Recursively generated B-spline surfaces on arbitrary topological meshes

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This paper describes a method for recursively generating surfaces that approximate points lying-on a mesh of arbitrary topology. The method is presented as a generalization of a recursive bicubic B-spline patch subdivision algorithm. For rectangular control-point meshes, the method generates a standard B-spline surface. For non-rectangular meshes, it generates surfaces that are shown to reduce to a standard B-spline surface except at a small number of points, called extraordinary points. Therefore, everywhere except at these points the surface is continuous in tangent and curvature. At the extraordinary points, the pictures of the surface indicate that the surface is at least continuous in tangent, but no proof of continuity is given. A similar algorithm for biquadratic B-splines is also presented.

Recursive patch subdivision algorithms have been used extensively in computer graphics since Catmull first devised them for rendering shaded pictures of curved surface patches[1]. The algorithm he devised recursively subdivides a surface patch into four subpatches until the resulting patch is roughly the size of a picture element (pixel) of the raster display on which it is to be rendered. At this point, the tests of its visibility and the respresentation of its shading properties are greatly simplified.

When Catmull's work was near completion, George Chaikin described in a seminar a method for generating smooth curves by recursively cutting the corners from a control polygon[2]. Motivated by this, Catmull invented a method for generating cubic surfaces for polyhedral nets of arbitrary topology. However, since he could not prove that the surface was well-behaved at all points on the surface, he did not implement it. Recently, Clark implemented the method to empirically determine if the surface is well behaved and generalized the rule for determining the new surface points. Presented in this paper is a set of subdivision rules that have been refined to the point where the pictures suggest that the generated surface is continuous in tangent and curvature. Doo and Sabin have analysed the behaviour of the surface in the neighbourhood of the extraordinary points[5], and the pictures presented here incorporate some tests of their predictions.

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The algorithm described herein is very useful for the purposes of making smooth pictures of three dimensional objects. The task of defining smooth approximations to objects is much simpler if the points in terms of which the object is defined do not have to lie on a topologically rectangular grid.

The basis of the method results from considering a standard bicubic B-spline patch on a rectangular control-point mesh. The shape of such a patch is governed by 16 control-points, as shown in Figure 1. The original points are circled. In subdividing this patch into 4 subpatches, 25 *subcontrol* points are generated. These are indicated in the figure by Xs. Note that some of the Xs lie in the middles of the squares of the original mesh; these are called new face points. Likewise, some of the new points lie on the edges connecting original control points; these are called new edge points. The points corresponding to the old control points are called new vertex points. In splitting the original patch, it is found that each new control point of a given type is computed from its neighbouring points by the same form of algebraic expression. For example, new face points are computed as the average of the four old vertices that define the face.

This paper describes a method for generalizing these subdivision rules to arbitrary control-points meshes. The method applies the same expressions that are generated in the rectangular case to faces, edges and points of

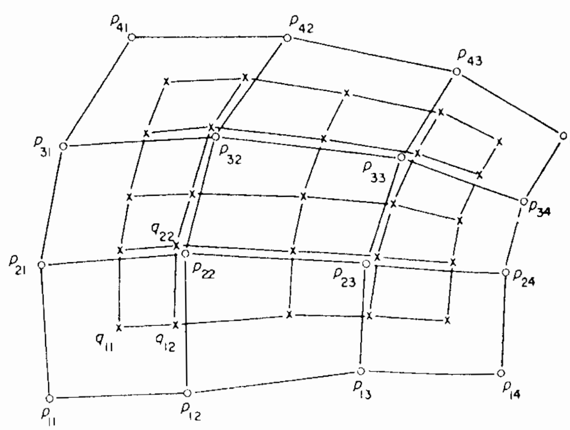

Figure 1. Standard bicubic B-spline patch on a rectangular control-point mesh

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arbitrary meshes. That is, new face points are computed as the average of the old points defining the face, etc., and the new vertex points depend upon the number of edges incident on a vertex in such a way that the correct expressions result when this number is 4, as in the rectangular case.

RECTANGULAR B-SPLINE PATCH SPLITTING

The bicubic B-spline patch can be expressed in matrix form by

$$S(u,v) = U\,M\,G\,M^t\,V^t \qquad (1)$$

where

$$M = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

is the B-spline basis matrix for cubics, and

$$G = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$$

is the set of control points, which are arranged on a topologically rectangular mesh according to their subscripts, and

$$U = [\,u^3 u^2 u\ 1\,] \text{ and } V = [v^3 v^2 v\ 1\,]$$

are the primitive basis vectors.

