Exercise on Continued Fractions

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This was one of several optional small computational projects assigned to undergraduate mathematics students at Cambridge University in 1993. I'm releasing my code and writeup in 2005 in case they are helpful to anyone—someone working in this area wrote to me asking for them.

Computing the right number of terms: There is an easy way to satisfy the accuracy requirement. If the true value of x is known to fall between $x - \epsilon$ and $x + \epsilon$, we can simply calculate the partial quotients of $x - \epsilon$ and $x + \epsilon$ until such point as they disagree.

I take a computationally distinct but mathematically equivalent approach. Instead of upper and lower bounds, I maintain a triple (x, δ, ϵ), which represents "some real number X between $x - \delta$ and $x + \epsilon$." When x changes, so do δ and ϵ. If $\lfloor x_n - \delta_n \rfloor \neq \lfloor x_n + \epsilon_n \rfloor$, the program declines to compute $a_n = \lfloor x_n \rfloor$.

The initial triple $(x_0, \delta_0, \epsilon_0)$ is given. (In the implementation, we assume that the accuracy of x_0 is the same as its floating-point precision.) We compute x_{n+1} as $1/(x_n - \lfloor x_n \rfloor)$. How can we find δ_{n+1} and ϵ_{n+1}?

We have $\lfloor x \rfloor = \lfloor X \rfloor$. Let $z = x - \lfloor x \rfloor, Z = X - \lfloor x \rfloor$; so the real number Z is z's intended value. Then $z - \delta \leq Z \leq z + \epsilon$,[1] and

$$\frac{1}{z} - \frac{\epsilon}{z(z+\epsilon)} = \frac{1}{z+\epsilon} \leq \frac{1}{Z} \leq \frac{1}{z-\delta} = \frac{1}{z} + \frac{\delta}{z(z-\delta)}$$

But $1/Z_n = X_{n+1}$. We can therefore take

$$\begin{aligned} \delta_{n+1} &= \epsilon_n/(z_n(z_n + \epsilon_n)) \\ \epsilon_{n+1} &= \delta_n/(z_n(z_n - \delta_n)). \end{aligned}$$

Note that we have not assumed $\delta^2 \approx 0 \approx \epsilon^2$. After all, we are expecting these quantities to become large!

[1] Note that $0 \leq z < x$, so the subtraction preserves precision as well as accuracy.

This approach is still subject to rounding errors: for we cannot guarantee that reciprocals are exact. However, in general it does produce the right number of continued fraction terms for a number known to a particular accuracy.[2]

Some results: The `continued-fraction` function prints the partial quotients and convergents as far as justified. For interest, I have also asked it to print the first *unjustified* partial quotient; but note that this one is *not* included in the list that the function returns. (The return value prints below the trace.)

Because Common Lisp can do rational arithmetic, the convergents are exactly correct. Moreover, the continued fraction approximation to a rational number will always be complete and correct in every term. For a floating-point number, the program finds the precision automatically.

"`pq`" stands for "partial quotient," and "`conv`" stands for "convergent."

```
> (continued-fraction pi)
Finding continued fraction for 3.141592653589793  (plus or minus 2.2204460492503136E-16).
pq 3    conv                3 = 3.0
pq 7    conv             22/7 = 3.142857142857143
pq 15   conv          333/106 = 3.141509433962264
pq 1    conv          355/113 = 3.1415929203539825
pq 292  conv     103993/33102 = 3.1415926530119025
pq 1    conv     104348/33215 = 3.141592653921421
pq 1    conv     208341/66317 = 3.1415926534674368
pq 1    conv     312689/99532 = 3.1415926536189365
pq 2    conv    833719/265381 = 3.141592653581078
pq 1    conv   1146408/364913 = 3.141592653591404
pq 3    conv  4272943/1360120 = 3.141592653589389
pq 1    conv  5419351/1725033 = 3.1415926535898153
pq 14   conv 80143857/25510582 = 3.1415926535897927
pq 3    conv 245850922/78256779 = 3.141592653589793 <--- not guaranteed; between about 2 and 6
(3 7 15 1 292 1 1 1 2 1 3 1 14)
> (continued-fraction (exp 1))
Finding continued fraction for 2.7182818284590456  (plus or minus 2.2204460492503136E-16).
pq 2    conv                2 = 2.0
pq 1    conv                3 = 3.0
pq 2    conv              8/3 = 2.6666666666666665
pq 1    conv             11/4 = 2.75
pq 1    conv             19/7 = 2.7142857142857144
pq 4    conv            87/32 = 2.71875
pq 1    conv           106/39 = 2.717948717948718
pq 1    conv           193/71 = 2.7183098591549295
pq 6    conv         1264/465 = 2.718279569892473
pq 1    conv         1457/536 = 2.718283582089552
```

