Secret Sharing and Visual Cryptography

Outline

- Secret Sharing
- Visual Secret Sharing
- Constructions
- Moiré Cryptography
- Issues

Secret Sharing

Secret Sharing

- **Threshold Secret Sharing (Shamir, Blakely 1979)**
- Motivation – increase confidentiality and availability
- *(k,n)* threshold scheme
  - Threshold *k*
  - Group Size *n*
- Confidentiality vs Availability

General Secret Sharing

- S – Secret to be shared
- $\mathcal{P}$ – Set of participants
- Qualified Subsets of $\mathcal{P}$ can reconstruct S
- Access Structure
  - Family of qualified subsets $\mathcal{A} \subseteq 2^{\mathcal{P}}$
  - Generally monotone
    - Superset of a qualified subset is also qualified

$$A \in \mathcal{A}, A \subseteq A' \subseteq \mathcal{P} \Rightarrow A' \in \mathcal{A}$$

Information Theoretically

- **Perfect Secret Sharing** scheme for S
  - **Qualified Subset G**

$$G \in \mathcal{A}, H(S|G) = 0$$

  - **Unqualified Subset B**

$$B \notin \mathcal{A}, H(S|B) = H(S)$$

- Information Rate of a scheme
  - $$\rho = \frac{log_2|Secret|}{max\, log_2|Share|}$$
  - $$\rho \leq 1$$
  - Measure of efficiency of the scheme

Size of Shares

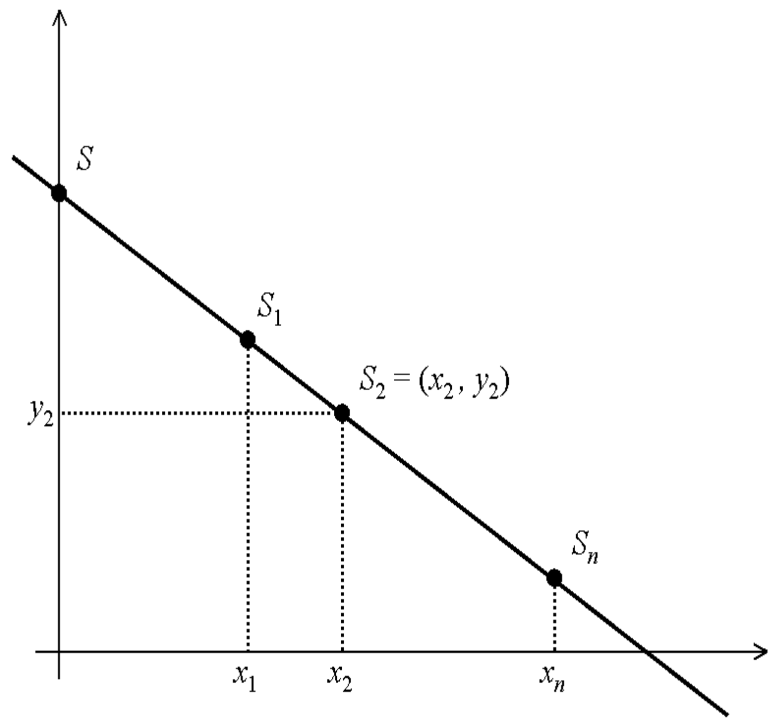
- Perfect Scheme
  - Size of share at least size of secret
  - Larger share size
    - More memory required
    - Lower efficiency
- Ideal Scheme
  - Share size = secret size
  - Information rate/efficiency is high

Shamir's Threshold Scheme

- *(k,n)* Threshold scheme
  - $s \epsilon F_q$ is the secret to be shared
  - $x_1, x_2 \ldots x_n$ are distinct non-zero elements chosen from F_q
  - Chose coefficients $f_1, \ldots f_{k-1}$ at random from F_q
  - Let $y = f(x) = s + \sum_{j=1}^{k-1} f_j x^j$
  - Share $s_i = (x_i, y_i)$

Lagrange's Interpolation

- Need k shares for reconstruction
- Figure shows *(2,n)* scheme
- Scheme is perfect and ideal
  - *2* shares: secret is defined
  - < *2* shares: secret can be any point on *y* axis



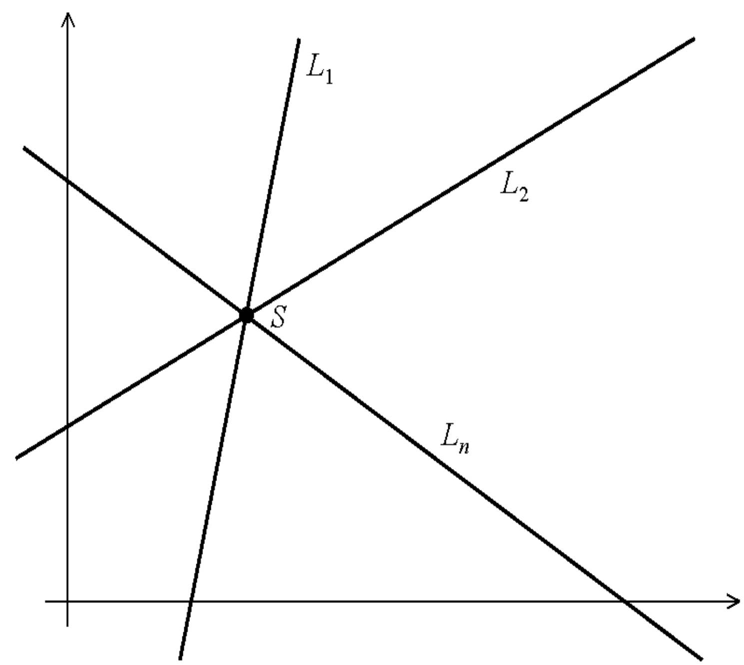
$$s = \sum_{j=1}^{k} \left(\prod_{1 \leq t \leq k, t \neq j} \frac{x_{i_k}}{x_{i_k} - x_{i_j}}\right) y_{i_j}$$

Blakely’s Secret Sharing

- Secret is point in *m*-dimensional space
- Share corresponds to a hyper plane
- Intersection of threshold planes gives the secret
- Less than threshold planes will not intersect to the secret

Blakely's Secret Sharing

- *2* dimensional plane
- Each share is a Line
- Intersection of *2* shares gives the secret



Non-perfect secret sharing scheme

- Motivation
- Semi-qualified subsets
  - Partial Information about Secret
  - Size of shares < Size of secret

