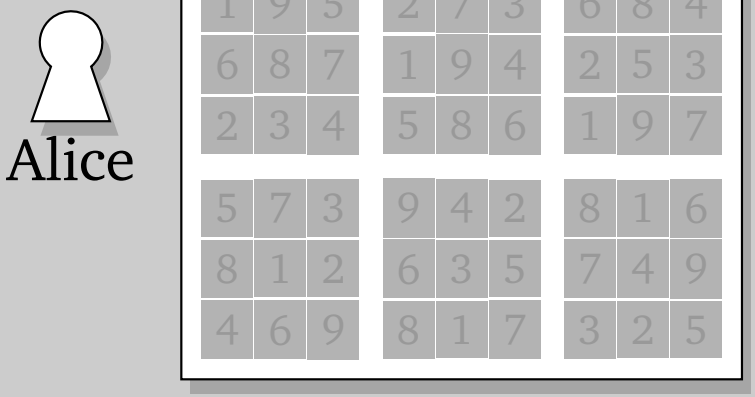

Alice

row 2

Bob

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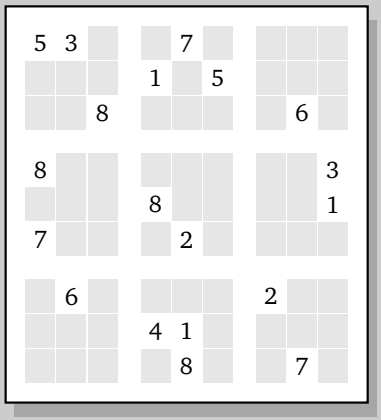

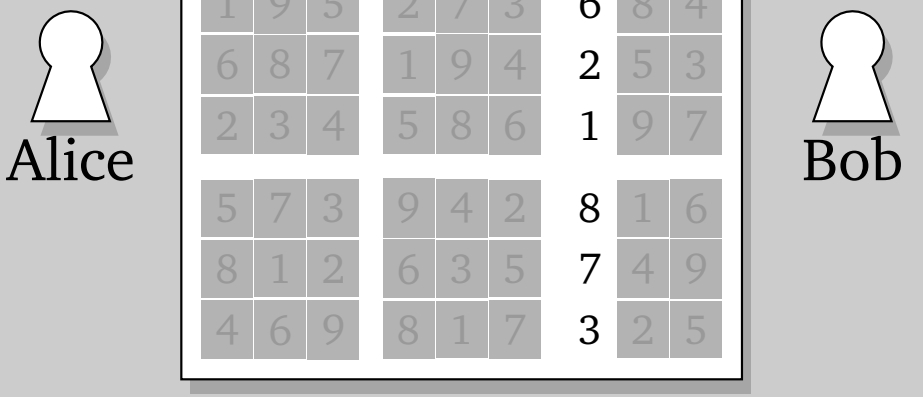

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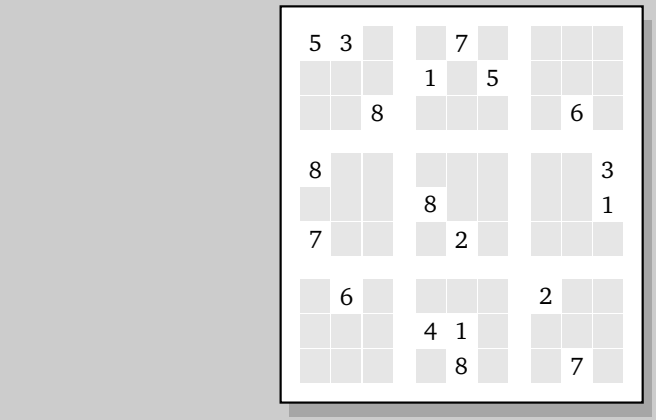