[2] This is not true of the earlier suggestion. The floating-point numbers $x - \epsilon$ and $x + \epsilon$ differ only in their two least significant bits; rounding errors can exaggerate or diminish this difference. Then the partial quotients will start to disagree either "too soon" or "too late."

```
pq 1    conv         2721/1001 = 2.7182817182817183
pq 8    conv        23225/8544 = 2.7182818352059926
pq 1    conv        25946/9545 = 2.7182818229439496
pq 1    conv       49171/18089 = 2.718281828735696
pq 10   conv     517656/190435 = 2.7182818284454014
pq 1    conv     566827/208524 = 2.718281828470584
pq 1    conv    1084483/398959 = 2.7182818284585633
pq 12   conv  13580623/4996032 = 2.718281828459065
pq 1    conv  14665106/5394991 = 2.718281828459028
pq 1    conv 28245729/10391023 = 2.718281828459046
pq 27   conv 777299789/285952612 = 2.7182818284590456 <--- not guaranteed; between about 16 and 88
(2 1 2 1 1 4 1 1 6 1 1 8 1 1 10 1 1 12 1 1)
> (continued-fraction (sqrt 2))
Finding continued fraction for 1.4142135623730952  (plus or minus 1.1102230246251568E-16).
pq 1    conv                 1 = 1.0
pq 2    conv               3/2 = 1.5
pq 2    conv               7/5 = 1.4
pq 2    conv             17/12 = 1.4166666666666668
pq 2    conv             41/29 = 1.4137931034482758
pq 2    conv             99/70 = 1.4142857142857144
pq 2    conv           239/169 = 1.4142011834319526
pq 2    conv           577/408 = 1.4142156862745099
pq 2    conv          1393/985 = 1.4142131979695432
pq 2    conv         3363/2378 = 1.4142136248948696
pq 2    conv         8119/5741 = 1.4142135516460548
pq 2    conv       19601/13860 = 1.4142135642135643
pq 2    conv       47321/33461 = 1.4142135620573204
pq 2    conv      114243/80782 = 1.4142135624272734
pq 2    conv     275807/195025 = 1.4142135623637995
pq 2    conv     665857/470832 = 1.4142135623746899
pq 2    conv   1607521/1136689 = 1.4142135623728214
pq 2    conv   3880899/2744210 = 1.414213562373142
pq 2    conv   9369319/6625109 = 1.414213562373087
pq 2    conv 22619537/15994428 = 1.4142135623730965
pq 2    conv 54608393/38613965 = 1.4142135623730947
pq 1    conv 77227930/54608393 = 1.4142135623730954 <--- not guaranteed; between about 1 and 2
(1 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2)
> (continued-fraction 144/89)     ;; represented as a RATIONAL, not floating-point
Finding continued fraction for 144/89  (plus or minus 0).
pq 1    conv                 1 = 1.0
pq 1    conv                 2 = 2.0
pq 1    conv               3/2 = 1.5
pq 1    conv               5/3 = 1.6666666666666668
pq 1    conv               8/5 = 1.6
pq 1    conv              13/8 = 1.625
pq 1    conv             21/13 = 1.6153846153846155
pq 1    conv             34/21 = 1.619047619047619
pq 1    conv             55/34 = 1.6176470588235295
pq 2    conv            144/89 = 1.6179775280898876
(1 1 1 1 1 1 1 1 1 2)
> (continued-fraction (float 144/89))      ;; same thing but in floating-point
Finding continued fraction for 1.6179775280898876  (plus or minus 1.1102230246251568E-16).
pq 1    conv                 1 = 1.0
pq 1    conv                 2 = 2.0
pq 1    conv               3/2 = 1.5
```

```
pq 1    conv                5/3 = 1.6666666666666668
pq 1    conv                8/5 = 1.6
pq 1    conv               13/8 = 1.625
pq 1    conv              21/13 = 1.6153846153846155
pq 1    conv              34/21 = 1.619047619047619
pq 1    conv              55/34 = 1.6176470588235295
pq 2    conv             144/89 = 1.6179775280898876 <--- not guaranteed; between about 1 and 2
(1 1 1 1 1 1 1 1 1 1)
> (continued-fraction (/ (1+ (sqrt 5)) 2))     ;; note 144/89 is a convergent
Finding continued fraction for 1.618033988749895  (plus or minus 1.1102230246251568E-16).
pq 1    conv                  1 = 1.0
pq 1    conv                  2 = 2.0
pq 1    conv                3/2 = 1.5
pq 1    conv                5/3 = 1.6666666666666668
pq 1    conv                8/5 = 1.6
pq 1    conv               13/8 = 1.625
pq 1    conv              21/13 = 1.6153846153846155
pq 1    conv              34/21 = 1.619047619047619
pq 1    conv              55/34 = 1.6176470588235295
pq 1    conv              89/55 = 1.6181818181818182
pq 1    conv             144/89 = 1.6179775280898876
pq 1    conv            233/144 = 1.6180555555555556
pq 1    conv            377/233 = 1.6180257510729615
pq 1    conv            610/377 = 1.6180371352785146
pq 1    conv            987/610 = 1.618032786885246
pq 1    conv           1597/987 = 1.618034447821682
pq 1    conv          2584/1597 = 1.6180338134001253
pq 1    conv          4181/2584 = 1.618034055727554
pq 1    conv          6765/4181 = 1.6180339631667065
pq 1    conv         10946/6765 = 1.6180339985218033
pq 1    conv        17711/10946 = 1.618033985017358
pq 1    conv        28657/17711 = 1.6180339901755972
pq 1    conv        46368/28657 = 1.618033988205325
pq 1    conv        75025/46368 = 1.618033988957902
pq 1    conv       121393/75025 = 1.6180339886704432
pq 1    conv      196418/121393 = 1.6180339887802427
pq 1    conv      317811/196418 = 1.618033988738303
pq 1    conv      514229/317811 = 1.6180339887543225
pq 1    conv      832040/514229 = 1.6180339887482036
pq 1    conv     1346269/832040 = 1.6180339887505409
pq 1    conv    2178309/1346269 = 1.6180339887496482
pq 1    conv    3524578/2178309 = 1.618033988749989
pq 1    conv    5702887/3524578 = 1.618033988749859
pq 1    conv    9227465/5702887 = 1.6180339887499087
pq 1    conv   14930352/9227465 = 1.6180339887498896
pq 1    conv  24157817/14930352 = 1.618033988749897
pq 1    conv  39088169/24157817 = 1.618033988749894
pq 1    conv  63245986/39088169 = 1.6180339887498952
pq 2    conv 165580141/102334155 = 1.618033988749895 <--- not guaranteed; between about 1 and 5
(1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1)
> (continued-fraction 235/71113)               ; a arbitrary rational
Finding continued fraction for 235/71113  (plus or minus 0).
pq 0    conv                  0 = 0.0
pq 302  conv              1/302 = 0.0033112582781456954
pq 1    conv              1/303 = 0.0033003300330033004
```

```
pq 1    conv            2/605 = 0.003305785123966942
pq 1    conv            3/908 = 0.003303964757709251
pq 1    conv           5/1513 = 0.003304692663582287
pq 4    conv          23/6960 = 0.0033045977011494253
pq 10   conv        235/71113 = 0.003304599721569896
(0 302 1 1 1 1 4 10)
> (continued-fraction 9173819257/13147251985)   ; a special rational
Finding continued fraction for 9173819257/13147251985  (plus or minus 0).
pq 0    conv                0 = 0.0
pq 1    conv                1 = 1.0
pq 2    conv              2/3 = 0.6666666666666666
pq 3    conv             7/10 = 0.7
pq 4    conv            30/43 = 0.6976744186046512
pq 5    conv          157/225 = 0.6977777777777778
pq 6    conv         972/1393 = 0.6977745872218234
pq 7    conv        6961/9976 = 0.6977746591820369
pq 8    conv      56660/81201 = 0.6977746579475622
pq 9    conv    516901/740785 = 0.6977746579641867
pq 10   conv   5225670/7489051 = 0.6977746579640064
pq 11   conv  57999271/83120346 = 0.697774657964008
pq 12   conv 701216922/1004933203 = 0.697774657964008
pq 13   conv 9173819257/13147251985 = 0.697774657964008
(0 1 2 3 4 5 6 7 8 9 10 11 12 13)
> (continued-fraction (float 9173819257/13147251985))  ; same in floating-point
Finding continued fraction for 0.697774657964008  (plus or minus 5.551115123125784E-17).
pq 0    conv                0 = 0.0
pq 1    conv                1 = 1.0
pq 2    conv              2/3 = 0.6666666666666666
pq 3    conv             7/10 = 0.7
pq 4    conv            30/43 = 0.6976744186046512
pq 5    conv          157/225 = 0.6977777777777778
pq 6    conv         972/1393 = 0.6977745872218234
pq 7    conv        6961/9976 = 0.6977746591820369
pq 8    conv      56660/81201 = 0.6977746579475622
pq 9    conv    516901/740785 = 0.6977746579641867
pq 10   conv   5225670/7489051 = 0.6977746579640064
pq 11   conv  57999271/83120346 = 0.697774657964008 <--- not guaranteed; between about 10 and 11
(0 1 2 3 4 5 6 7 8 9 10)
> (continued-fraction 16.0968)              ; my birthday in floating-point
Finding continued fraction for 16.0968  (plus or minus 1.776356839400251E-15).
pq 16   conv               16 = 16.0
pq 10   conv           161/10 = 16.1
pq 3    conv           499/31 = 16.096774193548388
pq 39   conv       19622/1219 = 16.096800656275637 <--- not guaranteed; between about 39 and 40
(16 10 3)
> (float (eval-finite-cf '(16 10 3 40)))    ; evaluate [16,10,3,40] -- right on the edge
16.0968
```

Time complexity: If we know x to an accuracy of $\pm\epsilon$, how long does it take to generate the continued fraction for x?

The answer does not depend entirely on ϵ. For example, for any $\epsilon > 0$, $x = 2$ will terminate immediately—we can't tell whether the first partial quotient should be 1 or 2. However, knowing ϵ certainly gives us an upper bound on running time.

A theorem from Part II Number Theory tells us how quickly the convergents converge: $|x - P_n/Q_n| \leq 1/Q_nQ_{n+1}$. Let n be such that the algorithm computes a_n. Then the continued fractions for $x-\epsilon$ and $x+\epsilon$ agree in their values for $a_0, a_1, \ldots a_n$, and so also in their values for P_n, Q_n, and Q_{n+1}. So

$$2\epsilon \leq |P_n/Q_n - (x-\epsilon)| + |(x+\epsilon) - P_n/Q_n| \leq 2/Q_nQ_{n+1}$$

and

$$Q_nQ_{n+1} \leq 1/\epsilon.$$

How quickly do the Q's increase? $(\forall k)Q_{k+1} = a_kQ_k + Q_{k-1} \geq Q_k + Q_{k-1}$, and $Q_0 = 1$, $Q_1 \geq 1$, so $Q_k \geq F_k$, where F_k is the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \ldots c\phi^k + c'\phi'^k, \ldots$$

for

$$c = \frac{5+\sqrt{5}}{10},\ c' = \frac{5-\sqrt{5}}{10},\ \phi = \frac{1+\sqrt{5}}{2},\ \phi' = \frac{1-\sqrt{5}}{2}.$$

Therefore $Q_n > c\phi^n$—this is really as good a lower bound as we can get, since ϕ'^n vanishes as $n \to \infty$—and $Q_nQ_{n+1} > (c^2\phi)\phi^{2n}$.

We now see that

$$(c^2\phi)\phi^{2n} < Q_nQ_{n+1} \leq 1/\epsilon$$

$$\log c^2\phi + 2n \log\phi < -\log\epsilon$$

$$n < \frac{1}{2}(-\log const - \frac{\log\epsilon}{\log\phi}) = O(\log 1/\epsilon)$$

for such n that we compute a_n. This means that we compute at most $O(\log 1/\epsilon)$ partial quotients. Under the usual convention that arithmetic operations are $O(1)$, each partial quotient is found in constant time, and the algorithm completes in time at most $O(\log 1/\epsilon)$.

A worst-case example: This upper bound is actually achieved when $1 = a_0 = a_1 = a_2 = \cdots$. This is the continued fraction for ϕ, as we can see by writing $\lfloor\phi\rfloor = 1$ and $1/(\phi - \lfloor\phi\rfloor) = \phi$, or by noting that for these values of a_n, the convergents P_n/Q_n take the form F_{n+1}/F_n, and so approach ϕ.