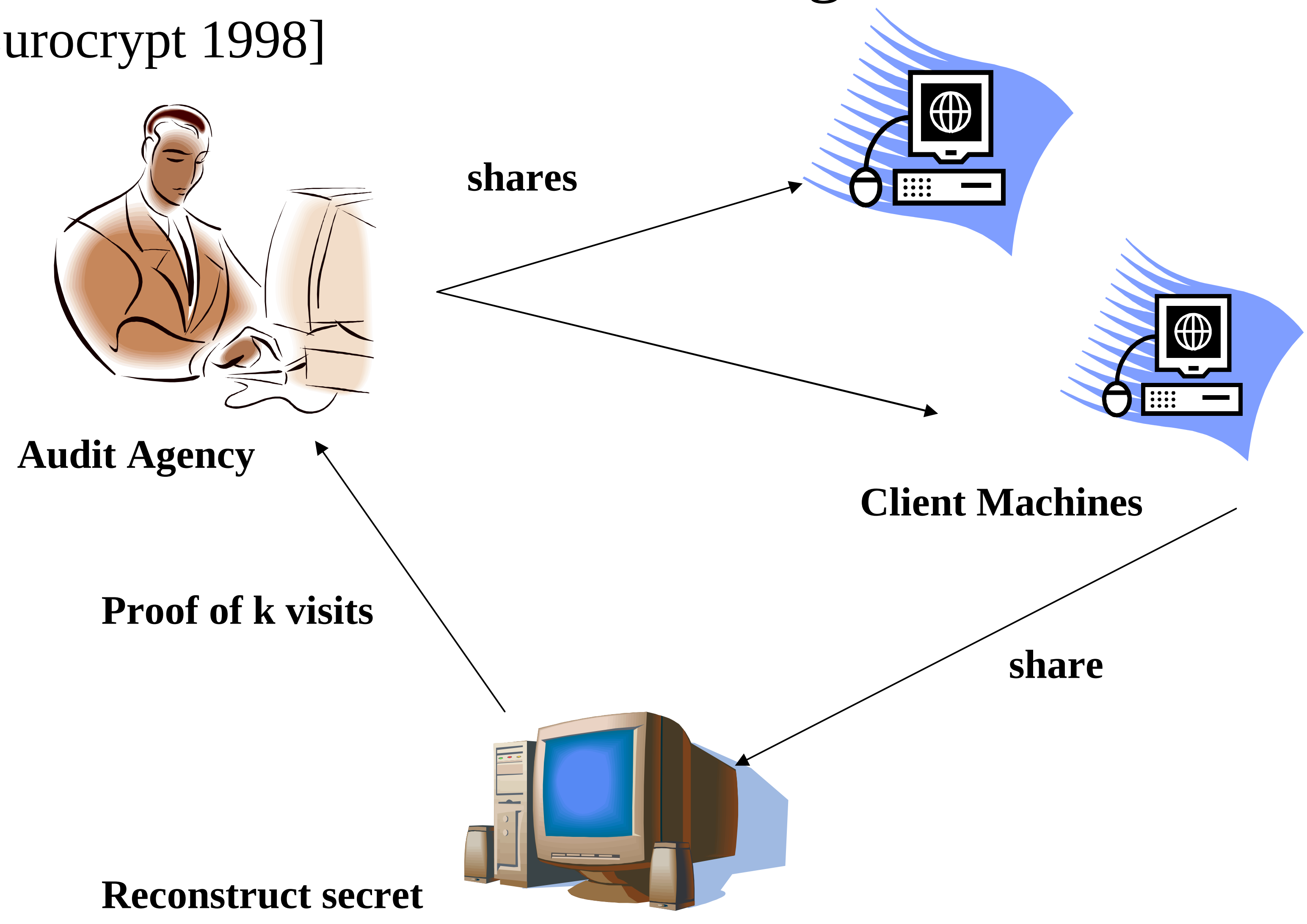
- (d,k,n) ramp scheme [Blakely, Medows Crypto 84]
  - Qualified subset A, $|A| \geq k$
    - $H(S|A)=0$
  - Unqualified subset U, $|U| \leq k\text{-}d$
    - $H(S|U)=H(S)$
  - Semi Qualified subset P, $k\text{-}d<|P|<k$
    - $0<H(S|P)<H(S)$

Making Shamir's scheme non-perfect

- Instead of one secret have a vector of secrets
- Each share is also a vector
- Each share reduces by the dimension of the secret space by 1
- Linear gain of information as you compromise more shares

Applications of Secret Sharing

- Secure and Efficient Metering [Naor and Pinkas, Eurocrypt 1998]