We will consider just the subpatch of this patch corresponding to $0 < u,v < \frac{1}{2}$. The other subpatches need not be considered due to the symmetry of the B-spline basis. This is the subpatch $S(u_1, v_1)$, where $u_1 = u/2$ and $v_1 = v/2$. Substituting these two expressions into (1)

$$S(u_1, v_1) = U\,S\,M\,G\,M^t S^t\,V^t \qquad (2)$$

is obtained, where

$$S = \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U = [u^3\ u^2\ u\ 1\,]$$

and

$$V = [v^3\ v^2\ v\ 1\,]$$

This patch must still be a bicubic B-spline with its own control-point mesh G_1, satisfying

$$S(u, v) = U\,M\,G_1\,M^t\,V^t$$

Requiring that this expression be equal to (2), this will be true for arbitrary values of u and v if and only if

$$M\,G_1\,M^t = S\,M\,G\,M^t\,S^t$$

Assuming that the basis matrix M is invertible, which is the case, it is found

$$G = [\,M^{-1}\,S\,M\,]\,G\,[\,M^t\,S\,M^{-t}\,]$$

$$= H_1\,G\,H_1^t$$

where

$$H_1 = M^{-1}\,S\,M$$

is called the splitting matrix. Carrying out the matrix multiplications, it is found

$$H_1 = \begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}$$

Hence the control point mesh corresponding to the subpatch in question is related to the old control point mesh by the expression

$$G_1 = H_1\,G\,H_1^t \qquad (3)$$

Referring now to Figure 1, the new face point labelled q is the (1,1) element of G. Carrying out the algebra of (3) gives

$$q_{11} = \frac{(p_{11} + p_{12} + p_{21} + p_{22})}{4} \qquad (4)$$

Likewise, the point q_{12}, a new edge point, is given by

$$q_{12} = \frac{\frac{(C + D)}{2} + \frac{(p_{12} + p_{22})}{2}}{2} \qquad (5)$$

where

$$q_{11} = C = \frac{(p_{11} + p_{12} + p_{21} + p_{22})}{4}$$

and

$$q_{13} = D = \frac{(p_{12} + p_{13} + p_{22} + p_{23})}{4}$$

The new vertex point, q_{22}, is given by

$$q_{22} = \frac{Q}{4} + \frac{R}{2} + \frac{p_{22}}{4} \qquad (6)$$

where

$$Q = \frac{(q_{11} + q_{13} + q_{31} + q_{33})}{4}$$

and

$$R = \frac{1}{4}\left[\frac{(p_{22} + p_{12})}{2} + \frac{(p_{22} + p_{21})}{2} + \frac{(p_{22} + p_{32})}{2} + \frac{(p_{22} + p_{23})}{2}\right]$$

It is easily verified that each of the elements of G satisfies an expression similar to one of (4, 5, 6). Since these expressions were deduced from the standard B-spline basis, they generate a bicubic B-spline surface.

ARBITRARY TOPOLOGY

For the purposes of generalizing the expressions (4, 5, 6) to arbitrary topologies, it is convenient to express them as a set of rules which are dependent on the number of points around a face and on the number of edges incident to a vertex. Of course the rules must yield the expressions (4, 5, 6) when that number is four. The rules are:

(A) New face points – the average of all of the old points defining the face.

(B) New edge points – the average of the midpoints of the old edge with the average of the two new face points of the faces sharing the edge.