Why is the bound achieved? The easiest demonstration (if not the prettiest) is in terms of ϵ and δ above. Because $z_n = z = 1/\phi$ for every n, and $z^2 + z - 1 = 0$, a little arithmetic allows us to write ϵ_{n+2} in terms of ϵ_n:

$$\begin{aligned}\epsilon_{n+2} &= \frac{\epsilon_n}{z^4 + z^3\epsilon_n - z\epsilon_n} \\ &= \frac{\epsilon_n}{z^4 - z^2\epsilon_n}\end{aligned}$$

So for $\epsilon_n < z^2/2$ say, $\epsilon_{n+2} \le 2\phi^4\epsilon_n$. In other words, until ϵ_n passes $z^2/2 \approx 0.19$, it grows no more quickly than $(\sqrt{2}\phi^2)^n$. Hence ϵ_0 takes at least $O(\log \epsilon_0)$ steps to reach $z^2/2$. But the algorithm does not stop until ϵ_n exceeds $2 - \phi > z^2/2$; so it does take at least $O(\log 1/\epsilon)$ steps in this case.

Periodic structure of $\sqrt{d}$ (floating-point): The repeating part of a continued fraction is displayed between /'s.

```
> (loop for d from 1 to 167
unless (square? d)
do (print-sqrt-cf (approx-sqrt-cf d) d))
sqrt(2)   = [1  /  2 / ...]
sqrt(3)   = [1  /  1  2 / ...]
sqrt(5)   = [2  /  4 / ...]
sqrt(6)   = [2  /  2  4 / ...]
sqrt(7)   = [2  /  1  1  1  4 / ...]
sqrt(8)   = [2  /  1  4 / ...]
sqrt(10)  = [3  /  6 / ...]
sqrt(11)  = [3  /  3  6 / ...]
sqrt(12)  = [3  /  2  6 / ...]
sqrt(13)  = [3  /  1  1  1  1  6 / ...]
sqrt(14)  = [3  /  1  2  1  6 / ...]
sqrt(15)  = [3  /  1  6 / ...]
sqrt(17)  = [4  /  8 / ...]
sqrt(18)  = [4  /  4  8 / ...]
sqrt(19)  = [4  /  2  1  3  1  2  8 / ...]
sqrt(20)  = [4  /  2  8 / ...]
sqrt(21)  = [4  /  1  1  2  1  1  8 / ...]
sqrt(22)  = [4  /  1  2  4  2  1  8 / ...]
sqrt(23)  = [4  /  1  3  1  8 / ...]
sqrt(24)  = [4  /  1  8 / ...]
sqrt(26)  = [5  / 10 / ...]
sqrt(27)  = [5  /  5 10 / ...]
sqrt(28)  = [5  /  3  2  3 10 / ...]
sqrt(29)  = [5  /  2  1  1  2 10 / ...]
sqrt(30)  = [5  /  2 10 / ...]
sqrt(31)  = [5  /  1  1  3  5  3  1  1 10 / ...]
sqrt(32)  = [5  /  1  1  1 10 / ...]
sqrt(33)  = [5  /  1  2  1 10 / ...]
sqrt(34)  = [5  /  1  4  1 10 / ...]
sqrt(35)  = [5  /  1 10 / ...]
sqrt(37)  = [6  / 12 / ...]
sqrt(38)  = [6  /  6 12 / ...]
sqrt(39)  = [6  /  4 12 / ...]
sqrt(40)  = [6  /  3 12 / ...]
sqrt(41)  = [6  /  2  2 12 / ...]
sqrt(42)  = [6  /  2 12 / ...]
sqrt(43)  = [6  /  1  1  3  1  5  1  3  1  1 12 / ...]
sqrt(44)  = [6  /  1  1  1  2  1  1  1 12 / ...]
sqrt(45)  = [6  /  1  2  2  2  1 12 / ...]
sqrt(46)  = [6  /  1  3  1  1  2  6  2  1  1  3  1 12 / ...]
sqrt(47)  = [6  /  1  5  1 12 / ...]
sqrt(48)  = [6  /  1 12 / ...]
sqrt(50)  = [7  / 14 / ...]
```

```
sqrt(51)  = [7  /  7 14 / ...]
sqrt(52)  = [7  /  4  1  2  1  4 14 / ...]
sqrt(53)  = [7  /  3  1  1  3 14 / ...]
sqrt(54)  = [7  /  2  1  6  1  2 14 / ...]
sqrt(55)  = [7  /  2  2  2 14 / ...]
sqrt(56)  = [7  /  2 14 / ...]
sqrt(57)  = [7  /  1  1  4  1  1 14 / ...]
sqrt(58)  = [7  /  1  1  1  1  1  1 14 / ...]
sqrt(59)  = [7  /  1  2  7  2  1 14 / ...]
sqrt(60)  = [7  /  1  2  1 14 / ...]
sqrt(61)  = [7  /  1  4  3  1  2  2  1  3  4  1 14 / ...]
sqrt(62)  = [7  /  1  6  1 14 / ...]
sqrt(63)  = [7  /  1 14 / ...]
sqrt(65)  = [8  / 16 / ...]
sqrt(66)  = [8  /  8 16 / ...]
sqrt(67)  = [8  /  5  2  1  1  7  1  1  2  5 16 / ...]
sqrt(68)  = [8  /  4 16 / ...]
sqrt(69)  = [8  /  3  3  1  4  1  3  3 16 / ...]
sqrt(70)  = [8  /  2  1  2  1  2 16 / ...]
sqrt(71)  = [8  /  2  2  1  7  1  2  2 16 / ...]
sqrt(72)  = [8  /  2 16 / ...]
sqrt(73)  = [8  /  1  1  5  5  1  1 16 / ...]
sqrt(74)  = [8  /  1  1  1  1 16 / ...]
sqrt(75)  = [8  /  1  1  1 16 / ...]
sqrt(76)  = [8  /  1  2  1  1  5  4  5  1  1  2  1 16 / ...]
sqrt(77)  = [8  /  1  3  2  3  1 16 / ...]
sqrt(78)  = [8  /  1  4  1 16 / ...]
sqrt(79)  = [8  /  1  7  1 16 / ...]
sqrt(80)  = [8  /  1 16 / ...]
sqrt(82)  = [9  / 18 / ...]
sqrt(83)  = [9  /  9 18 / ...]
sqrt(84)  = [9  /  6 18 / ...]
sqrt(85)  = [9  /  4  1  1  4 18 / ...]
sqrt(86)  = [9  /  3  1  1  1  8  1  1  1  3 18 / ...]
sqrt(87)  = [9  /  3 18 / ...]
sqrt(88)  = [9  /  2  1  1  1  2 18 / ...]
sqrt(89)  = [9  /  2  3  3  2 18 / ...]
sqrt(90)  = [9  /  2 18 / ...]
sqrt(91)  = [9  /  1  1  5  1  5  1  1 18 / ...]
sqrt(92)  = [9  /  1  1  2  4  2  1  1 18 / ...]
sqrt(93)  = [9  /  1  1  1  4  6  4  1  1  1 18 / ...]
sqrt(94)  = [9  /  1  2  3  1  1  5  1  8  1  5  1  1  3  2  1 18 / ...]
sqrt(95)  = [9  /  1  2  1 18 / ...]
sqrt(96)  = [9  /  1  3  1 18 / ...]
sqrt(97)  = [9  /  1  5  1  1  1  1  1  1  5  1 18 / ...]
sqrt(98)  = [9  /  1  8  1 18 / ...]
sqrt(99)  = [9  /  1 18 / ...]
sqrt(101) = [10 / 20 / ...]
sqrt(102) = [10 / 10 20 / ...]
sqrt(103) = [10 /  6  1  2  1  1  9  1  1  2  1  6 20 / ...]
sqrt(104) = [10 /  5 20 / ...]
sqrt(105) = [10 /  4 20 / ...]
sqrt(106) = [10 /  3  2  1  1  1  1  2  3 20 / ...]
sqrt(107) = [10 /  2  1  9  1  2 20 / ...]
sqrt(108) = [10 /  2  1  1  4  1  1  2 20 / ...]
```

```
sqrt(109) = [10 /  2  3  1  2  4  1  6  6  1  4  2  1  3  2 20 / ...]
sqrt(110) = [10 /  2 20 / ...]
sqrt(111) = [10 /  1  1  6  1  1 20 / ...]
sqrt(112) = [10 /  1  1  2  1  1 20 / ...]
sqrt(113) = [10 /  1  1  1  2  2  1  1  1 20 / ...]
sqrt(114) = [10 /  1  2 10  2  1 20 / ...]
sqrt(115) = [10 /  1  2  1  1  1  1  1  2  1 20 / ...]
sqrt(116) = [10 /  1  3  2  1  4  1  2  3  1 20 / ...]
sqrt(117) = [10 /  1  4  2  4  1 20 / ...]
sqrt(118) = [10 /  1  6  3  2 10  2  3  6  1 20 / ...]
sqrt(119) = [10 /  1  9  1 20 / ...]
sqrt(120) = [10 /  1 20 / ...]
sqrt(122) = [11 / 22 / ...]
sqrt(123) = [11 / 11 22 / ...]
sqrt(124) = ???   (insufficient floating-point precision)
sqrt(125) = [11 /  5  1  1  5 22 / ...]
sqrt(126) = [11 /  4  2  4 22 / ...]
sqrt(127) = [11 /  3  1  2  2  7 11  7  2  2  1  3 22 / ...]
sqrt(128) = [11 /  3  5  3 22 / ...]
sqrt(129) = [11 /  2  1  3  1  6  1  3  1  2 22 / ...]
sqrt(130) = [11 /  2  2 22 / ...]
sqrt(131) = [11 /  2  4 11  4  2 22 / ...]
sqrt(132) = [11 /  2 22 / ...]
sqrt(133) = [11 /  1  1  7  5  1  1  1  2  1  1  1  5  7  1  1 22 / ...]
sqrt(134) = [11 /  1  1  2  1  3  1 10  1  3  1  2  1  1 22 / ...]
sqrt(135) = [11 /  1  1  1  1  1  1  1 22 / ...]
sqrt(136) = [11 /  1  1  1 22 / ...]
sqrt(137) = [11 /  1  2  2  1  1  2  2  1 22 / ...]
sqrt(138) = [11 /  1  2  1 22 / ...]
sqrt(139) = ???   (insufficient floating-point precision)
sqrt(140) = [11 /  1  4  1 22 / ...]
sqrt(141) = [11 /  1  6  1 22 / ...]
sqrt(142) = [11 /  1 10  1 22 / ...]
sqrt(143) = [11 /  1 22 / ...]
sqrt(145) = [12 / 24 / ...]
sqrt(146) = [12 / 12 24 / ...]
sqrt(147) = [12 /  8 24 / ...]
sqrt(148) = [12 /  6 24 / ...]
sqrt(149) = [12 /  4  1  5  3  3  5  1  4 24 / ...]
sqrt(150) = [12 /  4 24 / ...]
sqrt(151) = ???   (insufficient floating-point precision)
sqrt(152) = [12 /  3 24 / ...]
sqrt(153) = [12 /  2  1  2  2  2  1  2 24 / ...]
sqrt(154) = [12 /  2  2  3  1  2  1  3  2  2 24 / ...]
sqrt(155) = [12 /  2  4  2 24 / ...]
sqrt(156) = [12 /  2 24 / ...]
sqrt(157) = [12 /  1  1  7  1  5  2  1  1  1  1  2  5  1  7  1  1 24 / ...]
sqrt(158) = [12 /  1  1  3 12  3  1  1 24 / ...]
sqrt(159) = [12 /  1  1  1  1  3  1  1  1  1 24 / ...]
sqrt(160) = [12 /  1  1  1  5  1  1  1 24 / ...]
sqrt(161) = [12 /  1  2  4  1  2  1  4  2  1 24 / ...]
sqrt(162) = [12 /  1  2  1  2 12  2  1  2  1 24 / ...]
sqrt(163) = ???   (insufficient floating-point precision)
sqrt(164) = [12 /  1  4  6  4  1 24 / ...]
sqrt(165) = [12 /  1  5  2  5  1 24 / ...]
```

```
sqrt(166) = ???   (insufficient floating-point precision)
sqrt(167) = [12 /  1 11  1 24 / ...]
NIL
```

