600.271 Automata & Computation Theory
Second Mid-Semester Examination
In-class, Closed Book
April 18, 2013
Time: 1 hr, 10 minutes

I. (10 pts.) Design a context-free grammar for the language
$\{a^n b^m a^k \mid n, m, k \geq 1,\ n \geq m,\ n \text{ and } k \text{ are even, and } m \text{ is odd}\}$.

$S \to Saa \mid Xaa$ $\quad$ k even

$X \to YZ$ $\quad$ Z: $a^m b^m$, m odd $\quad$ Y: a^{n-m}, $n-m$ odd

$Y \to Yaa \mid a$

$Z \to aaZbb \mid ab$

II. (10 pts.) Specify a decision procedure for one of the following problems:

1. The following variant of the Post Correspondence Problem: Given $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ and k, do there exist $i_1, i_2, \cdots, i_m$ s.t. $x_{i_1} x_{i_2} \cdots x_{i_m} = y_{i_1} y_{i_2} \cdots y_{i_m}$ and $|x_{i_1} x_{i_2} \cdots x_{i_m}| \leq k^2$?

2. Given a dfa M_1 and a dpda M_2 and a value k, do there exist strings x and y satisfying $|x| \leq k$, $x \in L(M_2)$, and $xy \in L(M_1)$.

1. Since each $|x_i|$ and $|y_i|$ is ≥ 1, m must be $\leq k^2$. Generate all sequences over $\{1, 2, \cdots, n\}$ of length $\leq k^2$ one after the other. For each such sequence $i_1, \cdots, i_m$ check whether $x_{i_1} x_{i_2} \cdots x_{i_m} = y_{i_1} \cdots y_{i_m}$ and $|x_{i_1} x_{i_2} \cdots x_{i_m}| \leq k^2$. If both the conditions are satisfied then output 'yes'. If all the sequences are exhausted, output 'no'. Then halt.

2. From the dfa M_1 construct a dfa M_1' s.t. $L(M_1') = \{x \mid (\exists y)(xy \in L(M_1))\}$:

In M_1 compute all the states from each of which there is a path to a final state. Make all such states as final states. The resulting dfa is M_1'.

Generate all strings of length $\leq k$ one after the other. For each such string x check whether x is accepted by M_2 and by M_1'. If some x is accepted by both then output "yes". If all the strings are exhausted, output 'no'. Then halt.

III. (10 pts.) Recursively enumerate the following set.
$\{[M] \mid$ Turing machine M halts on at least one of the inputs $[M]$ and $[M][M]\}$. ($[M][M]$ is $[M]$ concatenated with $[M]$.)

create an empty stack. $i = 1$.
At i^{th} iteration, generate $[M_i]$ and place ~~the~~ two simulations ~~of~~
~~TM M_i on input $[M_i][M_i]$ on the stack. Then perform~~
on the stack: TM M_i on input $[M_i]$ and TM M_i on input $[M_i][M_i]$. then perform one additional step of simulation for each process in the stack. If any $[M_j]$ halts (on input $[M_j]$ or $[M_j][M_j]$) output $[M_j]$ and remove that process from the stack.
This complete i^{th} iteration.
Then go ~~to~~ ~~i^{th} iteration~~ to the next iteration.

IV. (10 pts.) Prove the undecidability of one of the following problems. (Hint: Make use of the undecidability of the Post Correspondence Problem.)

1. Given two dpdas, is there a string accepted by both the dpdas?
2. Given a dlba M over the alphabet Σ, are both $L(M)$ and $\Sigma^* - L(M)$ infinite sets?

1. will show PCP $\leq_m$ two dpda problem

Typical instance $E = (x_1, y_1), \cdots (x_n, y_n)$ $[M_1], [M_2]$.

Given E transform it to $[M_1], [M_2]$ s.t. E has a solution iff $L(M_1) \cap L(M_2) \neq \phi$.

Construct dpda M_1 st. $L(M_1) = \{ x_{i_1} \cdots x_{i_m} c_{i_m} \cdots c_{i_1} \mid m \geq 1 \}$

& dpda M_2 st. $L(M_2) = \{ y_{i_1} \cdots y_{i_m} c_{i_m} \cdots c_{i_1} \mid m \geq 1 \}$

We have established several times that

$(\exists i_1, \cdots, i_m)(x_{i_1} \cdots x_{i_m} = y_{i_1} \cdots y_{i_m}) \iff L(M_1) \cap L(M_2) \neq \phi$.

Also, the transformation is computable.

2. will show PCP $\leq_m$ dlba problem

Typical inst: $E = (x_1, y_1), \cdots (x_n, y_n)$ $[M]$

Given E transform it to $[M]$ st. E has a solution iff $L(M)$ and $\Sigma^* - L(M)$ are both infinite sets.

M is not a dlba

M: wlog assume $\Sigma = \{a, b\}$.

M rejects every string that starts with symbol 'b'.

On any other string, in a separate part of the tape, M ~~checks~~ generates all possible strings over $\{1, 2, \cdots n\}$. On a typical string $i_1, \cdots, i_m$, M checks whether $x_{i_1} \cdots x_{i_m} = y_{i_1} \cdots y_{i_m}$. If one such equality holds then M accepts the input string.

Note that E has a solution $\Rightarrow L(M) = \{a, b\}^* - \{b\}\{a, b\}^*$

$\Rightarrow L(M)$ & $\{a, b\}^* - L(M)$ are infinite sets

E has no solution $\Rightarrow L(M) = \phi$

$\Rightarrow L(M)$ is finite.

Also, the transformation is computable.

2. will show PCP $\leq_m$ dlba problem.

Typical inst: $E = (x_1, y_1), \dots (x_n, y_n)$ $[M]$

Given E transform it to $[M]$ s.t. E has a solution iff $L(M)$ and $\Sigma^* - L(M)$ are both infinite sets.

M. wlog assume $\Sigma = \{a, b\}$.
M rejects every string that starts with symbol 'b', or ϵ string.
On any string that starts with symbol 'a', M starts checking whether $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n, x_1 x_1 = y_1 y_1, \dots$ one after the other.
When M runs out of the input space, M rejects the input.
If it detects that $x_{i_1} \dots x_{i_m} = y_{i_1} \dots y_{i_m}$, then M accepts the input.

Note that if E has a solution, then there exists a value k s.t. M accepts all inputs that start with 'a' and are of length $\geq k$. ~~In that case, ...~~ Since M rejects any string that starts with symbol 'b', M rejects an infinite set of strings.
Thus if E has a solution, $L(M)$ & $\{a,b\}^* - L(M)$ are both infinite sets.

If E has no solution, then $L(M) = \phi \Rightarrow L(M)$ is a finite set.

Also the transformation is computable.