(C) New vertex points – the average

$$\frac{Q}{n} + \frac{2R}{n} + \frac{S(n-3)}{n}$$

where

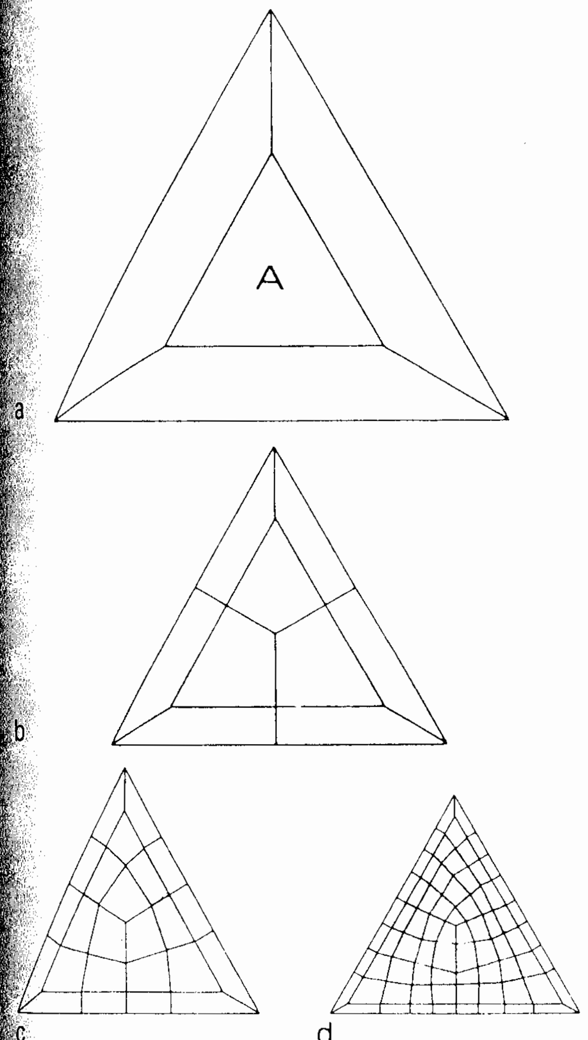


Figure 2. Result of applying rules to a simple nonrectangular topology

Q = the average of the new face points of all faces adjacent to the old vertex point.
R = the average of the midpoints of all old edges incident on the old vertex point.
S = old vertex point.

After these points have been computed, new edges are formed by

- connecting each new face point to the new edge points of the edges defining the old face
- connecting each new vertex point to the new edge points of all old edges incident on the old vertex point

New faces are then defined as those enclosed by new edges.

The results of applying these rules to a simple nonrectangular topology are shown in Figures 2(a,b,c,d). Figure 2a shows the original triangular region labelled A that will be approximated by a triangular surface *patch*. The other three regions around the perimeter of region A assist in defining the slope and curvature of the patch at its boundaries, as in a rectangular topology.

Figure 2b shows the result of applying the rules one time. Note that all new faces have four sides. However, now four vertex points have only 3 edges incident upon them. These are the three new vertex points corresponding to the original old vertices of the region A plus the new face point for the region. Following a suggestion by Coons, we refer to these points as *extraordinary* points because it is only at the final vertex points associated with these points that the resulting surface is not a standard B-spline surface.

Application of the rules once again yields Figure 2c. In this figure, six faces have emerged that have associated with them a set of 16 points that lie on a rectangular topology, as with the standard B-spline. Each of these faces has been shaded for clarity. Since the rules being applied generate a standard bicubic B-spline patch for points having this topology, these regions generate B-spline patches. Hence, a portion of the final triangular surface is now defined, and since bicubic B-splines that share vertices in this way are continuous in position, tangent and curvature, this portion of the surface is similarly continuous.

Applying the rules a third time results in further definition of B-spline surface patches near the extraordinary points, as shown in Figure 2d. The cross-hatched regions indicate where the new surface patches emerge with this application of the rules. Each of these new patches joins to the appropriate patches of Figure 2c with standard bicubic B-spline continuity. This is evident if we also subdivide the patches generated at that level; the points in common between patches dictate the continuity. However, since it is computationally more efficient to render standard patches by another algorithm, each time a standard B-spline patch is generated it is passed to a standard rendering algorithm.

It is clear that further application of the rules to the regions surrounding the extraordinary points will generate more standard patches near these points. In the limiting case, the entire triangular region, excluding the extraordinary points, is covered by a B-spline surface. Therefore, the triangular region is approximated by a surface that is continuous, except possibly at the extraordinary points. Since the rules hold for arbitrary topologies, the shape of the regions need not be simple triangles. Any number of sides will generate a B-spline surface except at the extraordinary points.

It should be noted that after one iteration all faces are four-sided, hence all new vertices created subsequently will have four incident edges. Therefore after one iteration the number of extraordinary points on the surface remains constant.