Periodic structure of $\sqrt{d}$ with integer arithmetic: This section is a good reminder that computers do not really do arithmetic operations in constant time. In principle Common Lisp can handle integers of any size, but some of *these* integers dwarf the googol: for example, evaluating the continued fraction for $\sqrt{94}$ involves numbers greater than 10^{62340}.

There is an easy way to avoid performing these ridiculously expensive calculations, which is to put the fraction $\frac{\alpha+\beta\sqrt{d}}{\gamma}$ into lowest terms after each step. It is easy to see that this must keep α, β, and γ bounded—the fraction itself is going to repeat, whatever may happen to our representation of it.

The Euclidean algorithm takes only logarithmic time, so this is a cheap operation. If we do *not* perform it, then $\gcd(a, b, c)$ will rapidly become gigantic: Subtracting an integer from the fraction cannot decrease the gcd, while taking reciprocals will square it. And of course a, b, and c are all at least as big as their gcd!

A few remarks should also be made about expressions of the form $x = \frac{\alpha+\beta\sqrt{d}}{\gamma}$. If β and γ are *positive*, we can compute $\lfloor x \rfloor$ by $\left\lfloor (\alpha + \left\lfloor \sqrt{\beta^2 d} \right\rfloor)/\gamma \right\rfloor$. Note that $\left\lfloor \sqrt{\beta^2 d} \right\rfloor$ can be computed in integer arithmetic, and in fact the built-in `isqrt` function does just that.

Indeed, we can arrange that β and γ *are* always positive. More strongly, our operations (when appropriately implemented) preserve the properties

$$\beta\sqrt{d} > |\alpha| \geq 0, \beta > 0, \gamma > 0.$$

It is possible to ensure these properties without ever testing the sign of these integers.

First we require $x' = x - \lfloor x \rfloor$ to have these properties if x does. Writing

$$x' = \frac{\alpha' + \beta'\sqrt{d}}{\gamma'} = \frac{(\alpha - \gamma\lfloor x \rfloor) + \beta\sqrt{d}}{\gamma},$$

we certainly get $\beta' > 0$, $\gamma' > 0$, $\beta'\sqrt{d} > \alpha'$. Moreover $x' \geq 0$ by its definition; but $\beta' > 0$ implies the numerator of x is irrational, whence $x \neq 0$. So $x' > 0$ with positive denominator; it must then have positive numerator, and $\beta'\sqrt{d} > -\alpha'$ also.

Now we arrange for $x' = 1/x$ to have these properties if x does. Put

$$x' = \frac{\alpha' + \beta'\sqrt{d}}{\gamma'} = \frac{-\alpha\gamma + \beta\gamma\sqrt{d}}{\beta^2 d - \alpha^2},$$

and the result is immediate.

Finally, dividing α, β, and γ by their gcd also leaves these properties intact. This proves the claim.