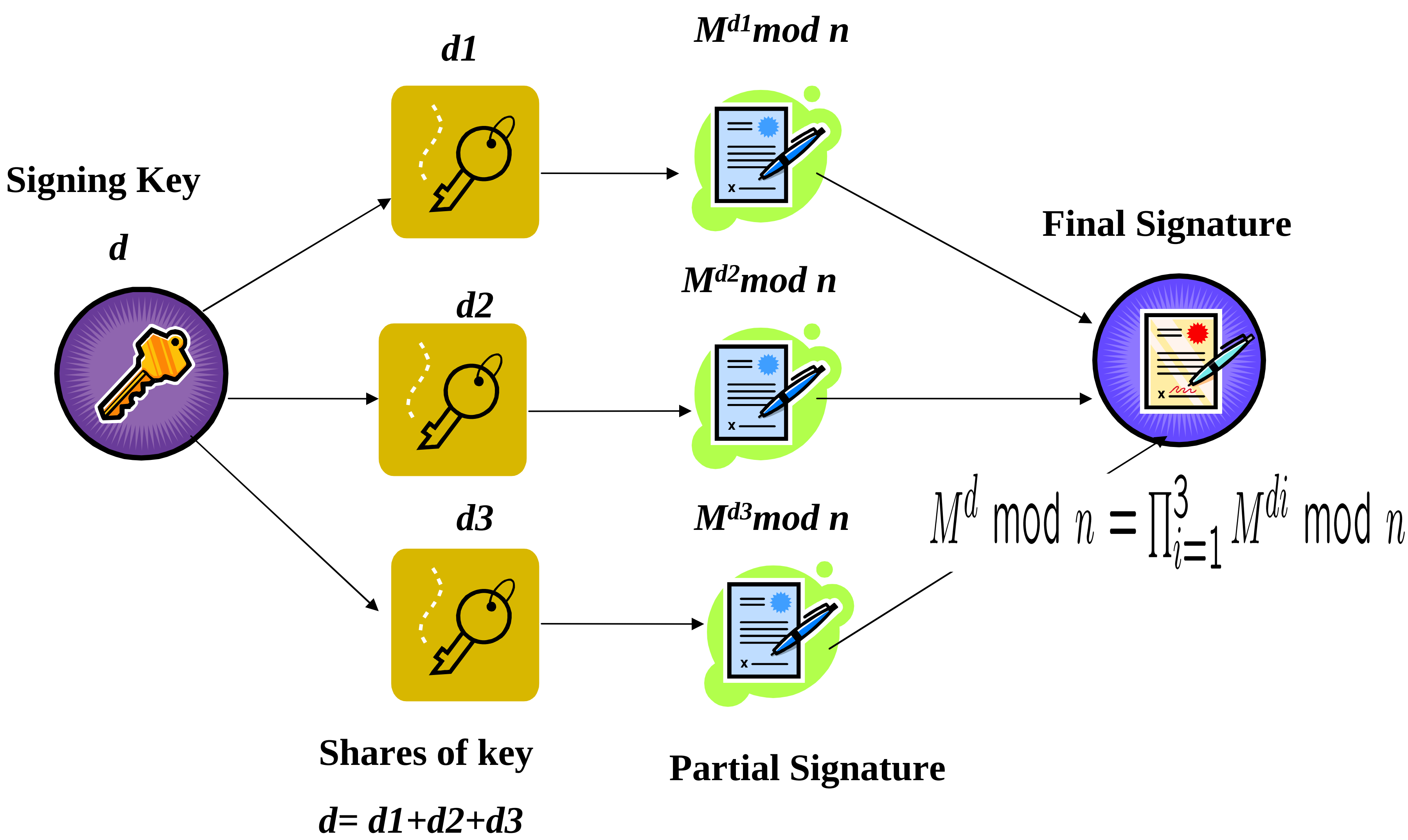


Applications of Secret Sharing

- Threshold Signature Sharing
  - Signing key with a single entity can be abused
  - Distribute the power to sign a document

- RSA Signatures
  - A Simplified Approach to Threshold and Proactive RSA [Rabin, CRYPTO 98]
    - Signing key shared at all times using additive method

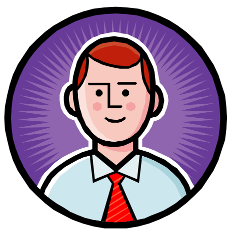
Basic Method of Signature Sharing



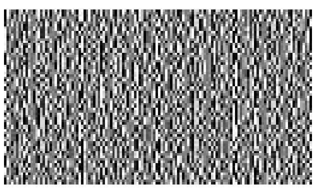
Visual Secret Sharing

Visual Secret Sharing

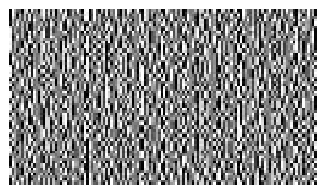
- Naor and Shamir [1994]

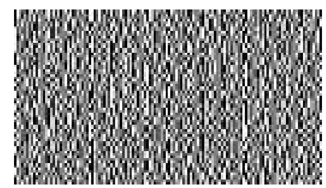


Bob faxes secret message

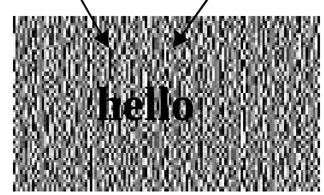


Ciphertext





Cipher text **Key**



No computer needed but other printer constraints involved

Visual Secret Sharing

- Encode secret image *S* in threshold shadow images (shares).
- Shares are represented on transparencies
- Secret is reconstructed visually
- ***(k,n)*** visual threshold scheme
  - *k* of the shares (transparencies) are superimposed reveal secret
  - <*k* shares do not reveal any information

Constructing a Threshold Scheme

- Consider (2,2) regular threshold scheme
  - Secret $K = s_1 \text{ } xor \text{ } s_2$
  - s_1, s_2 take values (0,1)
    - 0 xor 0 = 0, 1 xor 1 = 0
    - 0 xor 1 = 1, 1 xor 0 = 1
  - Neither s_1 nor s_2 reveal any information about K

