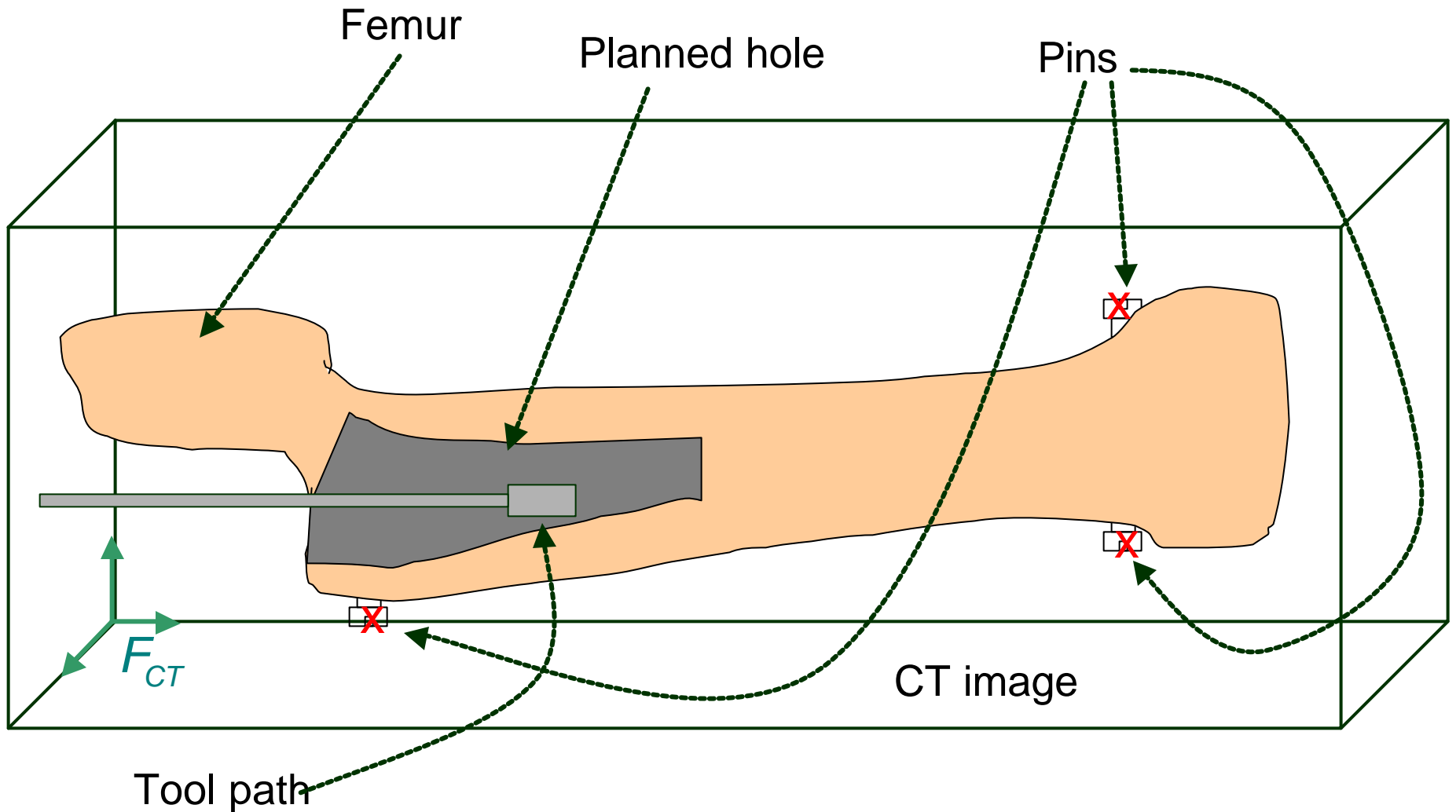


Introduction to Vectors and Frames

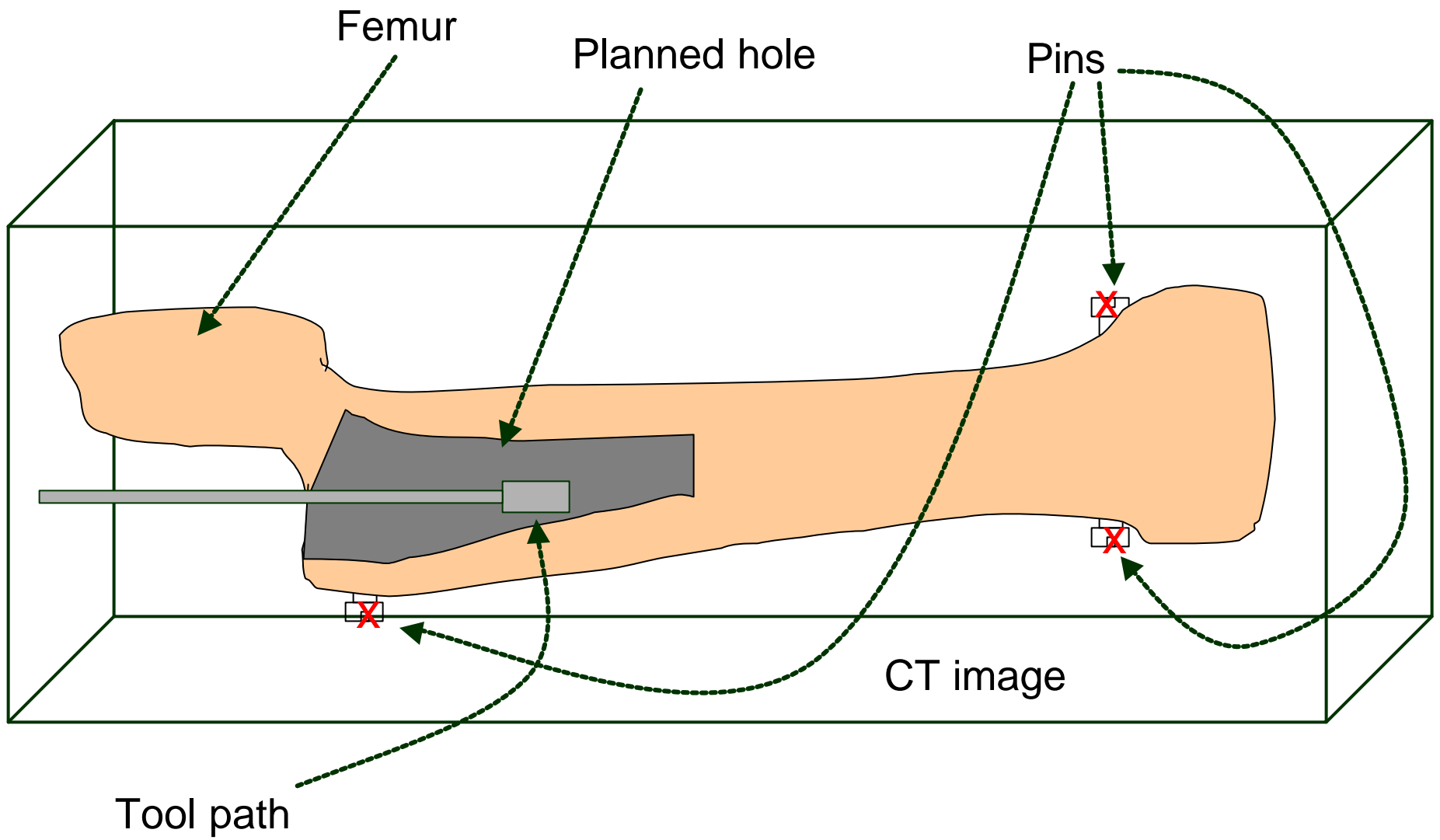