Results for periodic structure of $\sqrt{d}$ (integer arithmetic):

```
> (loop for d from 1 to 167
unless (square? d)
do (print-sqrt-cf (true-sqrt-cf d) d))
sqrt(2)   = [1  /  2 / ...]
sqrt(3)   = [1  /  1  2 / ...]
sqrt(5)   = [2  /  4 / ...]
sqrt(6)   = [2  /  2  4 / ...]
sqrt(7)   = [2  /  1  1  1  4 / ...]
sqrt(8)   = [2  /  1  4 / ...]
sqrt(10)  = [3  /  6 / ...]
sqrt(11)  = [3  /  3  6 / ...]
sqrt(12)  = [3  /  2  6 / ...]
sqrt(13)  = [3  /  1  1  1  1  6 / ...]
sqrt(14)  = [3  /  1  2  1  6 / ...]
sqrt(15)  = [3  /  1  6 / ...]
sqrt(17)  = [4  /  8 / ...]
sqrt(18)  = [4  /  4  8 / ...]
sqrt(19)  = [4  /  2  1  3  1  2  8 / ...]
sqrt(20)  = [4  /  2  8 / ...]
sqrt(21)  = [4  /  1  1  2  1  1  8 / ...]
sqrt(22)  = [4  /  1  2  4  2  1  8 / ...]
sqrt(23)  = [4  /  1  3  1  8 / ...]
sqrt(24)  = [4  /  1  8 / ...]
sqrt(26)  = [5  / 10 / ...]
sqrt(27)  = [5  /  5 10 / ...]
sqrt(28)  = [5  /  3  2  3 10 / ...]
sqrt(29)  = [5  /  2  1  1  2 10 / ...]
sqrt(30)  = [5  /  2 10 / ...]
sqrt(31)  = [5  /  1  1  3  5  3  1  1 10 / ...]
sqrt(32)  = [5  /  1  1  1 10 / ...]
sqrt(33)  = [5  /  1  2  1 10 / ...]
sqrt(34)  = [5  /  1  4  1 10 / ...]
sqrt(35)  = [5  /  1 10 / ...]
sqrt(37)  = [6  / 12 / ...]
sqrt(38)  = [6  /  6 12 / ...]
sqrt(39)  = [6  /  4 12 / ...]
sqrt(40)  = [6  /  3 12 / ...]
sqrt(41)  = [6  /  2  2 12 / ...]
sqrt(42)  = [6  /  2 12 / ...]
sqrt(43)  = [6  /  1  1  3  1  5  1  3  1  1 12 / ...]
sqrt(44)  = [6  /  1  1  1  2  1  1  1 12 / ...]
sqrt(45)  = [6  /  1  2  2  2  1 12 / ...]
sqrt(46)  = [6  /  1  3  1  1  2  6  2  1  1  3  1 12 / ...]
sqrt(47)  = [6  /  1  5  1 12 / ...]
sqrt(48)  = [6  /  1 12 / ...]
sqrt(50)  = [7  / 14 / ...]
sqrt(51)  = [7  /  7 14 / ...]
sqrt(52)  = [7  /  4  1  2  1  4 14 / ...]
sqrt(53)  = [7  /  3  1  1  3 14 / ...]
```

```
sqrt(54)  = [7  /  2  1  6  1  2 14 / ...]
sqrt(55)  = [7  /  2  2  2 14 / ...]
sqrt(56)  = [7  /  2 14 / ...]
sqrt(57)  = [7  /  1  1  4  1  1 14 / ...]
sqrt(58)  = [7  /  1  1  1  1  1  1 14 / ...]
sqrt(59)  = [7  /  1  2  7  2  1 14 / ...]
sqrt(60)  = [7  /  1  2  1 14 / ...]
sqrt(61)  = [7  /  1  4  3  1  2  2  1  3  4  1 14 / ...]
sqrt(62)  = [7  /  1  6  1 14 / ...]
sqrt(63)  = [7  /  1 14 / ...]
sqrt(65)  = [8  / 16 / ...]
sqrt(66)  = [8  /  8 16 / ...]
sqrt(67)  = [8  /  5  2  1  1  7  1  1  2  5 16 / ...]
sqrt(68)  = [8  /  4 16 / ...]
sqrt(69)  = [8  /  3  3  1  4  1  3  3 16 / ...]
sqrt(70)  = [8  /  2  1  2  1  2 16 / ...]
sqrt(71)  = [8  /  2  2  1  7  1  2  2 16 / ...]
sqrt(72)  = [8  /  2 16 / ...]
sqrt(73)  = [8  /  1  1  5  5  1  1 16 / ...]
sqrt(74)  = [8  /  1  1  1  1 16 / ...]
sqrt(75)  = [8  /  1  1  1 16 / ...]
sqrt(76)  = [8  /  1  2  1  1  5  4  5  1  1  2  1 16 / ...]
sqrt(77)  = [8  /  1  3  2  3  1 16 / ...]
sqrt(78)  = [8  /  1  4  1 16 / ...]
sqrt(79)  = [8  /  1  7  1 16 / ...]
sqrt(80)  = [8  /  1 16 / ...]
sqrt(82)  = [9  / 18 / ...]
sqrt(83)  = [9  /  9 18 / ...]
sqrt(84)  = [9  /  6 18 / ...]
sqrt(85)  = [9  /  4  1  1  4 18 / ...]
sqrt(86)  = [9  /  3  1  1  1  8  1  1  1  3 18 / ...]
sqrt(87)  = [9  /  3 18 / ...]
sqrt(88)  = [9  /  2  1  1  1  2 18 / ...]
sqrt(89)  = [9  /  2  3  3  2 18 / ...]
sqrt(90)  = [9  /  2 18 / ...]
sqrt(91)  = [9  /  1  1  5  1  5  1  1 18 / ...]
sqrt(92)  = [9  /  1  1  2  4  2  1  1 18 / ...]
sqrt(93)  = [9  /  1  1  1  4  6  4  1  1  1 18 / ...]
sqrt(94)  = [9  /  1  2  3  1  1  5  1  8  1  5  1  1  3  2  1 18 / ...]
sqrt(95)  = [9  /  1  2  1 18 / ...]
sqrt(96)  = [9  /  1  3  1 18 / ...]
sqrt(97)  = [9  /  1  5  1  1  1  1  1  1  5  1 18 / ...]
sqrt(98)  = [9  /  1  8  1 18 / ...]
sqrt(99)  = [9  /  1 18 / ...]
sqrt(101) = [10 / 20 / ...]
sqrt(102) = [10 / 10 20 / ...]
sqrt(103) = [10 /  6  1  2  1  1  9  1  1  2  1  6 20 / ...]
sqrt(104) = [10 /  5 20 / ...]
sqrt(105) = [10 /  4 20 / ...]
sqrt(106) = [10 /  3  2  1  1  1  1  2  3 20 / ...]
sqrt(107) = [10 /  2  1  9  1  2 20 / ...]
sqrt(108) = [10 /  2  1  1  4  1  1  2 20 / ...]
sqrt(109) = [10 /  2  3  1  2  4  1  6  6  1  4  2  1  3  2 20 / ...]
sqrt(110) = [10 /  2 20 / ...]
sqrt(111) = [10 /  1  1  6  1  1 20 / ...]
```

```
sqrt(112) = [10 /  1  1  2  1  1 20 / ...]
sqrt(113) = [10 /  1  1  1  2  2  1  1  1 20 / ...]
sqrt(114) = [10 /  1  2 10  2  1 20 / ...]
sqrt(115) = [10 /  1  2  1  1  1  1  1  2  1 20 / ...]
sqrt(116) = [10 /  1  3  2  1  4  1  2  3  1 20 / ...]
sqrt(117) = [10 /  1  4  2  4  1 20 / ...]
sqrt(118) = [10 /  1  6  3  2 10  2  3  6  1 20 / ...]
sqrt(119) = [10 /  1  9  1 20 / ...]
sqrt(120) = [10 /  1 20 / ...]
sqrt(122) = [11 / 22 / ...]
sqrt(123) = [11 / 11 22 / ...]
sqrt(124) = [11 /  7  2  1  1  1  3  1  4  1  3  1  1  1  2  7 22 / ...]
sqrt(125) = [11 /  5  1  1  5 22 / ...]
sqrt(126) = [11 /  4  2  4 22 / ...]
sqrt(127) = [11 /  3  1  2  2  7 11  7  2  2  1  3 22 / ...]
sqrt(128) = [11 /  3  5  3 22 / ...]
sqrt(129) = [11 /  2  1  3  1  6  1  3  1  2 22 / ...]
sqrt(130) = [11 /  2  2 22 / ...]
sqrt(131) = [11 /  2  4 11  4  2 22 / ...]
sqrt(132) = [11 /  2 22 / ...]
sqrt(133) = [11 /  1  1  7  5  1  1  1  2  1  1  1  5  7  1  1 22 / ...]
sqrt(134) = [11 /  1  1  2  1  3  1 10  1  3  1  2  1  1 22 / ...]
sqrt(135) = [11 /  1  1  1  1  1  1  1 22 / ...]
sqrt(136) = [11 /  1  1  1 22 / ...]
sqrt(137) = [11 /  1  2  2  1  1  2  2  1 22 / ...]
sqrt(138) = [11 /  1  2  1 22 / ...]
sqrt(139) = [11 /  1  3  1  3  7  1  1  2 11  2  1  1  7  3  1  3  1 22 / ...]
sqrt(140) = [11 /  1  4  1 22 / ...]
sqrt(141) = [11 /  1  6  1 22 / ...]
sqrt(142) = [11 /  1 10  1 22 / ...]
sqrt(143) = [11 /  1 22 / ...]
sqrt(145) = [12 / 24 / ...]
sqrt(146) = [12 / 12 24 / ...]
sqrt(147) = [12 /  8 24 / ...]
sqrt(148) = [12 /  6 24 / ...]
sqrt(149) = [12 /  4  1  5  3  3  5  1  4 24 / ...]
sqrt(150) = [12 /  4 24 / ...]
sqrt(151) = [12 /  3  2  7  1  3  4  1  1  1 11  1  1  1  4  3  1  7  2  3 24 / ...]
sqrt(152) = [12 /  3 24 / ...]
sqrt(153) = [12 /  2  1  2  2  2  1  2 24 / ...]
sqrt(154) = [12 /  2  2  3  1  2  1  3  2  2 24 / ...]
sqrt(155) = [12 /  2  4  2 24 / ...]
sqrt(156) = [12 /  2 24 / ...]
sqrt(157) = [12 /  1  1  7  1  5  2  1  1  1  1  2  5  1  7  1  1 24 / ...]
sqrt(158) = [12 /  1  1  3 12  3  1  1 24 / ...]
sqrt(159) = [12 /  1  1  1  1  3  1  1  1  1 24 / ...]
sqrt(160) = [12 /  1  1  1  5  1  1  1 24 / ...]
sqrt(161) = [12 /  1  2  4  1  2  1  4  2  1 24 / ...]
sqrt(162) = [12 /  1  2  1  2 12  2  1  2  1 24 / ...]
sqrt(163) = [12 /  1  3  3  2  1  1  7  1 11  1  7  1  1  2  3  3  1 24 / ...]
sqrt(164) = [12 /  1  4  6  4  1 24 / ...]
sqrt(165) = [12 /  1  5  2  5  1 24 / ...]
sqrt(166) = [12 /  1  7  1  1  1  2  4  1  3  2 12  2  3  1  4  2  1  1  1  7  1 24 / ...]
sqrt(167) = [12 /  1 11  1 24 / ...]
NIL
```