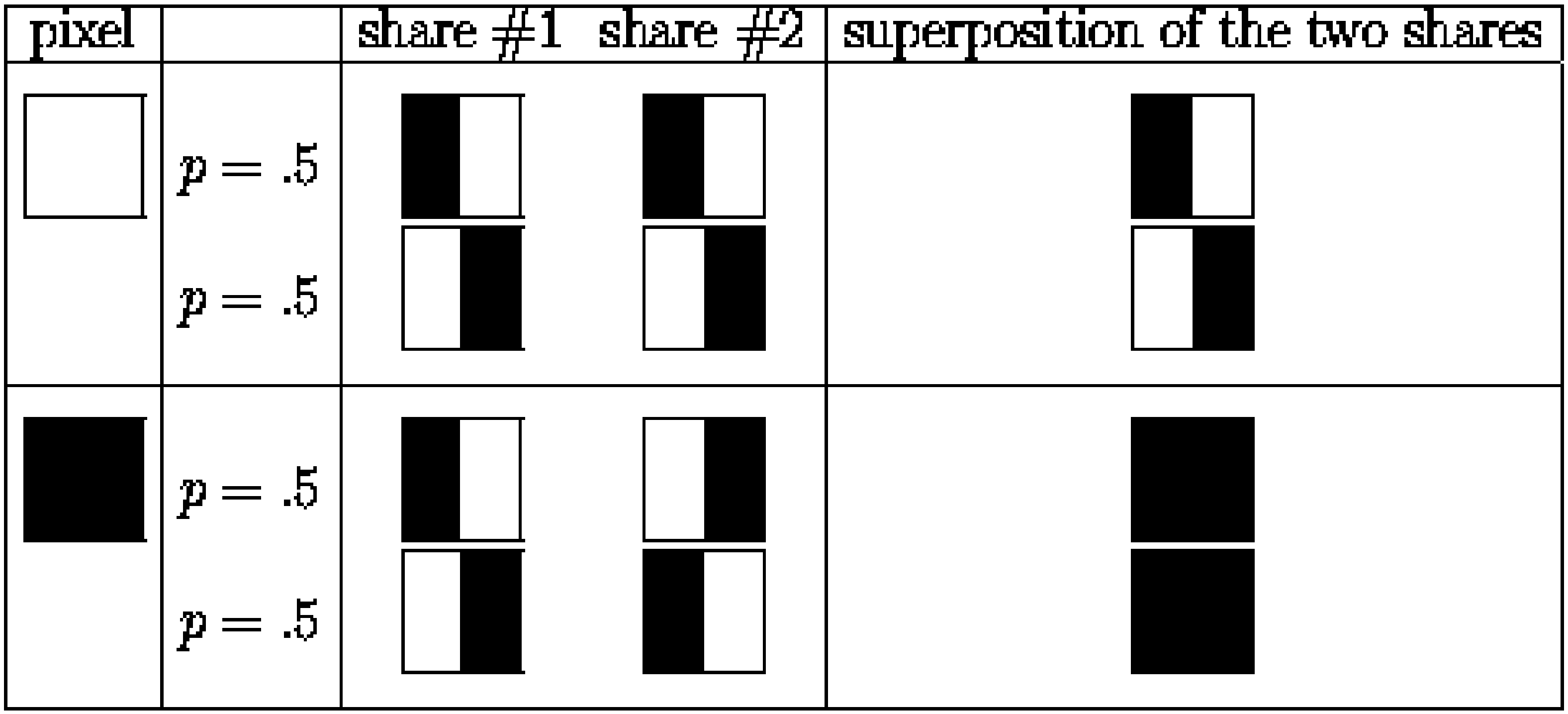
Constructing a Visual Threshold Scheme

- Associate black pixel with binary digit 1
- Associate white pixel with binary digit 0
  - 0 on 0 = 0 (good)
  - 0 on 1 = 1 (good)
  - 1 on 0 = 1 (good)
  - 1 on 1 = 1 (oops!)
- Visual system performs Boolean OR instead of XOR

Naor and Shamir Constructions

- Basic Idea
  - Replace a pixel with $m > 1$ subpixels in each share
  - Gray level of superimposed pixels decides the color (black or white)
- Less than threshold shares do not convey any information about a pixel in final image

Naor and Shamir Construction
(2,2) Scheme



Note the difference in gray levels of white and black pixels

Example

- (2,2) Threshold Scheme – Mona Lisa image
- This is like a one time pad scheme
- Original Picture

- Superimposed picture has 50% loss in contrast

Further Naor Shamir Constructions

- Will be considering
  - $(3,n)$
  - (k,k)
  - (k,n)
- Each has a different properties in terms of pixel expansion and contrast

Preliminary Notation

- ***n*** $\rightarrow$ Group Size
- ***k*** $\rightarrow$ Threshold
- ***m*** $\rightarrow$ Pixel Expansion
- α $\rightarrow$ Relative Contrast
- $\boldsymbol{C_0}$ $\rightarrow$ Collection of *n* x *m* boolean matrices for shares of White pixel
- $\boldsymbol{C_1}$ $\rightarrow$ Collection of *n* x *m* boolean matrices for shares of Black pixel
- ***V*** $\rightarrow$ OR'ed k rows
- ***H(V)*** $\rightarrow$ Hamming weight of *V*
- ***d*** $\rightarrow$ number in $[1,m]$
- ***r*** $\rightarrow$ Size of collections C_0 and C_1

Properties of *(k,n)* scheme

- Contrast
  - For S in C_0 (WHITE): $H(V) \leq d - \alpha m$
  - For S in C_1 (BLACK): $H(V) \geq d$
- Security
  - The two collections of $\boldsymbol{q}$ **x** $\boldsymbol{m}$ ($1 \leq q < k$) matrices, formed by restricting $\boldsymbol{n}$ **x** $\boldsymbol{m}$ matrices in $\boldsymbol{C_0}$ and $\boldsymbol{C_1}$ *to* any $\boldsymbol{q}$ rows, are indistinguishable
- Their constructions are uniform
  - There is a function $\boldsymbol{f}$ such that the for any matrix in $\boldsymbol{C_0}$ or $\boldsymbol{C_1}$ the hamming weight of OR'ed $\boldsymbol{q}$ rows is $\boldsymbol{f(q)}$

Constructing a $(3,n)$, $n \geq 3$ scheme

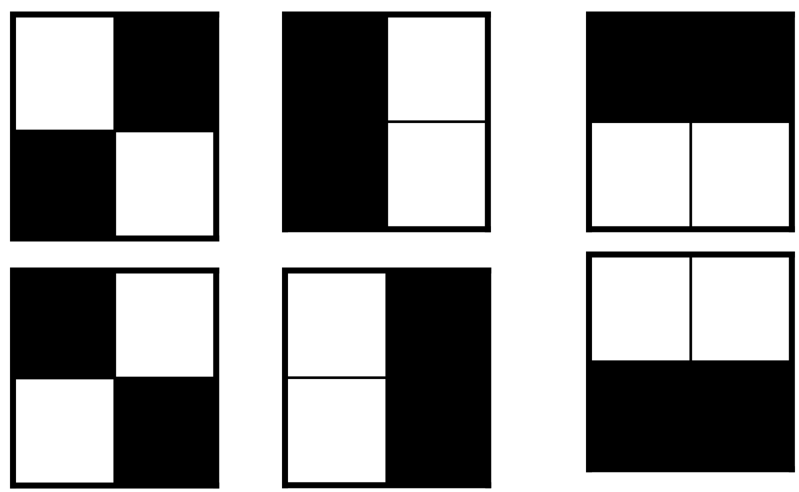
- $m=2n-2$
- $\alpha=1/2n-2$
- ***B*** is a n x $(n-2)$ matrix containing 1's
- ***I*** is a n x n identity matrix
- ***BI*** is a n x $(2n-2)$ concatenated matrix
- ***c(BI)*** is the complement of *BI*
- $\boldsymbol{C_0}$ contains matrices obtained by permuting columns of *c(BI)*
- $\boldsymbol{C_1}$ contains matrices obtained by permuting columns of *BI*