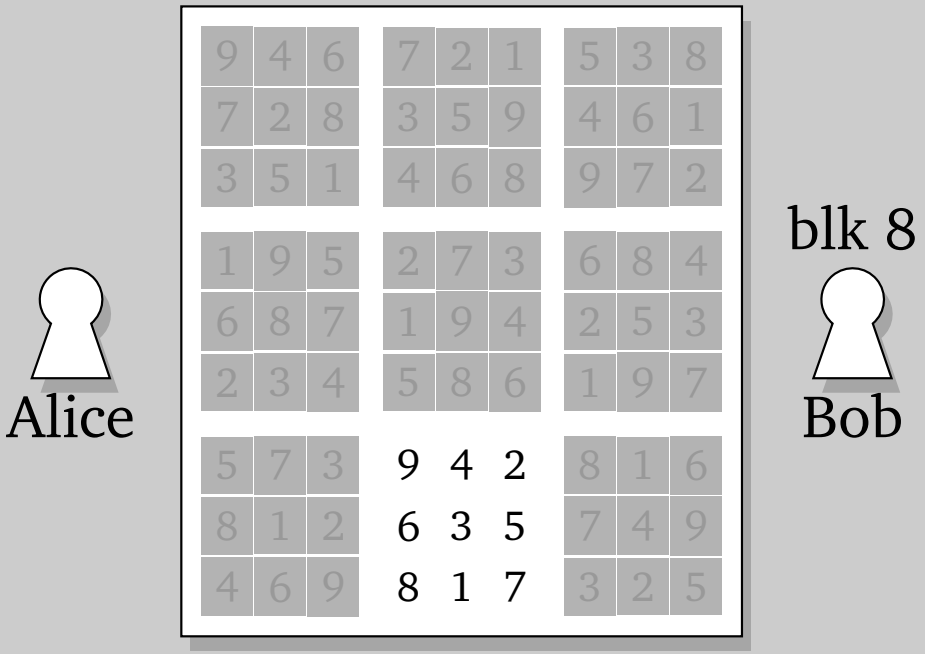

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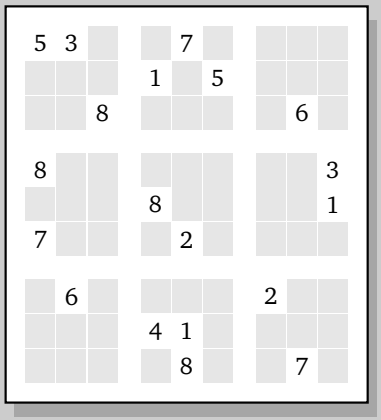

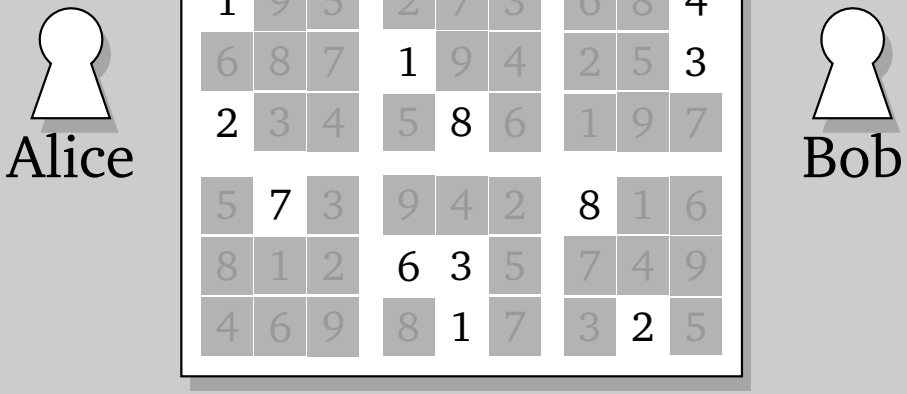

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Zero-knowledge protocol:

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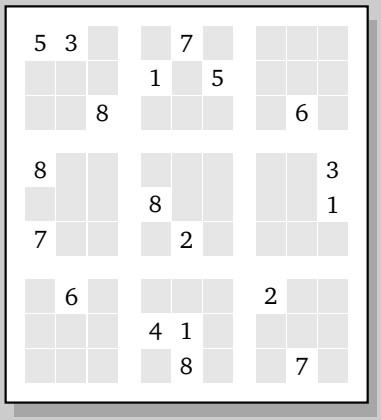
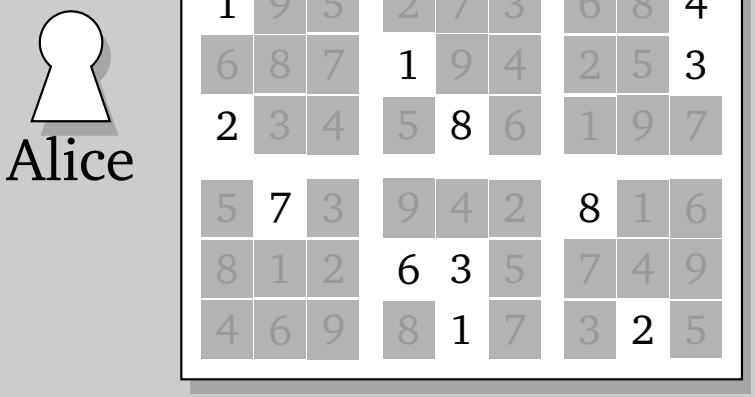
2. Alice writes relabeled solution on scratch card, shows to Bob
3. Bob asks Alice to scratch off either:
   - A particular row
   - A particular column
   - A particular 3×3 block
   - Initial positions

   and checks consistency
4. Repeat n times





Sudoku Zero-Knowledge Proof, Analysis

Observation

If Alice can answer all challenges successfully, her scratch card satisfies:

- Every row, column, block is permutation of $\{1, \ldots, 9\}$
- Initial positions consistent with relabeling of $\{1, \ldots, 9\}$ in original puzzle

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$\Rightarrow$ No scratch card can correctly answer all challenges.

Sudoku Zero-Knowledge Proof, Analysis

Suppose Alice tries to prove an incorrect statement. Let c be a challenge that is bad for Alice's scratch card.

- Bob picks random challenge (28 choices)
- With probability 1/28, Bob chooses c and Alice is caught!
- With probability $\leq 27/28$, Alice's cheating undetected

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Repeat protocol n times. Alice cheats undetected in all rounds with probability $(27/28)^n \approx (1/2)^{0.05n}$

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When $n = 2500$, Alice caught with 99% probability.

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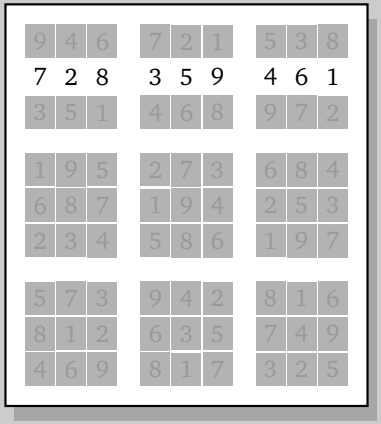
Sudoku Zero-Knowledge Proof, Analysis



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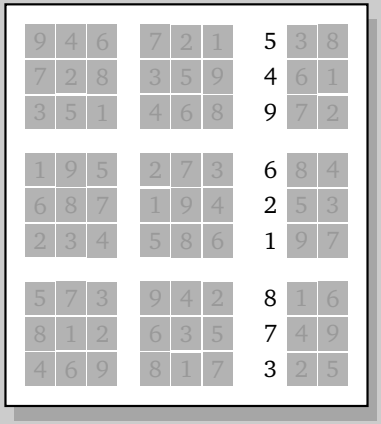

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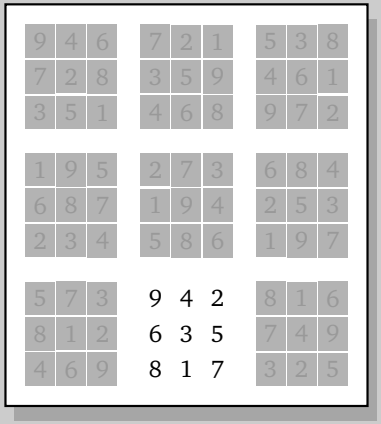

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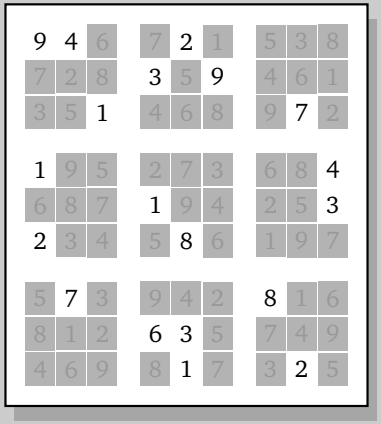

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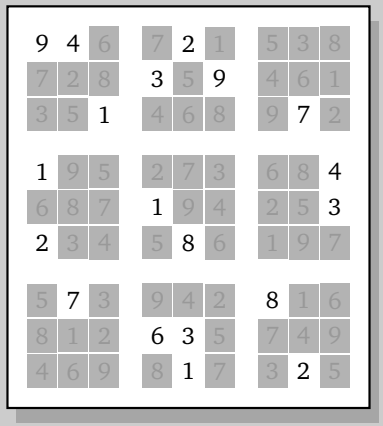

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Each of these Bob could have generated himself (without the solution)!

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Theorem

Every NP statement can be proven in zero-knowledge.

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Disclaimer: ZK proofs **very bad** for teaching courses:

- Students convinced that professor knows a lot
- Students gained no additional knowledge

Like Cryptography?

Take 600.442!

Thanks for your attention!

fin.

I hope this was a “talk about zero knowledge,”
not a “zero-knowledge talk.”