Pell's equation: This is a special case of the Diophantine equation

$$x^2 - dy^2 = r,$$

whose solutions are convergents to $\sqrt{d}$ whenever $d \geq r^2 + |r|$. We proved this fact in Part II Number Theory, from the fact that such solutions satisfy

$$|x/y - \sqrt{d}| \leq 1/(2y^2);$$

such a close rational approximation is necessarily a convergent.

```
> (loop for d from 1 to 167
unless (square? d)
do (print-Pell (Pell d) d))
Pell with d =   2: fundamental soln x = 3,             y = 2
Pell with d =   3: fundamental soln x = 2,             y = 1
Pell with d =   5: fundamental soln x = 9,             y = 4
Pell with d =   6: fundamental soln x = 5,             y = 2
Pell with d =   7: fundamental soln x = 8,             y = 3
Pell with d =   8: fundamental soln x = 3,             y = 1
Pell with d =  10: fundamental soln x = 19,            y = 6
Pell with d =  11: fundamental soln x = 10,            y = 3
Pell with d =  12: fundamental soln x = 7,             y = 2
Pell with d =  13: fundamental soln x = 649,           y = 180
Pell with d =  14: fundamental soln x = 15,            y = 4
Pell with d =  15: fundamental soln x = 4,             y = 1
Pell with d =  17: fundamental soln x = 33,            y = 8
Pell with d =  18: fundamental soln x = 17,            y = 4
Pell with d =  19: fundamental soln x = 170,           y = 39
Pell with d =  20: fundamental soln x = 9,             y = 2
Pell with d =  21: fundamental soln x = 55,            y = 12
Pell with d =  22: fundamental soln x = 197,           y = 42
Pell with d =  23: fundamental soln x = 24,            y = 5
Pell with d =  24: fundamental soln x = 5,             y = 1
Pell with d =  26: fundamental soln x = 51,            y = 10
Pell with d =  27: fundamental soln x = 26,            y = 5
Pell with d =  28: fundamental soln x = 127,           y = 24
Pell with d =  29: fundamental soln x = 9801,          y = 1820
Pell with d =  30: fundamental soln x = 11,            y = 2
Pell with d =  31: fundamental soln x = 1520,          y = 273
Pell with d =  32: fundamental soln x = 17,            y = 3
Pell with d =  33: fundamental soln x = 23,            y = 4
Pell with d =  34: fundamental soln x = 35,            y = 6
Pell with d =  35: fundamental soln x = 6,             y = 1
Pell with d =  37: fundamental soln x = 73,            y = 12
Pell with d =  38: fundamental soln x = 37,            y = 6
Pell with d =  39: fundamental soln x = 25,            y = 4
Pell with d =  40: fundamental soln x = 19,            y = 3
Pell with d =  41: fundamental soln x = 2049,          y = 320
Pell with d =  42: fundamental soln x = 13,            y = 2
Pell with d =  43: fundamental soln x = 3482,          y = 531
Pell with d =  44: fundamental soln x = 199,           y = 30
Pell with d =  45: fundamental soln x = 161,           y = 24
```

```
Pell with d =  46: fundamental soln x = 24335,          y = 3588
Pell with d =  47: fundamental soln x = 48,             y = 7
Pell with d =  48: fundamental soln x = 7,              y = 1
Pell with d =  50: fundamental soln x = 99,             y = 14
Pell with d =  51: fundamental soln x = 50,             y = 7
Pell with d =  52: fundamental soln x = 649,            y = 90
Pell with d =  53: fundamental soln x = 66249,          y = 9100
Pell with d =  54: fundamental soln x = 485,            y = 66
Pell with d =  55: fundamental soln x = 89,             y = 12
Pell with d =  56: fundamental soln x = 15,             y = 2
Pell with d =  57: fundamental soln x = 151,            y = 20
Pell with d =  58: fundamental soln x = 19603,          y = 2574
Pell with d =  59: fundamental soln x = 530,            y = 69
Pell with d =  60: fundamental soln x = 31,             y = 4
Pell with d =  61: fundamental soln x = 1766319049,     y = 226153980
Pell with d =  62: fundamental soln x = 63,             y = 8
Pell with d =  63: fundamental soln x = 8,              y = 1
Pell with d =  65: fundamental soln x = 129,            y = 16
Pell with d =  66: fundamental soln x = 65,             y = 8
Pell with d =  67: fundamental soln x = 48842,          y = 5967
Pell with d =  68: fundamental soln x = 33,             y = 4
Pell with d =  69: fundamental soln x = 7775,           y = 936
Pell with d =  70: fundamental soln x = 251,            y = 30
Pell with d =  71: fundamental soln x = 3480,           y = 413
Pell with d =  72: fundamental soln x = 17,             y = 2
Pell with d =  73: fundamental soln x = 2281249,        y = 267000
Pell with d =  74: fundamental soln x = 3699,           y = 430
Pell with d =  75: fundamental soln x = 26,             y = 3
Pell with d =  76: fundamental soln x = 57799,          y = 6630
Pell with d =  77: fundamental soln x = 351,            y = 40
Pell with d =  78: fundamental soln x = 53,             y = 6
Pell with d =  79: fundamental soln x = 80,             y = 9
Pell with d =  80: fundamental soln x = 9,              y = 1
Pell with d =  82: fundamental soln x = 163,            y = 18
Pell with d =  83: fundamental soln x = 82,             y = 9
Pell with d =  84: fundamental soln x = 55,             y = 6
Pell with d =  85: fundamental soln x = 285769,         y = 30996
Pell with d =  86: fundamental soln x = 10405,          y = 1122
Pell with d =  87: fundamental soln x = 28,             y = 3
Pell with d =  88: fundamental soln x = 197,            y = 21
Pell with d =  89: fundamental soln x = 500001,         y = 53000
Pell with d =  90: fundamental soln x = 19,             y = 2
Pell with d =  91: fundamental soln x = 1574,           y = 165
Pell with d =  92: fundamental soln x = 1151,           y = 120
Pell with d =  93: fundamental soln x = 12151,          y = 1260
Pell with d =  94: fundamental soln x = 2143295,        y = 221064
Pell with d =  95: fundamental soln x = 39,             y = 4
Pell with d =  96: fundamental soln x = 49,             y = 5
Pell with d =  97: fundamental soln x = 62809633,       y = 6377352
Pell with d =  98: fundamental soln x = 99,             y = 10
Pell with d =  99: fundamental soln x = 10,             y = 1
Pell with d = 101: fundamental soln x = 201,            y = 20
Pell with d = 102: fundamental soln x = 101,            y = 10
Pell with d = 103: fundamental soln x = 227528,         y = 22419
Pell with d = 104: fundamental soln x = 51,             y = 5
```

```
Pell with d = 105: fundamental soln x = 41,             y = 4
Pell with d = 106: fundamental soln x = 32080051,       y = 3115890
Pell with d = 107: fundamental soln x = 962,            y = 93
Pell with d = 108: fundamental soln x = 1351,           y = 130
Pell with d = 109: fundamental soln x = 15807067198624 9, y = 15140424455100
Pell with d = 110: fundamental soln x = 21,             y = 2
Pell with d = 111: fundamental soln x = 295,            y = 28
Pell with d = 112: fundamental soln x = 127,            y = 12
Pell with d = 113: fundamental soln x = 1204353,        y = 113296
Pell with d = 114: fundamental soln x = 1025,           y = 96
Pell with d = 115: fundamental soln x = 1126,           y = 105
Pell with d = 116: fundamental soln x = 9801,           y = 910
Pell with d = 117: fundamental soln x = 649,            y = 60
Pell with d = 118: fundamental soln x = 306917,         y = 28254
Pell with d = 119: fundamental soln x = 120,            y = 11
Pell with d = 120: fundamental soln x = 11,             y = 1
Pell with d = 122: fundamental soln x = 243,            y = 22
Pell with d = 123: fundamental soln x = 122,            y = 11
Pell with d = 124: fundamental soln x = 4620799,        y = 414960
Pell with d = 125: fundamental soln x = 930249,         y = 83204
Pell with d = 126: fundamental soln x = 449,            y = 40
Pell with d = 127: fundamental soln x = 4730624,        y = 419775
Pell with d = 128: fundamental soln x = 577,            y = 51
Pell with d = 129: fundamental soln x = 16855,          y = 1484
Pell with d = 130: fundamental soln x = 6499,           y = 570
Pell with d = 131: fundamental soln x = 10610,          y = 927
Pell with d = 132: fundamental soln x = 23,             y = 2
Pell with d = 133: fundamental soln x = 2588599,        y = 224460
Pell with d = 134: fundamental soln x = 145925,         y = 12606
Pell with d = 135: fundamental soln x = 244,            y = 21
Pell with d = 136: fundamental soln x = 35,             y = 3
Pell with d = 137: fundamental soln x = 6083073,        y = 519712
Pell with d = 138: fundamental soln x = 47,             y = 4
Pell with d = 139: fundamental soln x = 77563250,       y = 6578829
Pell with d = 140: fundamental soln x = 71,             y = 6
Pell with d = 141: fundamental soln x = 95,             y = 8
Pell with d = 142: fundamental soln x = 143,            y = 12
Pell with d = 143: fundamental soln x = 12,             y = 1
Pell with d = 145: fundamental soln x = 289,            y = 24
Pell with d = 146: fundamental soln x = 145,            y = 12
Pell with d = 147: fundamental soln x = 97,             y = 8
Pell with d = 148: fundamental soln x = 73,             y = 6
Pell with d = 149: fundamental soln x = 25801741449,    y = 2113761020
Pell with d = 150: fundamental soln x = 49,             y = 4
Pell with d = 151: fundamental soln x = 1728148040,     y = 140634693
Pell with d = 152: fundamental soln x = 37,             y = 3
Pell with d = 153: fundamental soln x = 2177,           y = 176
Pell with d = 154: fundamental soln x = 21295,          y = 1716
Pell with d = 155: fundamental soln x = 249,            y = 20
Pell with d = 156: fundamental soln x = 25,             y = 2
Pell with d = 157: fundamental soln x = 46698728731849, y = 3726964292220
Pell with d = 158: fundamental soln x = 7743,           y = 616
Pell with d = 159: fundamental soln x = 1324,           y = 105
Pell with d = 160: fundamental soln x = 721,            y = 57
Pell with d = 161: fundamental soln x = 11775,          y = 928
```

```
Pell with d = 162: fundamental soln x = 19601,          y = 1540
Pell with d = 163: fundamental soln x = 64080026,       y = 5019135
Pell with d = 164: fundamental soln x = 2049,           y = 160
Pell with d = 165: fundamental soln x = 1079,           y = 84
Pell with d = 166: fundamental soln x = 1700902565,     y = 132015642
Pell with d = 167: fundamental soln x = 168,            y = 13
NIL
```