$m=4$, $\alpha=1/4$, $(3,3)$ Scheme Example

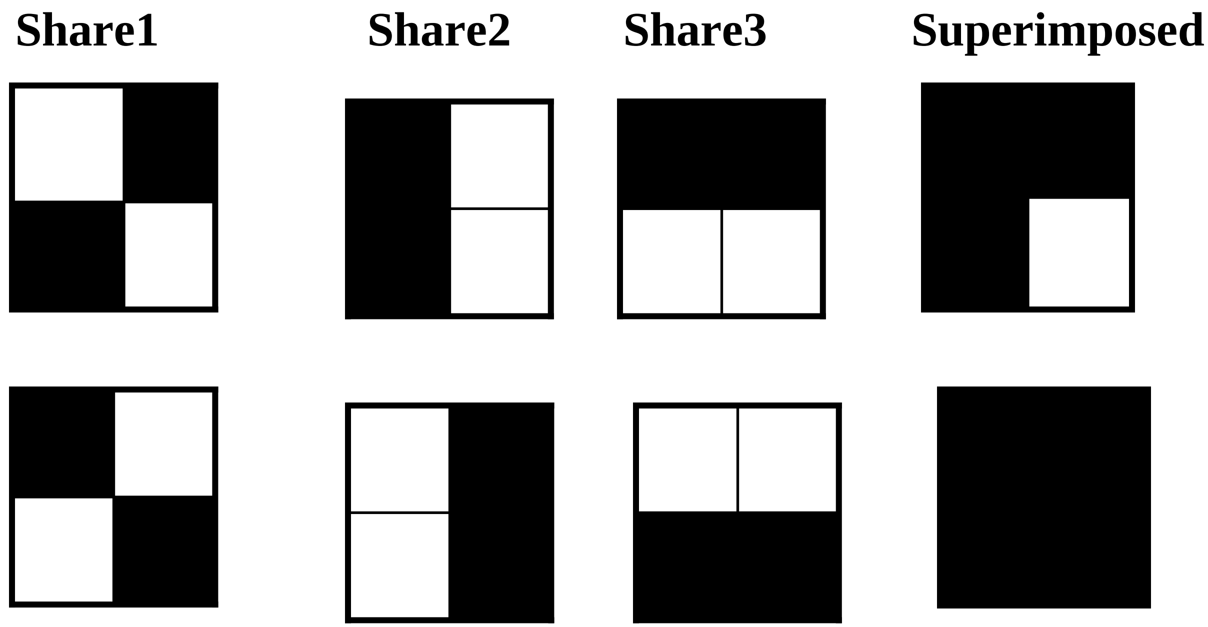
- B: $\begin{pmatrix}1\\1\\1\end{pmatrix}$ I: $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ BI: $\begin{array}{c}\textbf{BLACK}\\ \begin{pmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}\end{array}$ $c(BI)$: $\begin{array}{c}\textbf{WHITE}\\ \begin{pmatrix}0&0&1&1\\0&1&0&1\\0&1&1&0\end{pmatrix}\end{array}$

- Say permutation is {2,3,4,1}
- Shares
  - White Pixel
  - Black Pixel

share1 share2 share3



Contrast for *(3,3) m=4, α=1/4*



- White
- Black
- Can also be seen by Hamming weight

Black $H(V) = 4$

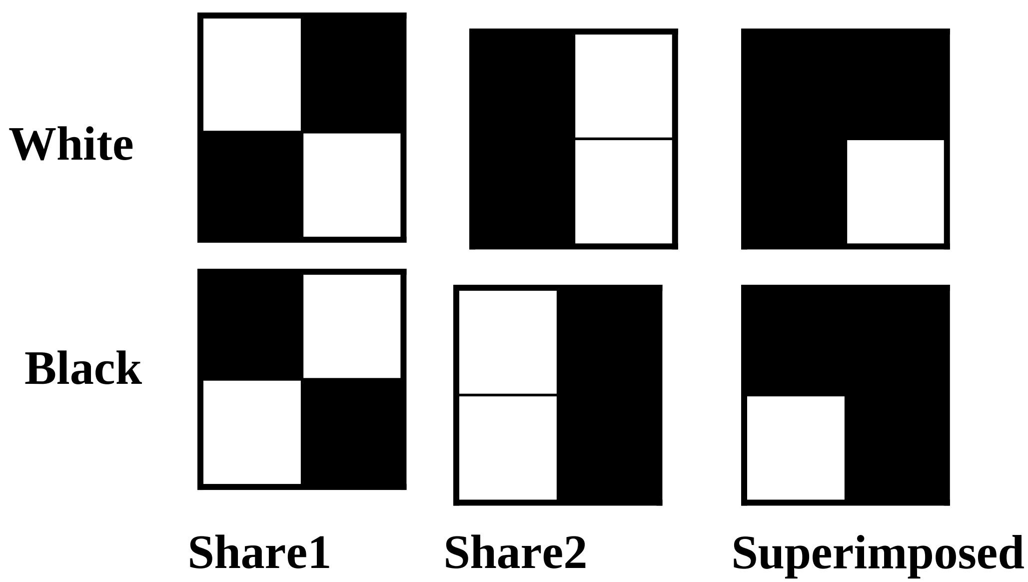
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

White $H(V) = 3$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Security for *(3,3)* Scheme

- Security
  - Superimposing < *3* shares does not reveal if secret pixel is white or black
  - Hamming weight of *2* superimposed shares is always *3*



Constructing *(k,k)* scheme

- $m = 2^{k-1}, \alpha = 1/2^{k-1}$
- Base Set $W = \{e_1 \ldots e_k\}$
- Even cardinality subsets $\pi_1 \ldots \pi_{2^{k-1}}$
- Odd cardinality subsets $\sigma_1 \ldots \sigma_{2^{k-1}}$
- k x 2^{k-1} matrix S^0, S^1
- $S^0[i,j] = 1$, if $e_i \in \pi_j$
- $S^1[i,j] = 1$ if $e_i \in \sigma_j$

Example $m=8$ $\alpha=1/8$, $(4,4)$

- $W = \{1,2,3,4\}$
- Even cardinality subsets
  - $\{\{\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3,4\}\}$
- Odd cardinality subsets
  - $\{\{1\},\{2\},\{3\},\{4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$
- Contrast
  - ***H(V)*** for $S_0 = 7$
  - ***H(V)*** for $S_1 = 8$
- Security
  - Restrict to ***q<4*** rows (***Say q=3***)
  - The ***3* x *8*** collections of matrices will be indistinguishable

$$S_0 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Moving to *(k,n)* scheme

- C is *(k,k)* scheme
  - Parameters *m* ,*r*,α
  - $C_0 = T_1^0, T_2^0, \ldots T_r^0$
  - $C_1 = T_1^1, T_2^1, \ldots T_r^1$
- *H* is collection of *l* functions

$$\forall h \in H, h : \{1 \ldots n\} \longrightarrow \{1 \ldots k\}$$

- *B* subset of *{1..n}* of size *k*
- β_q is probability that randomly chosen function $h \in H$ yields *q* different values on *B*, *1*$\leq$ *q* $\leq$ *k*