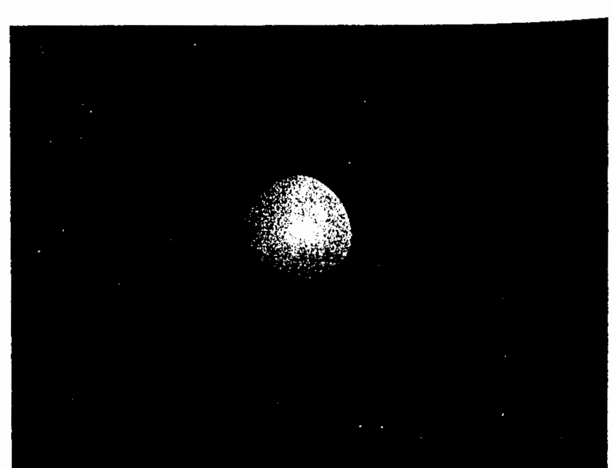


Figure 3. Surface generated from a tetrahedron

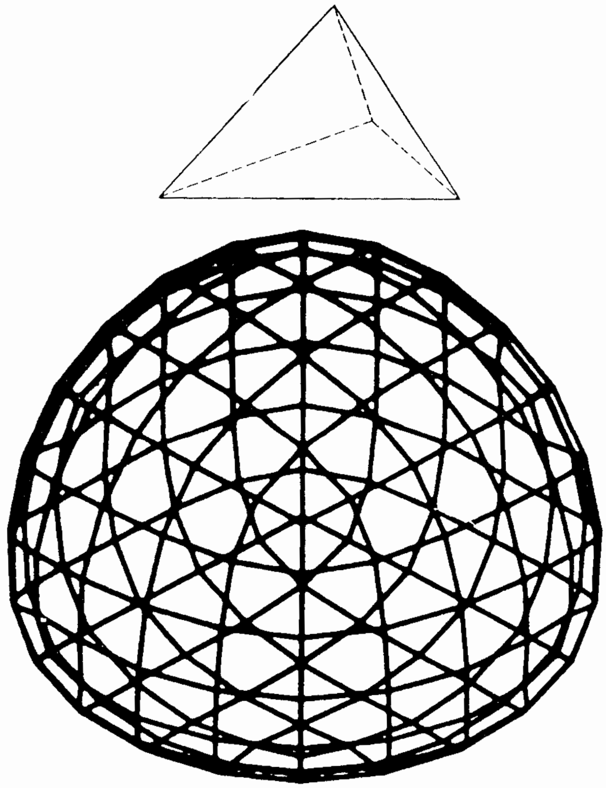


Figure 4. Original tetrahedron used to generate Figure 3 with line drawing of the generated surface

The authors would like to know the behaviour of the surface at the extraordinary points. At the present time they have not made an analytical proof of the continuity at these points. However, the pictures of the surfaces generated by these rules suggest that the surface is at least continuous in tangent everywhere.

Figure 3 shows a view of a surface generated using as a starting shape a tetrahedron, which is the smallest volume element that does not have a rectangular grid of control points. The surface is closed and has 8 extraordinary points, one for each original degree 3 vertex of the tetrahedron and one for each face, since each face yields a degree 3 vertex after the first application of the rules. The original tetrahedron used to generate Figure 3 is shown in Figure 4, along with a line drawing of the generated surface. It is evident from Figure 3 that the surface is continuous in tangent at the final vertex points corresponding to the original vertices of the tetrahedron.

Another grid of points is shown in Figure 5. This grid generates the closed volume shown in Figure 6.

The set of rules presented above is somewhat arbitrary. In fact initially a different rule was tried for (*C*). The new vertex point was

(*C*) (alternate) $\frac{Q}{4} + \frac{R}{2} + \frac{S}{4}$

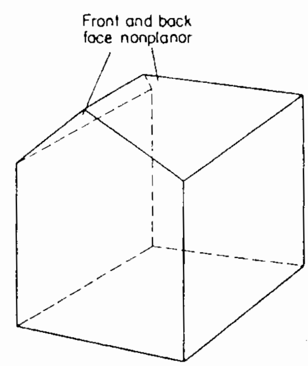
The results using that rule were unsatisfactory in that the surface became too *pointy* for the tetrahedron. The pictures made using that rule motivated us to find a better set of rules, the best of which was presented above. A better set of rules, indeed, a better criterion for judging the rules than the qualitative appearance of a picture, is yet to be devised.

At the suggestion of M Sabin a net of points taken from the saddle $z = xy$ has been made. The centre polygon has eight sides. Figure 7 shows an orthogonal view along the z axis. After one iteration of the algorithm there is a vertex in the centre with 8 edges attached to it (Figures 8, 9). A shaded picture of the saddle shows that with a large number of edges around a saddle, the centre is not well behaved (Figure 10). The saddle demonstrates that the authors have not found the best set of rules.