Time required to solve Pell's equation: To solve Pell's equation by brute-force search, we would have to check whether each $dy^2 + 1$ is a square, for $y = 1, 2, 3, \ldots$ as far as necessary. The search is guaranteed to succeed eventually, but for $d = 109$, say, it would clearly be a poor idea.

Instead of searching through $y = 1, 2, 3, \ldots$, our method is to consider only $y = Q_1, Q_2, Q_3, \ldots$. These are, after all, the only potential denominators! Remember that Q_n grows exponentially with n—at least as fast as ϕ^n. So for $d = 109$, we will find the solution in at most $\log_\phi 15140424455100 = 63$ steps, rather than in 15140424455100 steps. (In fact, since the continued fraction for $\sqrt{109}$ has period 15, we need 30 steps.)

The moral of the story is that for any particular instance of Pell's equation (any value of d), the continued fraction algorithm only takes about the *logarithm* of the number of steps required by brute force.

How long do the steps take? In the brute-force algorithm, each step has to take the integer square root of a large number (size $O(y)$). My version of the continued-fraction algorithm only needs to take the integer square root of a small number, and run the Euclidean algorithm on some other small numbers (to put $\frac{\alpha+\beta\sqrt{d}}{\gamma}$ into lowest terms). Since `isqrt` and `gcd` are both worst-case logarithmic on their arguments, the steps in the continued-fraction algorithm are at least as fast as the steps in the brute-force search—and of course there are far fewer of them.