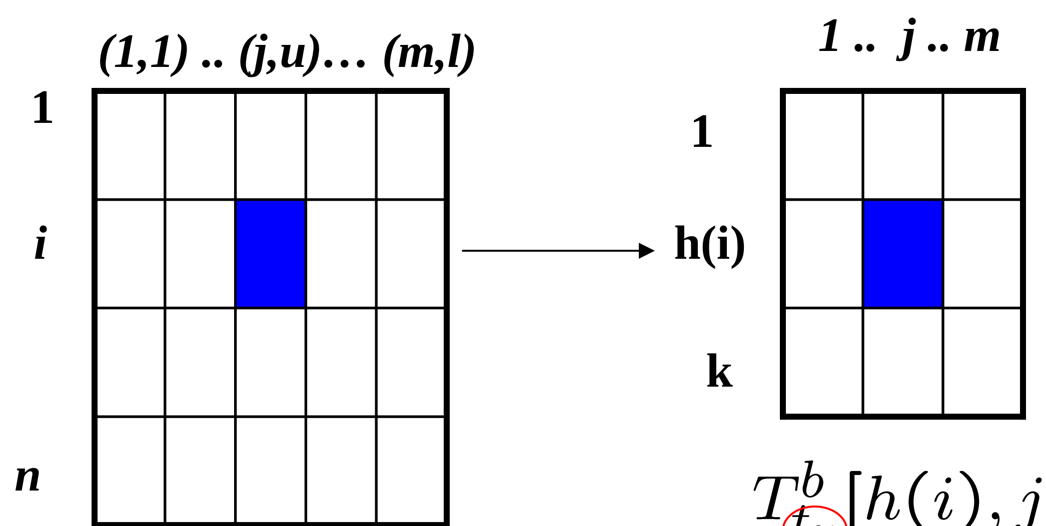
(k,n) scheme

- $m' = ml$, $\alpha' \geq \beta_k \alpha$, $r' = r^l$
- Each $S_t^b, 1 \leq t \leq r^l, b \in \{0,1\}$
  - Indexed by $t = (t_1, \ldots t_u \ldots t_l), 1 \leq t_i \leq r$
  - $S_t^b[i,(j,u)] = T_{t_u}^b[h(i), j]$
  - $1 \leq i \leq n$

    $1 \leq u \leq l$

    $1 \leq j \leq m$

    $1 \leq h(i) \leq k$



$T_{t_u}^b[h(i), j]$

$S_t^b[i,(j,u)]$

Contrast $\geq \beta_k \alpha$

- k rows is S_t^b mapped to $q < k$ different values by h
- Hamming weight of OR of q rows is $f(q)$
- Difference αm white and black pixels occurs when h is one to one and happens at β_k
- WHITE:

$$H(V) \leq l(\beta_k(d - \alpha m) + \sum_{q=1}^{k-1} \beta_q . f(q))$$

- BLACK:

$$H(V) \geq l(\beta_k d + \sum_{q=1}^{k-1} \beta_q . f(q))$$

Security

- You are using *(k,k)* scheme to create *(k,n)* scheme
- Security properties of the *(k,k)* scheme implies the security of *(k,n)* scheme
- Expected Hamming weight of OR of *q* rows, $q<k$ is $l\sum_{q=1}^{k-1}\beta_q f(q)$ irrespective of WHITE or BLACK pixel

Visual Cryptography for General Access Structures [Ateniese *et al* ‘96]

- Goal:
  - Create a scheme such that qualified combinations of participants can reconstruct secret
  - Unqualified combinations of participants gain no information about the secret
- For a *(2,n)* scheme access structure can be represented as Graph
  - Share s_i and s_j reveal secret image if ij is edge in Graph

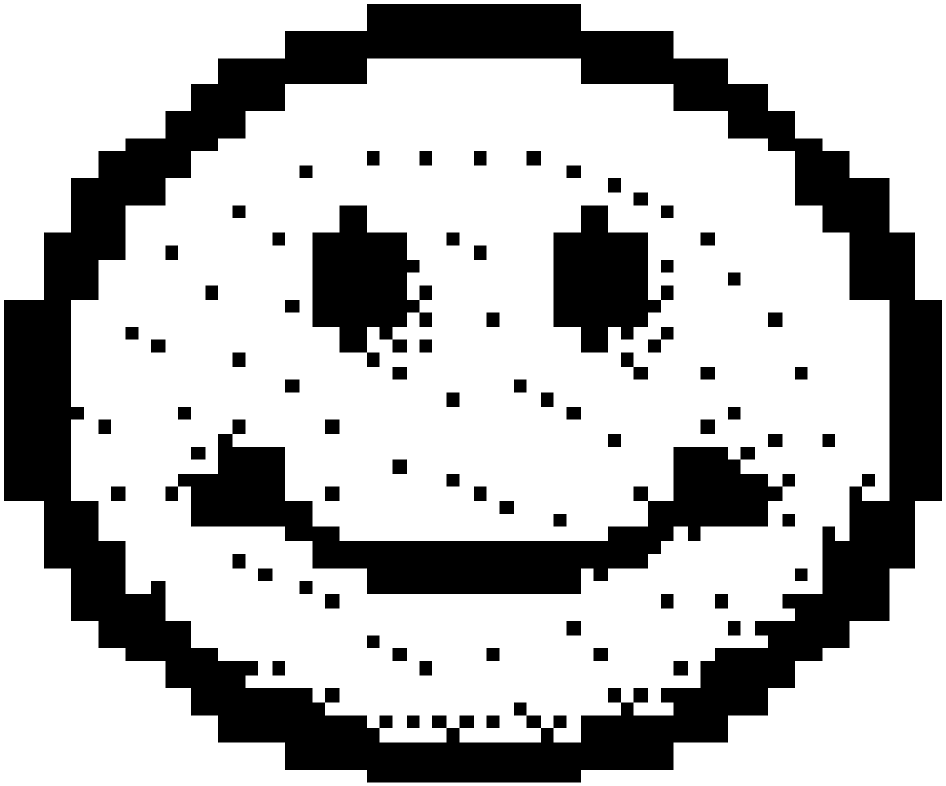
Example (2,4) scheme

- 

- Qualified Subsets *{{1,2},{2,3},{3,4}}*
- Forbidden Subsets *{{1,3},{1,4},{2,4}}*
- Matrices for the scheme
- Some Shares Darker
- $S_0 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Example

- Original Image



- Is superset of qualified subset also qualified?

