I. (10 pts.) Design a context-free grammar for the language
\( \{a^n b^m a^k \mid n, m, k \geq 1, n \geq m, n \text{ and } k \text{ are even, and } m \text{ is odd}\}.

\[
S \rightarrow Saa \mid Xaa \quad \text{k even}
\]
\[
X \rightarrow YZ
\]
\[
Y \rightarrow Yaa \mid a
\]
\[
Z \rightarrow aazb \mid ab
\]
II. (10 pts.) Specify a decision procedure for one of the following problems:

1. The following variant of the Post Correspondence Problem:
   Given \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) and \(k\), do there exist \(i_1, i_2, \ldots, i_m\) s.t. \(x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_m\) and \(|x_1 x_2 \cdots x_m| \leq k^2\)?

2. Given a dfa \(M_1\) and a dpda \(M_2\) and a value \(k\), do there exist strings \(x\) and \(y\) satisfying \(|x| \leq k\), \(x \in L(M_2)\), and \(xy \in L(M_1)\).

1. Since each \(|x_i| \text{ and } |y_i| \geq 1\), \(m \) must be \( \leq k^2\). One after the other, generate all sequences over \(\{1, 2, \ldots, n\}\) of length \( \leq k^2\). For each such sequence \(i_1, i_2, \ldots, i_m\), check whether \(x_1 x_2 \cdots x_m = y_1 y_2 \cdots y_m\) and \(|x_1 x_2 \cdots x_m| \leq k^2\). If both of the conditions are satisfied, then output "yes". If all the sequences are exhausted, output "no". Then halt.

2. From the dfa \(M\), construct a dfa \(M'\) s.t. \(L(M') = \{x \mid \exists y (xy \in L(M))\}\).

In \(M\), compute all the states from each of which there is a path to a final state. Make all states as final states. The resulting dfa is \(M'\).

Generate all strings of length \( \leq k\) one after the other. For each such string \(z\), check whether \(z\) is accepted by \(M_2\) and \(M'\). If some \(z\) is accepted by both, then output "yes". If all the strings are exhausted, output "no". Then halt.
III. (10 pts.) Recursively enumerate the following set.
\( [\{M\}] \) Turing machine \( M \) halts on at least one of the inputs \( [M] \) and \( [M][M] \). (\( [M][M] \) is \( [M] \) concatenated with \( [M] \)).

Create an empty stack \( i = 1 \).

At \( i \)th iteration, generate \( [M_i] \) and place the simulation \( \overline{TM} \).

\( TM \ M_i \) on input \( [M_i][M_i] \) on the stack. Then perform on the stack: \( TM \ M_i \) on input \( [M_i] \) and \( TM \ M_i \) on input \( [M_i][M_i] \). Then perform one additional step of simulation for each process in the stack. If any \( [M_j] \) halts (on input \( [M_j] \) or \( [M_j][M_j] \)) output \( [M_j] \) and remove that process from the stack.

This completes \( i \)th iteration.

Then go to \( i+1 \)th iteration to the next iteration.
IV. (10 pts.) Prove the undecidability of one of the following problems. (Hint: Make use of the undecidability of the Post Correspondence Problem.)

1. Given two dpdas, is there a string accepted by both the dpdas?

2. Given a dlla $M$ over the alphabet $\Sigma$, are both $L(M)$ and $\Sigma^* - L(M)$ infinite sets?

1. will show \( PCP \leq_m \) two dpda problem

   Typical instance: \( E = (E_1, E_2) \), \( (M_1, M_2) \).

   Given $E$ transform it to \( (M_1, M_2) \) s.t. $E$ has a solution.

   If $L(M_1) \cap L(M_2) \neq \emptyset$.

   Construct dpda $M_1$ s.t. $L(M_1) = \{ x_1 \ldots x_n c_1 \ldots c_i \mid m_1 \}$

   Construct dpda $M_2$ s.t. $L(M_2) = \{ y_1 \ldots y_n c_1 \ldots c_i \mid m_2 \}$

   We have several times that

   \( \exists \vec{x}, \ldots, \vec{y} \) \( (x_i \neq y_i) \Rightarrow L(M_1) \cap L(M_2) = \emptyset \).

2. will show \( PCP \leq_m \) dlla problem

   Typical instance: \( E = (E_1, E_2) \), \( (M_1, M_2) \).

   Given $E$ transform it to \( (M_1, M_2) \) s.t. $E$ has a solution.

   $L(M_1)$ and $\Sigma^* - L(M_1)$ are both infinite sets.

M: who assume $\Sigma = \{a, b\}$

M rejects every string that starts with symbol 'b'.

On any other string, in a separate part of the tape, M generates all possible strings over \( \{1, 2, \ldots, n\} \). On a typical string $x_1 \ldots x_n$, M checks whether

\[ x_i = y_i \ldots y_n \]

If one such equality holds, then M accepts the input string.

Note that $E$ has a solution \( \Rightarrow L(M) = \{a, b\}^* - \{b\} \Rightarrow L(M) \neq \{a, b\}^* - L(M) \) are infinite.

If $E$ has no solution \( \Rightarrow L(M) = \emptyset \Rightarrow L(M)$ is finite.

Also, the transformation is computable.
2. will show $PCP \leq m$ diffuse problem.

Typical $M = (Q, \Sigma, \Gamma, \delta, q_0, \delta_1, \delta_2)$

Given $E$ transform it to $\langle M \rangle$ s.t. $E$ has a solution if $L(M)$ and $\Sigma^* \subseteq L(M)$ are both infinite sets.

$M$, why assume $\Sigma = \{a, b\}$

$M$ rejects every string that starts with symbol $a$ on $e$ string.

On any string that starts with symbol $a$, $M$ starts checking whether $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n, y_{n+1} = y_{n+1}, \ldots$ one after the other.

When $M$ runs out of the input space, $M$ rejects the input.

If it detects that $x_i \neq y_i$, $y_i$, then $M$ accepts the input.

Note that if $E$ has a solution, then there exists a value $k$ s.t. $M$ accepts all inputs that start with $a$, and are of length $\geq k$. Since $M$ rejects any string that starts with symbol $b$, $M$ rejects an infinite set of strings.

Thus if $E$ has a solution, $L(M) = \emptyset \implies L(M) \in \#SAT$.

Thus if $E$ has no solution, then $L(M) = \emptyset \implies L(M) \in \#SAT$.

Also the transformation is computable.