CIS - 600.445

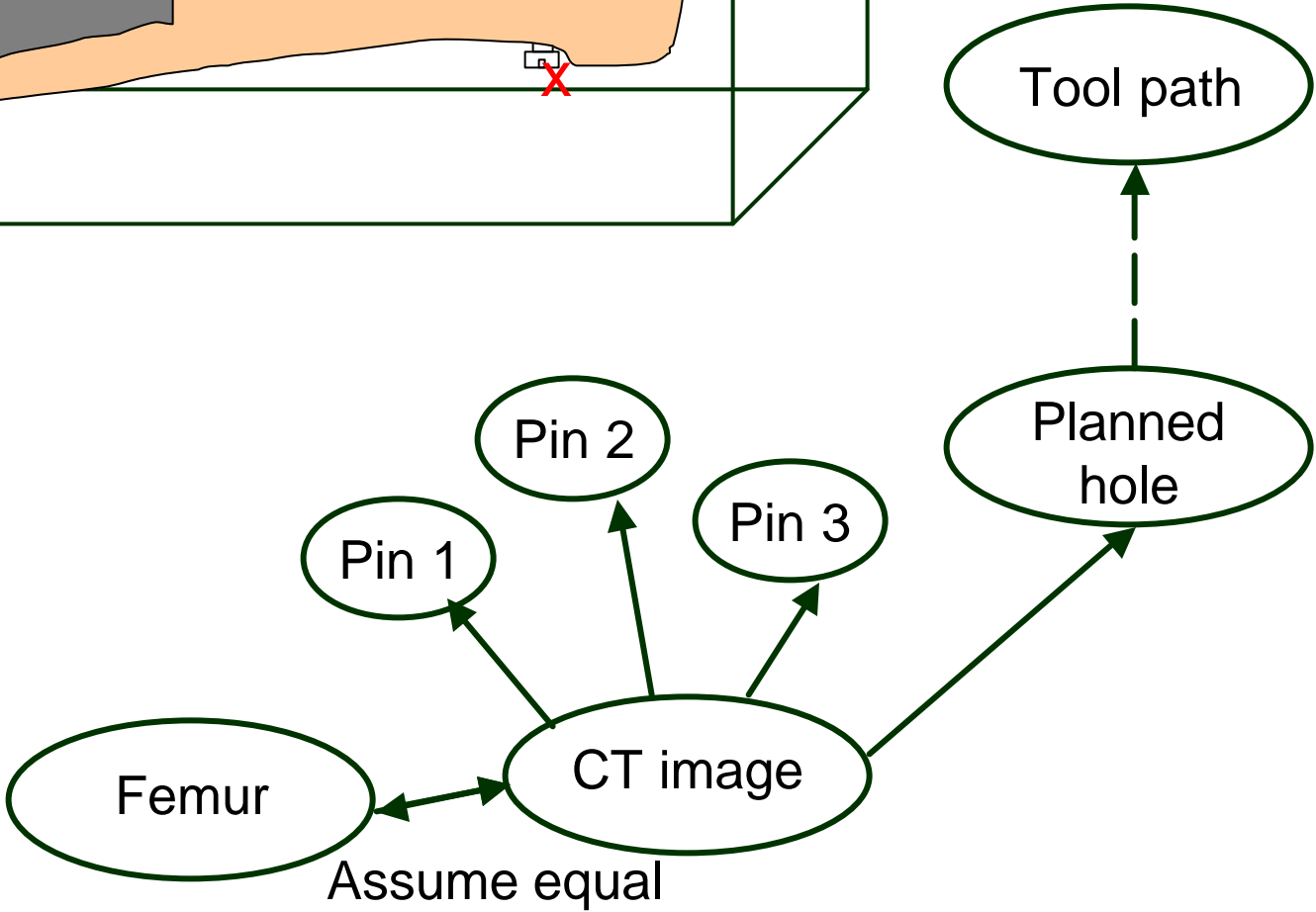
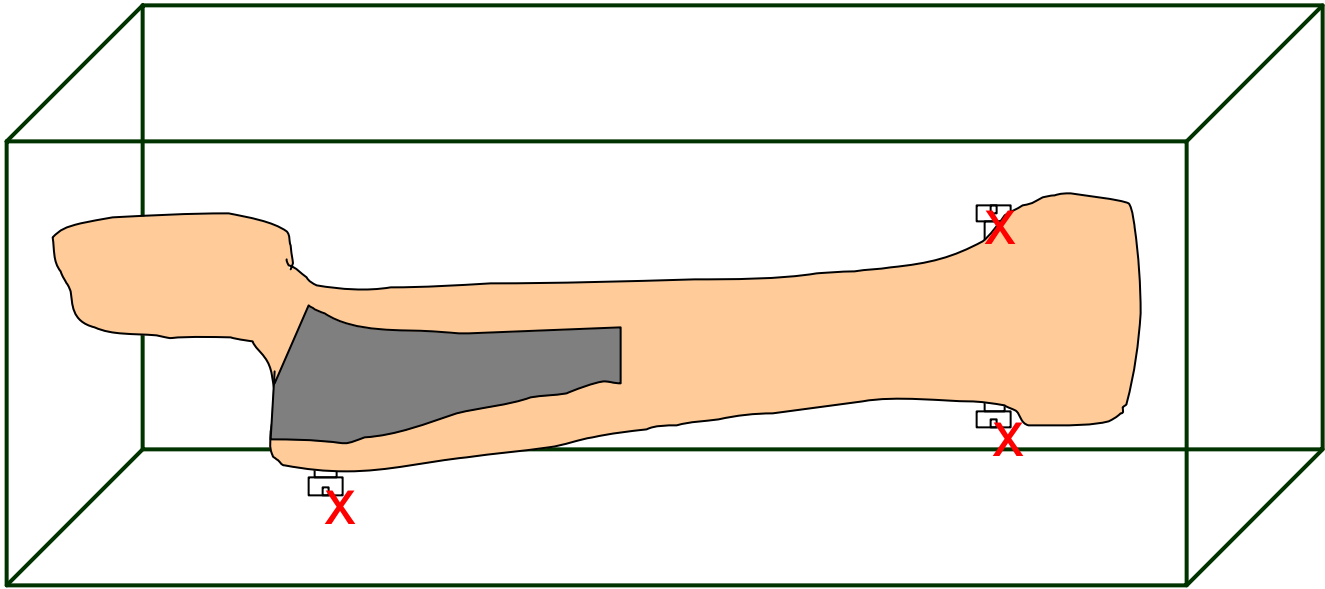
