600.271 Automata & Computation Theory *
Second Mid-Semester Examination
April 12, 2011
In-class, Closed Book
Time: 1 hr, 10 minutes

I. (10 pts.) Design a context-free grammar for the language
$\{xx^R yy^R \mid x, y \in \{a,b\}^+, abb \text{ is a substring of } x, \text{ and } |y| \text{ is odd}\}$

$S \to S_1 S_2$ $\quad$ S_1 for xx^R $\quad$ S_2 for yy^R

~~$S_2 \to S_2' a \mid S_2' b$~~

$S_2 \to a S_2' a \mid b S_2' b \mid aa \mid bb$

$S_2' \to a S_2 a \mid b S_2 b$

$S_1 \to a S_1 a \mid b S_1 b \mid abbXbba \mid \cancel{aXa}$

$X \to aXa \mid bXb \mid \epsilon$

* I was not planning to write solutions to these problems. But the TA & a couple of students in the class to post solutions to these problems.
4/15/2013

II. (10 pts.) Establish the decidability of one of the following problems by designing an appropriate algorithm.

- Given a CFG G, is there an $x \in L(G)$ such that $|x|$ is even?
- Given $[M_1]$ and $[M_2]$, M_1 and M_2 being dfa language recognizers, is $L(M_1) \cap L(M_2)$ an infinite set?

1) For simplicity of description, we assume that the CFG is in Chomsky normal form. But the proof can be extended to any form. For each nonterminal A, we create A_{ev} & A_{od}, and we create a set Δ ~~s.t.~~ s.t.

$A_{ev} \in \Delta$ iff $A \overset{*}{\Rightarrow} x$ & $|x|$ is even, and
$A_{od} \in \Delta$ iff $A \overset{*}{\Rightarrow} x$ & $|x|$ is odd.

Initially Δ is empty.
For each $A \to a$, add A_{od} to Δ.
For each $A \to \epsilon$, add A_{ev} to Δ.
Repeat until the size of Δ doesn't increase
For each $A \to BC$, and
B_{ev}, C_{ev} in $\Delta \Rightarrow$ add A_{ev} to Δ
B_{ev}, C_{od} in $\Delta \Rightarrow$ add A_{od} to Δ
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B_{od}, C_{od} in $\Delta \Rightarrow$ add A_{ev} to Δ.

Since the max size of Δ is $2N$ if N is the number of nonterminals, the process stops in a finite number of steps. Finally check whether $S_{ev} \in \Delta$, if so output 'yes' else output 'no'.

Claim:
2) If M_1 and M_2 have s_1 and s_2 states, then $L(M_1) \cap L(M_2)$ is infinite iff M_1 & M_2 accepts a string x s.t. $s \le |x| \le 2s$.
Then check ~~every~~ whether a string of length in the interval $[s, 2s]$ is accepted by both M_1 & M_2.
~~Pf outline of claim: If no Since $L(M_1) \cap L(M_2)$~~
An additional sheet is attached.

Pf: If M_1 & M_2 accept a string of length in $[s, 2s]$, let one such string be $a_1 a_2 \ldots a_l$ s.t. $s \le l \le 2s$

let the pairs of states reached by M_1 & M_2 as this string is accepted be $(p_0, q_0), (p_1, q_1), \ldots (p_{l+1}, q_{l+1})$. ($p$'s of M_1 & q's for M_2)

Since there are only $s_1 s_2$ distinct ~~st~~ pairs, 2 of the pairs are equal. let they be (p_i, q_i) & (p_j, q_j) $i<j$

Then the string $a_1 \ldots a_i (a_{i+1} \ldots a_j)^k a_{j+1} \ldots a_l$, $k \ge 0$, are accepted by both M_1 & M_2. This is a set of infinite strings.

If $L(M_1)$ ~~∩~~ & $L(M_2)$ is an infinite set, it must contain a string of length $\ge s$. let a shortest such string be $a_1 \ldots a_l$. If $l \le 2s$, we are done.

If $l > 2s$, track the pairs as above and cut off a substring of length $\le s_1 s_2$ resulting in a shorter string of length $\ge s$ that is accepted. This violates the minimality assumption.

III. (10 pts.) Design a Turing machine for computing the following function.

$$f(x,y,z) = \begin{cases} x + (y - z) & \text{if } y \geq z \\ x & \text{otherwise} \end{cases}$$

I am skipping this simple problem which almost every student got it correct.

IV. (10 pts.) Solve one of the following problems.

- Prove the undecidablity of the following problem. Given CFGs G_1 and G_2, is there an $x \in L(G_1) \cap (L(G_2))^R$, and $|x|$ is even? (Hint: Reduce the Post Correspondence Problem to this problem.
- Recall that the Uniform Halting Problem (UHP) asks whether a given Turing machine halts on every input. Reduce the UHP to the problem of testing whether a given TM computes the identity function; i.e. the function f s.t. $f(x) = x$ for every x.

i) PCP $\leq_m$ this problem

Typical ind. $E = (x_1, y_1), \ldots (x_n, y_n)$ CFGs G_1 and G_2

Given E we want to transform it (G_1, G_2) s.t. E has a solution iff $\exists x \in L(G_1) \cap (L(G_2))^R$ & $|x|$ is even.

G_1: $S \to x_1 S c_1 \mid x_2 S c_2 \mid \ldots \mid x_n S c_n \mid x_1 c_1 \mid x_2 c_2 \mid \ldots \mid x_n c_n$

G_2: $S \to y_1^R S c_1 \mid y_2^R S c_2 \mid \ldots \mid y_n^R S c_n \mid y_1^R c_1 \mid y_2^R c_2 \mid \ldots \mid y_n^R c_n$.

It is easily seen that $x_{i_1} \cdots x_{i_m} = y_{i_1} \cdots y_{i_m}$ iff $x_{i_1} \cdots x_{i_m} x_{i_1} \cdots x_{i_m} c_{i_m} \cdots c_{i_2} c_{i_1} c_{i_m} \cdots c_{i_2} c_{i_1} \in L(G_1) \cap (L(G_2))^R$.

Note also that $|(x_{i_1} \cdots x_{i_m} x_{i_1} \cdots x_{i_m} c_{i_m} \cdots c_{i_2} c_{i_1})^2|$ is of even length.

Note that the transformation from E to (G_1, G_2) is computable.

ii) UHP $\leq_m$ this prob

Typical inst $[M]$ $[M']$

Transform $[M]$ to $[M']$ s.t. TM M halts on every input iff TM M' compute $f(x) = x$ for every x.

M': On any input x, M' ~~stop~~ simulate M on input x on a separate part of the tape. If M halts, M' erases the computation & leaves x on tape.

The rest is easy.