Problem with various schemes

- The shares in the schemes are random transparencies
- A person carrying around these shares is obviously suspicious
- Need to hide the share in innocent looking images

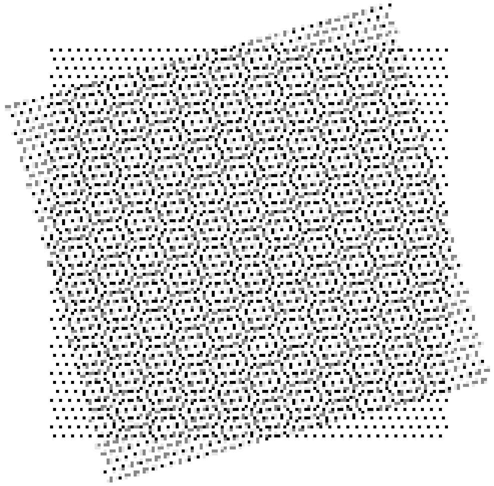
Related works with Natural Images

- M. Nakajima. Y. Yamaguchi.
  - *Extended Visual Cryptography for natural Images* [2002]
- Y. Desmedt and Van. Le.
  - *Moire Cryptography*. [CCS 2000]

Moiré Cryptography

Moiré effect

- Interference of two or more regular structures with different frequencies
- High frequency lattices combined produce a low frequency pattern

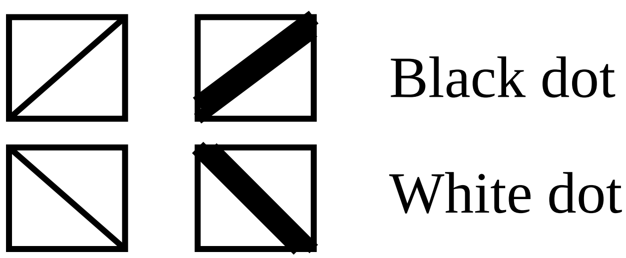


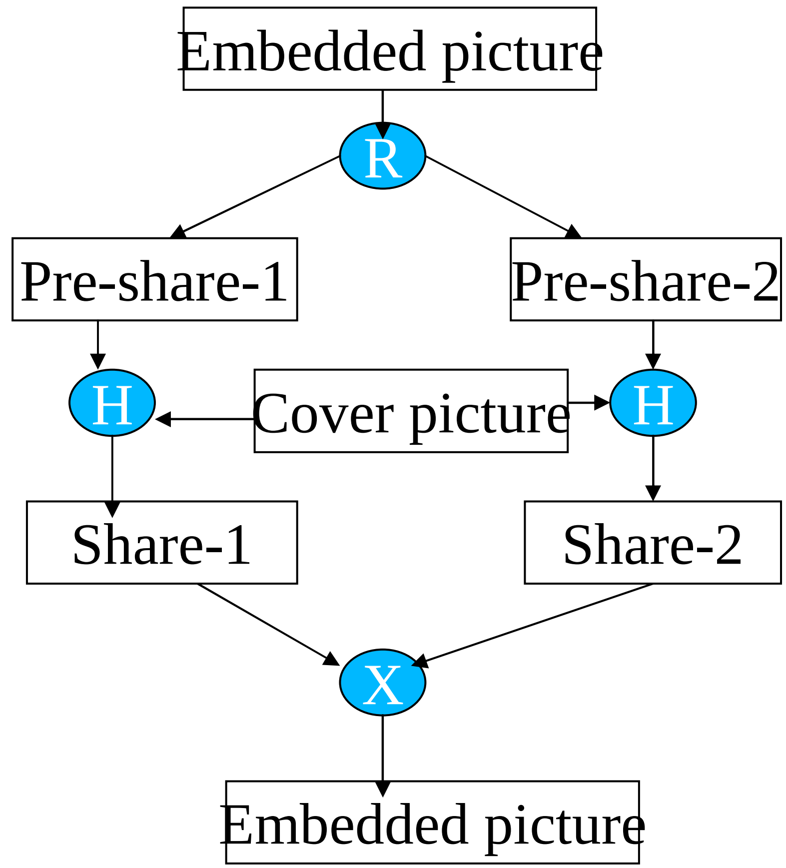
Moiré Cryptography [Demedt, Van Le (2000)]

- Use steganography to create secret sharing schemes
- Shares are realistic images
- Utilize moiré patterns to create the images

Moiré Cryptography process

- Randomize Embedded Picture into pre-shares
- Hide the pre-shares in cover picture





- Note the cryptography lies in X

Moiré Effect …

- For 0 bit
  - Superimposed shares whose dots are oriented at same angle
- For 1 bit
  - Superimposed shares where dots are oriented with different angles

- Moire pattern forms the embedded picture and not gray level of shares as in visual cryptography

- Superimposing shares results
  - Two moire patterns with different textures
  - Since textures are visually different we see picture