BIQUADRATIC SURFACES

The method can also be applied to biquadratic B-splines. The subdivided net is generated by creating a new face for each face, edge, and vertex of the original net. In Figure 11 the heavy lines are the original net and the light lines are the new net. The rule for finding each new point is dependent on the corner it is near. After analysing the biquadratic subdivision in a manner similar to that described above it can be seen that in Figure 11

$$q_{11} = \frac{(9P_{11} + 3P_{12} + 3P_{21} + P_{22})}{16}$$



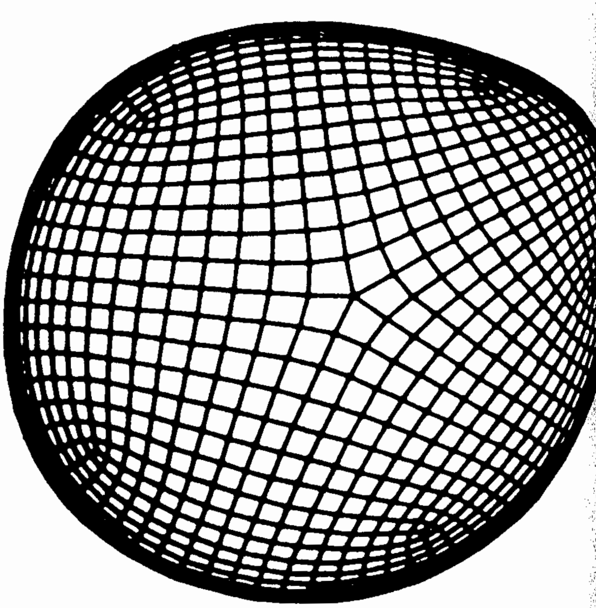


Figure 5. Grid of points

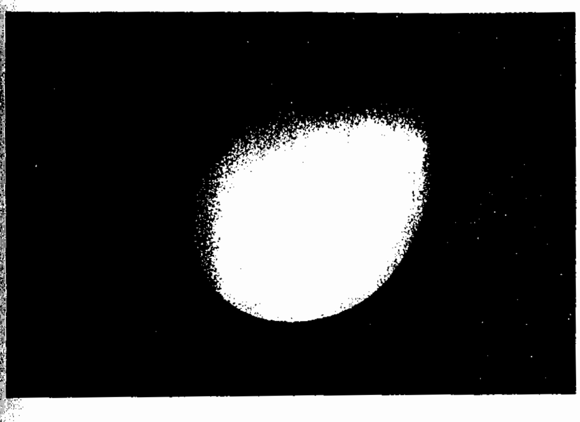


Figure 6. Closed volume generated from grid in Figure 5

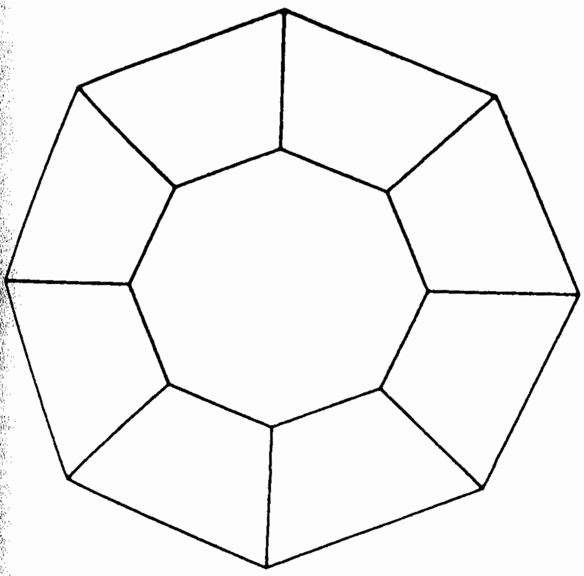


Figure 7. Orthogonal view along z axis of z=xy

This can be rewritten as a rule to find a new vertex q near an old vertex p

$$q = \frac{F}{n} + \frac{2E}{n} + \frac{P(n-3)}{n}$$

where

n = number of vertices in the face
F = the average of the vertices in the face
E = the average of the two edges incident on P

This rule can likewise be applied to any topology. In this case, after one iteration the number of non-four faces remains constant while all vertices have four incident-edges.

CONCLUSIONS

The methods presented in this paper generate B-spline surfaces on arbitrary meshes that are continuous except at a small number of *extraordinary* points. The pictures generated indicate that the surface is also continuous at these points, although no analytical proof of continuity is given.