Russell Taylor

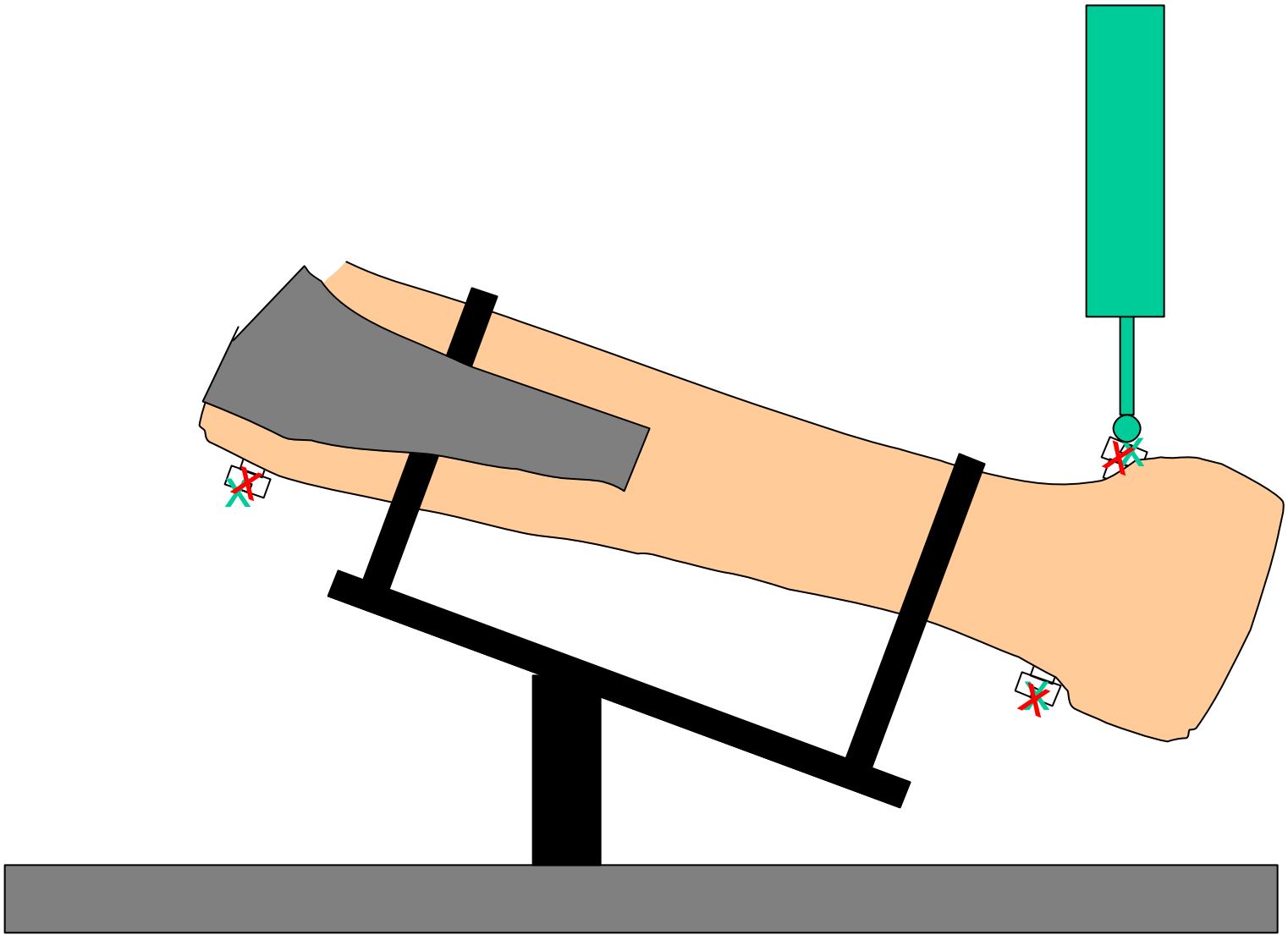
Sarah Graham