Example

- FSU Moiré Example
- Robustness against misplacement or orientation

Comparison and Issues

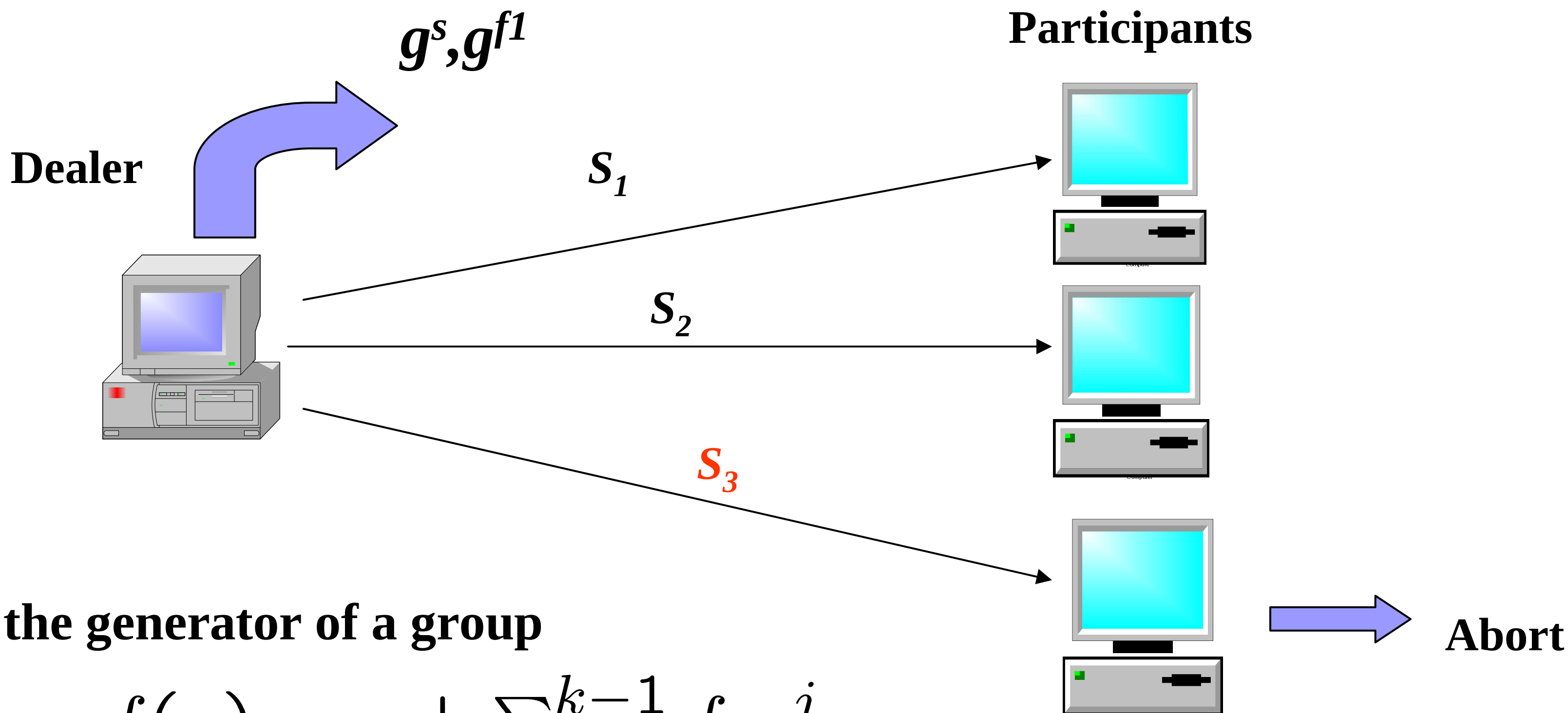
Visual Schemes Seen So Far

- Perfect secrecy ☺
- No expensive computer operations ☺
- Size of shares large ☹
  - If secret contains *p* pixels share contains *pm* pixels
  - Cannot have ideal visual scheme
- Superimposed secret - loss in contrast 😐
- Tedious ☹

Honest Dealer Issue

- Honest dealer assumed
- Verifiable Secret Sharing schemes tolerate a faulty dealer
  - Security is computational

Verifiable Secret Sharing for Shamir's scheme [Feldman87] (2,3) VSS scheme



g is the generator of a group

$$y = f(x) = s + \sum_{j=1}^{k-1} f_j x^j$$

$$s_i = (x_i, y_i)$$

$$g^{y_i} = g^s (g^{f_1})^{x_i}$$

- **Can visual VSS schemes be created?**

Dynamic Groups

- Old share holder leaves
- New share holder joins
- Threshold changes
- Need to refresh the sharing (k,n) to (k',n')
- Is there any way to do that visually without requiring an online dealer ?

Related Works

- Proactive Secret Sharing and public key cryptosystems [Jarecki, 1995]
- Verifiable Secret Redistribution for threshold sharing schemes [Wong *et. al*. 2002]
- Asynchronous verifiable secret sharing and proactive cryptosystems [Cachin *et. al* CCS 2002]

Questions?

Visual Cryptography: Hadamard BIBDs

- Constructions for optimal contrast and minimal pixel expansion [Blundo *et. al.'98*]
- *(v,p,λ)*- Balanced Incomplete Block Design (BIBD)
  - Pair *(X,A)*
  - *X* is set of *v* elements called **points**
  - *A* is collection of subsets of *X* called **blocks**
  - Each block has *p* points
  - Every pair of distinct points is contained in λ blocks

Hadamard Matrices

- n x n matrix H
- Every entry is ± 1 and $HH^T = nI_n$
- Example Hadamard Matrix of order 4

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Hadmard and BIBD equivalence

- *(4t-1,2t-1,t-1)*–BIBD exists if and only if Hadamard matrix of order *4t* exists

- Blundo *et. al*. show
  - if $n \equiv 3 \bmod 4$, there exists a *(2,n)* visual scheme with optimal α and optimal *m* if and only if Hadamard matrix of order *n+1* exists

Construction $(2,n)$ $(n \equiv 3 \bmod 4)$

- Blocks
  - $A_0 = \{i^2 \bmod n : 1 \le i \le (n-1)/2\}$
  - $A_i = A_0 + i \bmod n,\ 1 \le i \le n-1$
- Points Z_n
- Point Block Incidence matrix M
  - Rows indexed by points and columns indexed by Blocks
  - $M[i,j]=1$ if $i \in A_j$
- M is the basis matrix S^1

Construction *(2,11)*

- $m=11, \alpha=3/11$
- Basis matrix S^1
- Basis matrix S^0
  - Each row is **(11111000000)**
- Contrast
  - Black $H(V) = 8$
  - White $H(V) = 5$
- Security
  - *1*x*11* matrix collections are indistinguishable

$$\begin{pmatrix} 0&1&0&1&1&1&0&0&0&1&0 \\ 0&0&1&0&1&1&1&0&0&0&1 \\ 1&0&0&1&0&1&1&1&0&0&0 \\ 0&1&0&0&1&0&1&1&1&0&0 \\ 0&0&1&0&0&1&0&1&1&1&0 \\ 0&0&0&1&0&0&1&0&1&1&1 \\ 1&0&0&0&1&0&0&1&0&1&1 \\ 1&1&0&0&0&1&0&0&1&0&1 \\ 1&1&1&0&0&0&1&0&0&1&0 \\ 0&1&1&1&0&0&0&1&0&0&1 \\ 1&0&1&1&1&0&0&0&1&0&0 \end{pmatrix}$$

S^1

$m = 2^k, \alpha = 1/2^k$ (k,k) scheme

- Two lists of vectors each of length k over GF[2]
- $J_1^0 \ldots J_k^0$
  - k -1 linearly independent, k are not independent
  - $J_i^0 = 0^{i-1}10^{k-i}, 1 \leq i \leq k, J_k^0 = 1^{k-1}0$
- $J_1^1 \ldots J_k^1$
  - Linearly independent
- $S^t[i,x] =< J_i^t, x >, t \in \{0,1\}$
  - Indexing the columns of *S* with a vector *x* of length *k* over GF[2]

Example m=8, α =1/8, (3,3) scheme

- $J_1^0 = [1\ 0\ 0], J_2^0 = [0\ 1\ 0], J_3^0 = [1\ 1\ 0]$
- $J_1^1 = [1\ 0\ 0], J_2^1 = [0\ 1\ 0], J_3^1 = [0\ 0\ 1]$
- $x = [0\ 0\ 0], \ldots [1\ 1\ 1]$
- S^0

$$\begin{pmatrix} 0 & 0 & 0 & \boxed{0} & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad [1\ 0\ 0] \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$