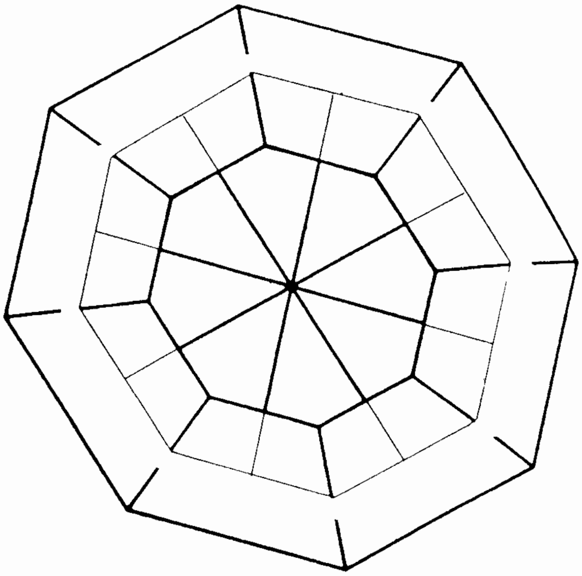


Figure 8. View of Figure 7 after one iteration

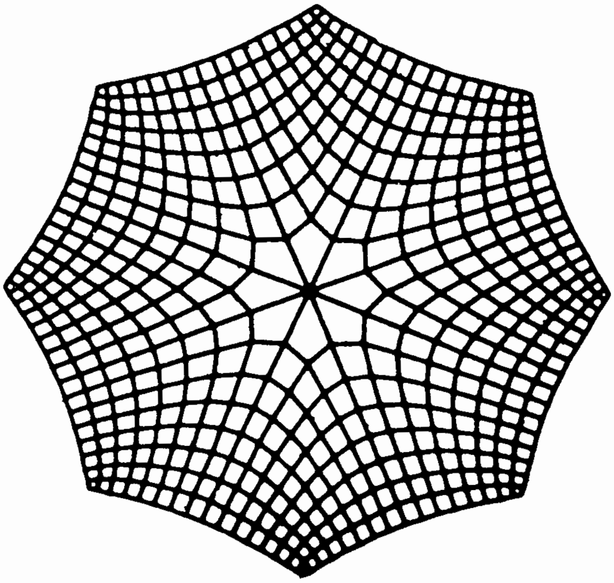


Figure 9. View of Figure 7 after many iterations

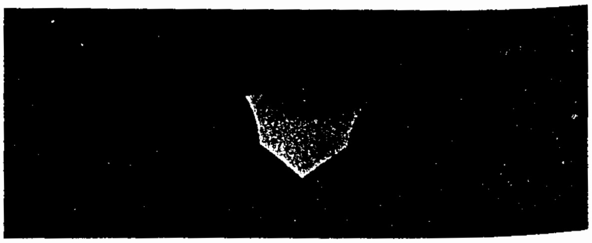


Figure 10. Shaded picture of the saddle

Other methods have been developed for approximating non-rectangular control-point meshes. For example, Lane and Riesenfeld[3] have presented an approach that is formulated in terms of a generalized basis function of two parametric variables. Also, Barnhill[4] describes a triangular patch approximation scheme. Neither of these approaches is the same as the method described in this paper.

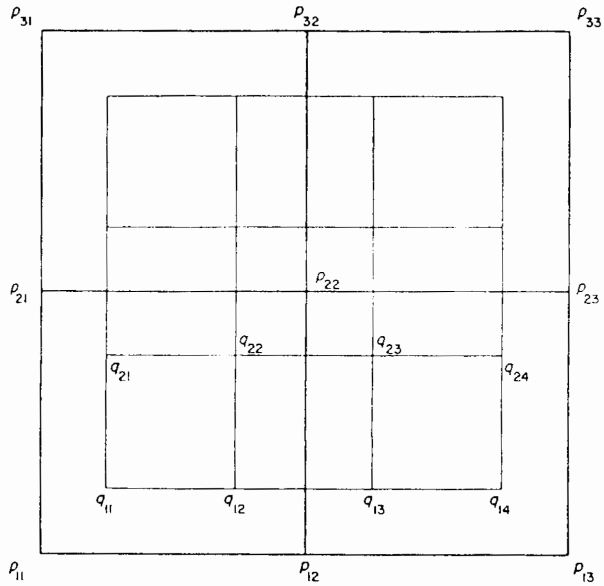


Figure 11. Biquadratic B-spline

In addition to the need for a proof of continuity at the extraordinary points of these surfaces, there is also a need for a coherent mathematical treatment of approximation schema on arbitrary topological meshes; such a treatment should encompass all of these approaches. It is hoped that this paper might stimulate such investigations.

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