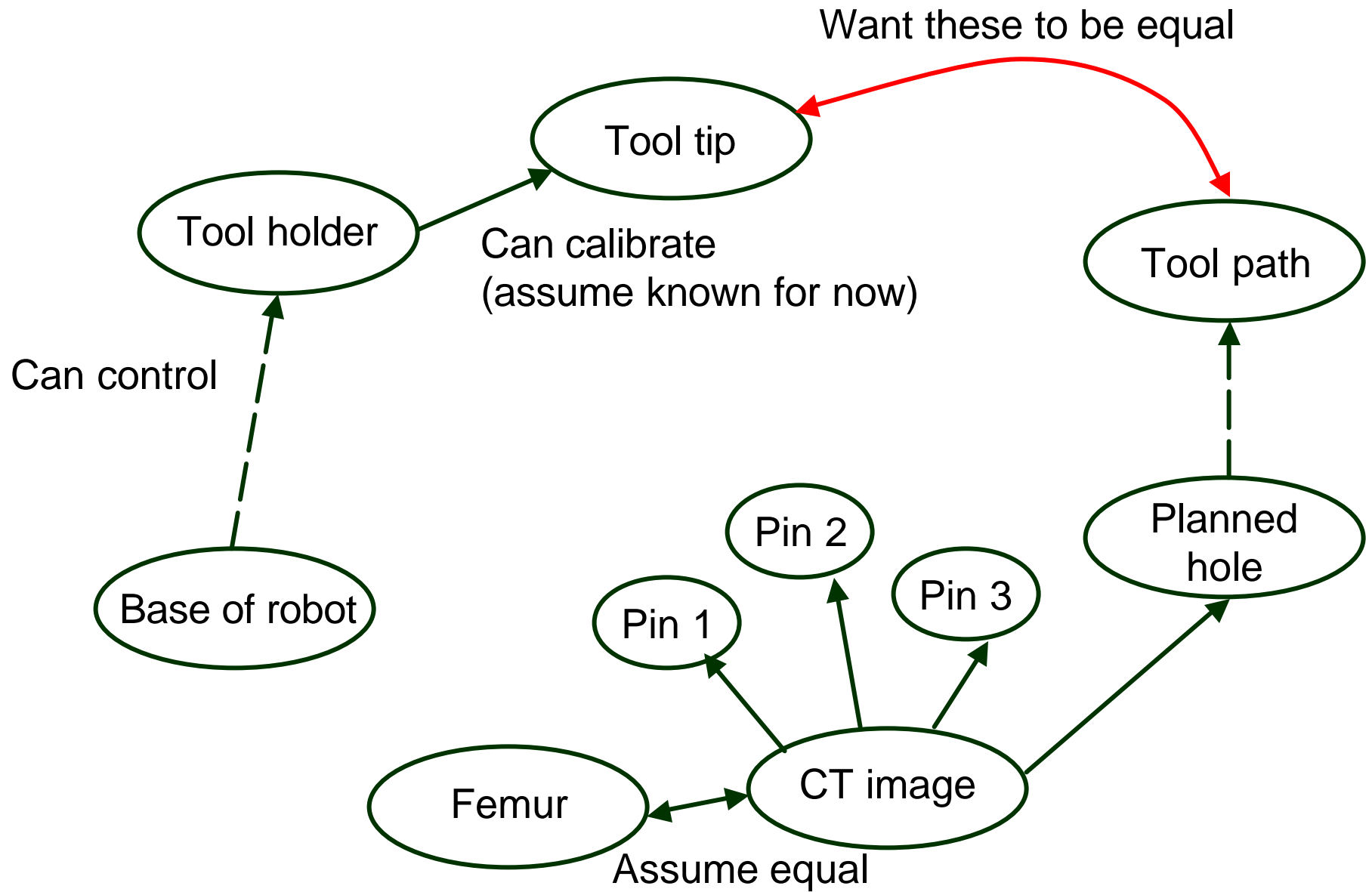


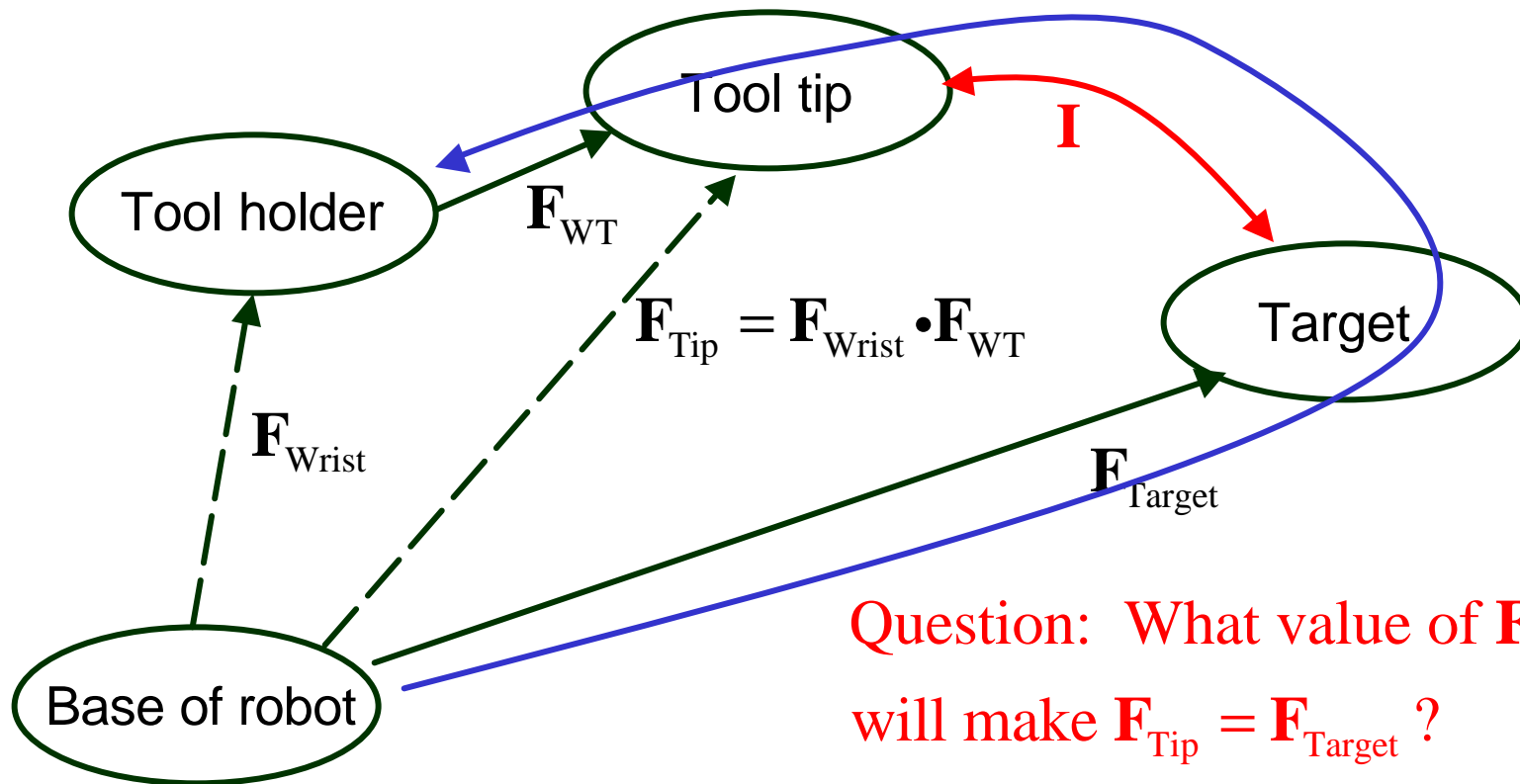
COMMON NOTATION: Use the notation F_{obj} to represent a coordinate system or the position and orientation of an object (relative to some unspecified coordinate system). Use $F_{x,y}$ to mean position and orientation of y relative to x .







