$L_1 = \{a^i b^j a^k b^l \mid i,j,k,l \geq 1\}$

The matching indicates the proper matchings:

- $a, e, VA$
- $b, e, VB$
- $a, B$
- $b, \lambda, VB$
- $a, B, a$
- $b, A, VB$
- $a, B, a$
- $b, \lambda, VB$
- $a, B, a$
- $b, \lambda, VB$

$L_{10} = \{x \in R \mid a b a \text{ is a substring of } x\}$

a, z^k relationship via the pushdown, substring via the finite control.

$L_8 = \{x \mid \#a = \#_e, \text{ last symbol of } x = b \}$

- $\#a = \#_e$ by pushdown, last symbol by the finite control.

Let $A_a = \{a, A, VA \}$ and $A_b = \{b, A, VB \}$

$\quad$ $\quad$ $\quad$ $\quad$ $\quad$ $\quad$

$L_2 = \{a^i b^j a^k b^l \mid i, j, k, l \geq 1, \text{i.e. } i = k \text{ or } j = l\}$

Guess $i = k$ and $j = l$ and verify.
$L_3 = \{ x \mid x = x^k \}$

$x = x^k \iff \begin{cases} (\exists y \in \Sigma^*, e \in \Sigma) \left( x = y y^R \wedge x = y cy^R \right) \\ \end{cases}$

On the $y$ part push, on $y^R$ $\uparrow$ match $\times$ pop.

Push $x$, guess $\times$ through away $x$ part, then match. After the pd becomes empty throw away the $y$ part.

$F_{13} = \{ x cy \mid x^k$ is a substring of $y^k \}$

$y = u z^k v$

For any $k$, let $z = a^k b^k c^k$. Note that $z \in L \delta$

For any way of writing $d^k e^{k+1} f^k g^k = uu w w x y z^t$. If $w = k \wedge \varepsilon = 0$.

Since $|w w z| \leq k$, wux and hence $ux$ cannot contain all (b's.

If $ux$ does not contain $c$'s then $u v w y^k \notin L$ since $i < j < k$ cannot be maintained.

If $ux$ does contain $c$'s then $u v w y^k \notin L$ since $i < j < k$ cannot be maintained.

Hence $L$ is not an npda lang.
(m) \( L = \{ x \mid \# x = \# z^2 \geq \# x^2 \} \)

Let \( S = \{ s \mid \text{all} b, c \} \). As before, \( V \) cannot contain all \( b, c \).

If \( V \) does not contain \( c \)'s, then \( 2u \not\in V \) \# \( y \# L \) since \( d \)'s and/or \( b \)'s will be reduced \( \# y \leq \# z \), \( \# y \geq \# z \) cannot hold.

If \( V \) does not contain \( a \)'s, then \( 2u \not\in V \) \# \( y \# L \) since \( \# y = \# x^2 \) and/or \( \# y \geq \# z \) cannot hold.

Hence \( L \) is not an np-hard lang.

II (a) \( \{ x \mid \# x = \# b = \# c \} \).

Repeatedly cross off one \( a \), one \( b \), one \( c \).

(c) \( \{ a_i b^j c^k \mid i, j \geq 1, i = j \} \) and \( (k = i \vee k = 2i) \)

Check from \( L \) \( a_i b^j c^k \), check \( i = j \); then \( \# = \# b \) or \( \# = \# + \# c \).

See next page.