The negative Pell equation:

```
> (loop for d from 1 to 167
unless (square? d)
do (print-Pell (Pell d t) d t))
Negative Pell with d =   2: fundamental soln x = 1,        y = 1
Negative Pell with d =   3: (no solutions)
Negative Pell with d =   5: fundamental soln x = 2,        y = 1
Negative Pell with d =   6: (no solutions)
Negative Pell with d =   7: (no solutions)
Negative Pell with d =   8: (no solutions)
Negative Pell with d =  10: fundamental soln x = 3,        y = 1
Negative Pell with d =  11: (no solutions)
Negative Pell with d =  12: (no solutions)
Negative Pell with d =  13: fundamental soln x = 18,       y = 5
Negative Pell with d =  14: (no solutions)
Negative Pell with d =  15: (no solutions)
```

```
Negative Pell with d =  17: fundamental soln x = 4,        y = 1
Negative Pell with d =  18: (no solutions)
Negative Pell with d =  19: (no solutions)
Negative Pell with d =  20: (no solutions)
Negative Pell with d =  21: (no solutions)
Negative Pell with d =  22: (no solutions)
Negative Pell with d =  23: (no solutions)
Negative Pell with d =  24: (no solutions)
Negative Pell with d =  26: fundamental soln x = 5,        y = 1
Negative Pell with d =  27: (no solutions)
Negative Pell with d =  28: (no solutions)
Negative Pell with d =  29: fundamental soln x = 70,       y = 13
Negative Pell with d =  30: (no solutions)
Negative Pell with d =  31: (no solutions)
Negative Pell with d =  32: (no solutions)
Negative Pell with d =  33: (no solutions)
Negative Pell with d =  34: (no solutions)
Negative Pell with d =  35: (no solutions)
Negative Pell with d =  37: fundamental soln x = 6,        y = 1
Negative Pell with d =  38: (no solutions)
Negative Pell with d =  39: (no solutions)
Negative Pell with d =  40: (no solutions)
Negative Pell with d =  41: fundamental soln x = 32,       y = 5
Negative Pell with d =  42: (no solutions)
Negative Pell with d =  43: (no solutions)
Negative Pell with d =  44: (no solutions)
Negative Pell with d =  45: (no solutions)
Negative Pell with d =  46: (no solutions)
Negative Pell with d =  47: (no solutions)
Negative Pell with d =  48: (no solutions)
Negative Pell with d =  50: fundamental soln x = 7,        y = 1
Negative Pell with d =  51: (no solutions)
Negative Pell with d =  52: (no solutions)
Negative Pell with d =  53: fundamental soln x = 182,      y = 25
Negative Pell with d =  54: (no solutions)
Negative Pell with d =  55: (no solutions)
Negative Pell with d =  56: (no solutions)
Negative Pell with d =  57: (no solutions)
Negative Pell with d =  58: fundamental soln x = 99,       y = 13
Negative Pell with d =  59: (no solutions)
Negative Pell with d =  60: (no solutions)
Negative Pell with d =  61: fundamental soln x = 29718,    y = 3805
Negative Pell with d =  62: (no solutions)
Negative Pell with d =  63: (no solutions)
Negative Pell with d =  65: fundamental soln x = 8,        y = 1
Negative Pell with d =  66: (no solutions)
Negative Pell with d =  67: (no solutions)
Negative Pell with d =  68: (no solutions)
Negative Pell with d =  69: (no solutions)
Negative Pell with d =  70: (no solutions)
Negative Pell with d =  71: (no solutions)
Negative Pell with d =  72: (no solutions)
Negative Pell with d =  73: fundamental soln x = 1068,     y = 125
Negative Pell with d =  74: fundamental soln x = 43,       y = 5
Negative Pell with d =  75: (no solutions)
```

```
Negative Pell with d =  76: (no solutions)
Negative Pell with d =  77: (no solutions)
Negative Pell with d =  78: (no solutions)
Negative Pell with d =  79: (no solutions)
Negative Pell with d =  80: (no solutions)
Negative Pell with d =  82: fundamental soln x = 9,         y = 1
Negative Pell with d =  83: (no solutions)
Negative Pell with d =  84: (no solutions)
Negative Pell with d =  85: fundamental soln x = 378,       y = 41
Negative Pell with d =  86: (no solutions)
Negative Pell with d =  87: (no solutions)
Negative Pell with d =  88: (no solutions)
Negative Pell with d =  89: fundamental soln x = 500,       y = 53
Negative Pell with d =  90: (no solutions)
Negative Pell with d =  91: (no solutions)
Negative Pell with d =  92: (no solutions)
Negative Pell with d =  93: (no solutions)
Negative Pell with d =  94: (no solutions)
Negative Pell with d =  95: (no solutions)
Negative Pell with d =  96: (no solutions)
Negative Pell with d =  97: fundamental soln x = 5604,      y = 569
Negative Pell with d =  98: (no solutions)
Negative Pell with d =  99: (no solutions)
Negative Pell with d = 101: fundamental soln x = 10,        y = 1
Negative Pell with d = 102: (no solutions)
Negative Pell with d = 103: (no solutions)
Negative Pell with d = 104: (no solutions)
Negative Pell with d = 105: (no solutions)
Negative Pell with d = 106: fundamental soln x = 4005,      y = 389
Negative Pell with d = 107: (no solutions)
Negative Pell with d = 108: (no solutions)
Negative Pell with d = 109: fundamental soln x = 8890182,   y = 851525
Negative Pell with d = 110: (no solutions)
Negative Pell with d = 111: (no solutions)
Negative Pell with d = 112: (no solutions)
Negative Pell with d = 113: fundamental soln x = 776,       y = 73
Negative Pell with d = 114: (no solutions)
Negative Pell with d = 115: (no solutions)
Negative Pell with d = 116: (no solutions)
Negative Pell with d = 117: (no solutions)
Negative Pell with d = 118: (no solutions)
Negative Pell with d = 119: (no solutions)
Negative Pell with d = 120: (no solutions)
Negative Pell with d = 122: fundamental soln x = 11,        y = 1
Negative Pell with d = 123: (no solutions)
Negative Pell with d = 124: (no solutions)
Negative Pell with d = 125: fundamental soln x = 682,       y = 61
Negative Pell with d = 126: (no solutions)
Negative Pell with d = 127: (no solutions)
Negative Pell with d = 128: (no solutions)
Negative Pell with d = 129: (no solutions)
Negative Pell with d = 130: fundamental soln x = 57,        y = 5
Negative Pell with d = 131: (no solutions)
Negative Pell with d = 132: (no solutions)
Negative Pell with d = 133: (no solutions)
```

```
Negative Pell with d = 134: (no solutions)
Negative Pell with d = 135: (no solutions)
Negative Pell with d = 136: (no solutions)
Negative Pell with d = 137: fundamental soln x = 1744,      y = 149
Negative Pell with d = 138: (no solutions)
Negative Pell with d = 139: (no solutions)
Negative Pell with d = 140: (no solutions)
Negative Pell with d = 141: (no solutions)
Negative Pell with d = 142: (no solutions)
Negative Pell with d = 143: (no solutions)
Negative Pell with d = 145: fundamental soln x = 12,        y = 1
Negative Pell with d = 146: (no solutions)
Negative Pell with d = 147: (no solutions)
Negative Pell with d = 148: (no solutions)
Negative Pell with d = 149: fundamental soln x = 113582,    y = 9305
Negative Pell with d = 150: (no solutions)
Negative Pell with d = 151: (no solutions)
Negative Pell with d = 152: (no solutions)
Negative Pell with d = 153: (no solutions)
Negative Pell with d = 154: (no solutions)
Negative Pell with d = 155: (no solutions)
Negative Pell with d = 156: (no solutions)
Negative Pell with d = 157: fundamental soln x = 4832118,   y = 385645
Negative Pell with d = 158: (no solutions)
Negative Pell with d = 159: (no solutions)
Negative Pell with d = 160: (no solutions)
Negative Pell with d = 161: (no solutions)
Negative Pell with d = 162: (no solutions)
Negative Pell with d = 163: (no solutions)
Negative Pell with d = 164: (no solutions)
Negative Pell with d = 165: (no solutions)
Negative Pell with d = 166: (no solutions)
Negative Pell with d = 167: (no solutions)
NIL
```

Fundamental solution to Pell's equation: Suppose (x, y) solves the negative Pell equation for some value of d. Then $(1 + 2x^2, 2xy)$ solves the corresponding positive Pell equation:

$$\begin{aligned}(1+2x^2)^2 - d(2xy)^2 &= 1 + 4x^2 + 4x^4 - 4dx^2y^2 \\ &= 1 + 4x^2(1 + x^2 - dy^2) \\ &= 1 + 4x^2(1 + (-1)) \\ &= 1.\end{aligned}$$

Suppose (x, y) is actually the fundamental solution to the negative equation. We would like to show that $(1 + 2x^2, 2xy)$ is the fundamental solution to the positive equation.

Write $(x, y) = (P_{m-1}, Q_{m-1})$, where

$$\frac{P_{m-1}}{Q_{m-1}} = [a_0, a_1, \ldots a_{m-1}].$$

Since the negative equation is soluble, m is odd, and the fundamental solution to the positive equation is (P_{2m-1}, Q_{2m-1}) where

$$\begin{aligned}
\frac{P_{2m-1}}{Q_{2m-1}} &= [a_0, a_1, \ldots a_{m-1}, a_m, a_{m+1}, \ldots a_{2m-1}] \\
&= [a_0, a_1, \ldots a_{m-1}, 2a_0, a_1, \ldots a_{m-1}] \\
&= [a_0, a_1, \ldots a_{m-1}, a_0 + [a_0, a_1, \ldots a_{m-1}]] \\
&= [a_0, a_1, \ldots a_{m-1}, a_0 + \frac{P_{m-1}}{Q_{m-1}}] \\
&= \frac{P_{m-2} + P_{m-1}(a_0 + P_{m-1}/Q_{m-1})}{Q_{m-2} + Q_{m-1}(a_0 + P_{m-1}/Q_{m-1})} \\
&= \frac{P_{m-2}Q_{m-1} + a_0 P_{m-1}Q_{m-1} + P_{m-1}^2}{Q_{m-2}Q_{m-1} + a_0 Q_{m-1}^2 + P_{m-1}Q_{m-1}} \\
&= \frac{1 + P_{m-1}Q_{m-2} + a_0 P_{m-1}Q_{m-1} + P_{m-1}^2}{Q_{m-2}Q_{m-1} + a_0 Q_{m-1}^2 + P_{m-1}Q_{m-1}} \\
&= \frac{1 + P_{m-1}(Q_{m-2} + a_0 Q_{m-1} + P_{m-1})}{Q_{m-1}(Q_{m-2} + a_0 Q_{m-1} + P_{m-1})}
\end{aligned}$$

We need to show that $Q_{m-2} + Q_{m-1}a_0 = P_{m-1}$. This will make the numerator and denominator $(1 + 2x^2)^2$ and $2xy$, respectively; which means that they are a solution to the positive Pell equation, therefore coprime, and therefore equal to P_{2m-1} and Q_{2m-1}—the fundamental solution.

Unfortunately, I can't see right now how to demonstrate this!

There is, however, an easy converse. By reversing the first derivation in this section, we see that if the positive Pell equation has a solution of the form $(1 + 2x^2, 2xy)$, $x > 0$, then the negative equation is soluble with solution (x, y).

Moreover, if $(1 + 2x^2, 2xy)$ is the *fundamental* solution to the positive equation, then (x, y) must be the fundamental solution to the negative one—since a smaller solution to the negative equation would give a smaller solution to the positive one.[3]

Solutions to the negative equation: The negative Pell equation $x^2 - dy^2 = -1$ is soluble if and only if the fundamental unit in I_{4d} has norm -1. This is true for $d = k^2 + 1$, for example.

[3]What does a "smaller" solution mean? The solutions inherit an order from the series of convergents. Since their numerators form a subsequence of P_n, the values of x increase; similarly, since their denominators form a subsequence of Q_n, the values of y increase. In particular, the fundamental solution has the smallest values of x and y.