All the subproblems carry equal weight. There are 5 subproblems in this examination.

I. Design the specified automaton for every one of the following languages.

1. An NFA for the language $L_1 = \{xy | x, y \in \{a, b\}^*, \text{aba is a substring of both } x \text{ and } y, \text{ and the last symbol of } x = \text{last symbol of } y = b\}.$

2. An NPDA for the language $L_2 = \{a^{i+j}b^ka^\ell | i, j, k, \ell \geq 1, \text{ and } (k = i \text{ or } \ell = 2i \text{ or } k = \ell\}.$
3. A dLBA for the language $L_3 = \{xcyxycy | x, y \in \{a, b\}^*, |x| = |y|\}$.

Match one symbol of $x$'s then one symbol of $y$'s. Repeat.

4. A CFG for the language $L_2$, which is specified:

$L_2 = \{a^i b^j a^k b^l | i, j, k, l \geq 1, \text{ and } (k = i \text{ or } l = 2i \text{ or } k = l)\}$.

$$
S \rightarrow X | Y | Z

\begin{align*}
X & \rightarrow Xb | Ub \\
U & \rightarrow aUa | aVa \\
V & \rightarrow aVb | ab
\end{align*}

\begin{align*}
Y & \rightarrow aYbb | aWbb \\
W & \rightarrow Wa | Pa \\
P & \rightarrow aPb | ab
\end{align*}

\begin{align*}
Z & \rightarrow QR \\
R & \rightarrow aRb | ab \\
Q & \rightarrow aQ | \epsilon \text{ or } AM \\
M & \rightarrow aMb | ab
\end{align*}
II. Prove that the following language is not an FA language by applying the pumping lemma.

$L_5 = \{a^ib^j | i, j \geq 1, (i \geq j) \text{ or } (i < j \text{ and } j \text{ is a multiple of } i)\}$.

Let $Z = a^{3k}b^{3k}$. Note that $|Z| = 6k > k$, and $Z \in L_5$ (i.e., condition is satisfied).

For any way of writing $a^{3k}b^{3k} = uvwxy$ with $|vy| > 0$:

Since $|uv| \leq k$, $uv$ & hence $v$ contains only $a$'s.

Let $v = a^l$, $1 \leq l \leq k$.

Then $uv^0w = a^{3k-l}b^{3k} \not\in L_5$ since $3k-l$ is not a multiple of $3k$.

Hence $L_5$ is not an FA language.