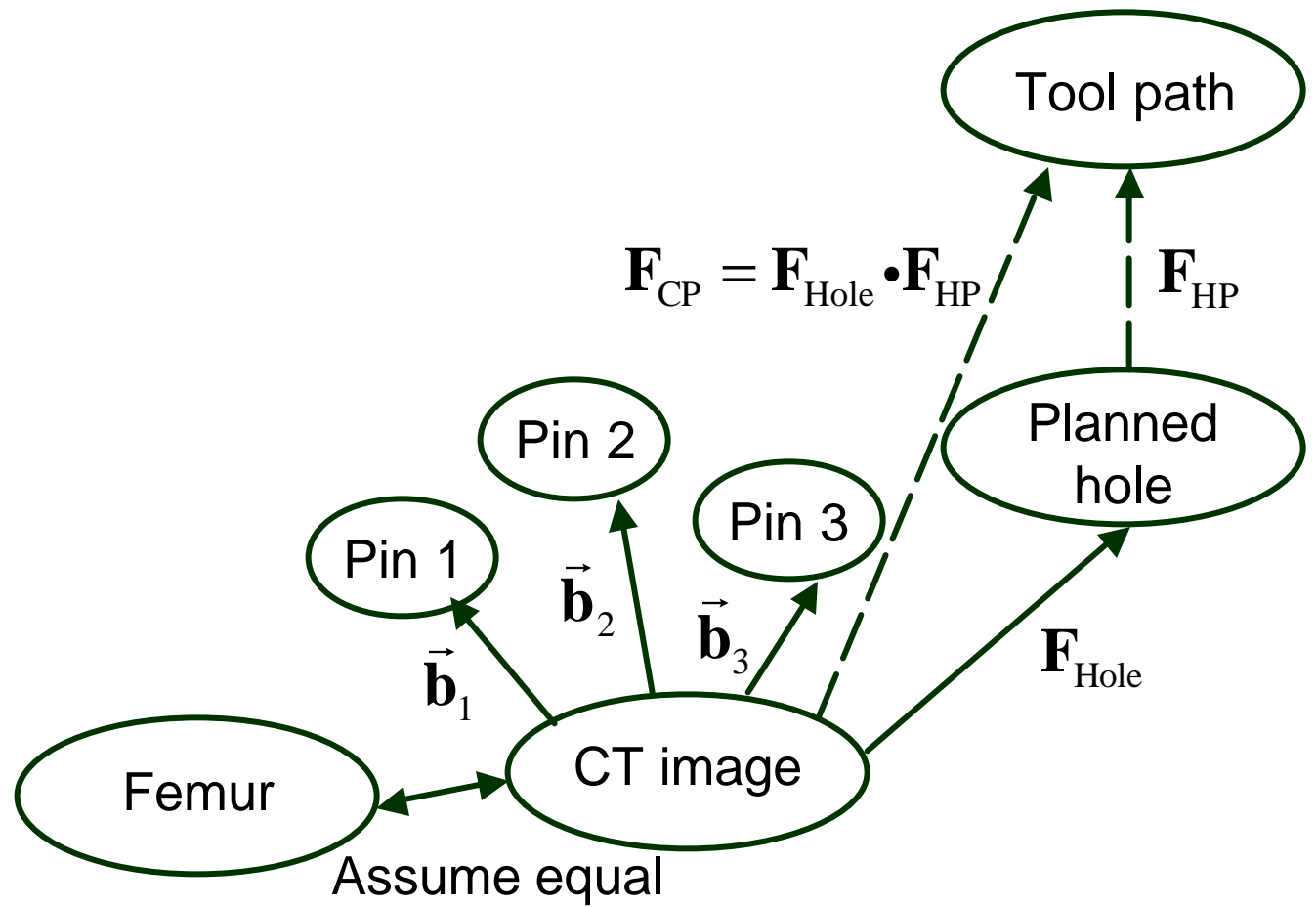




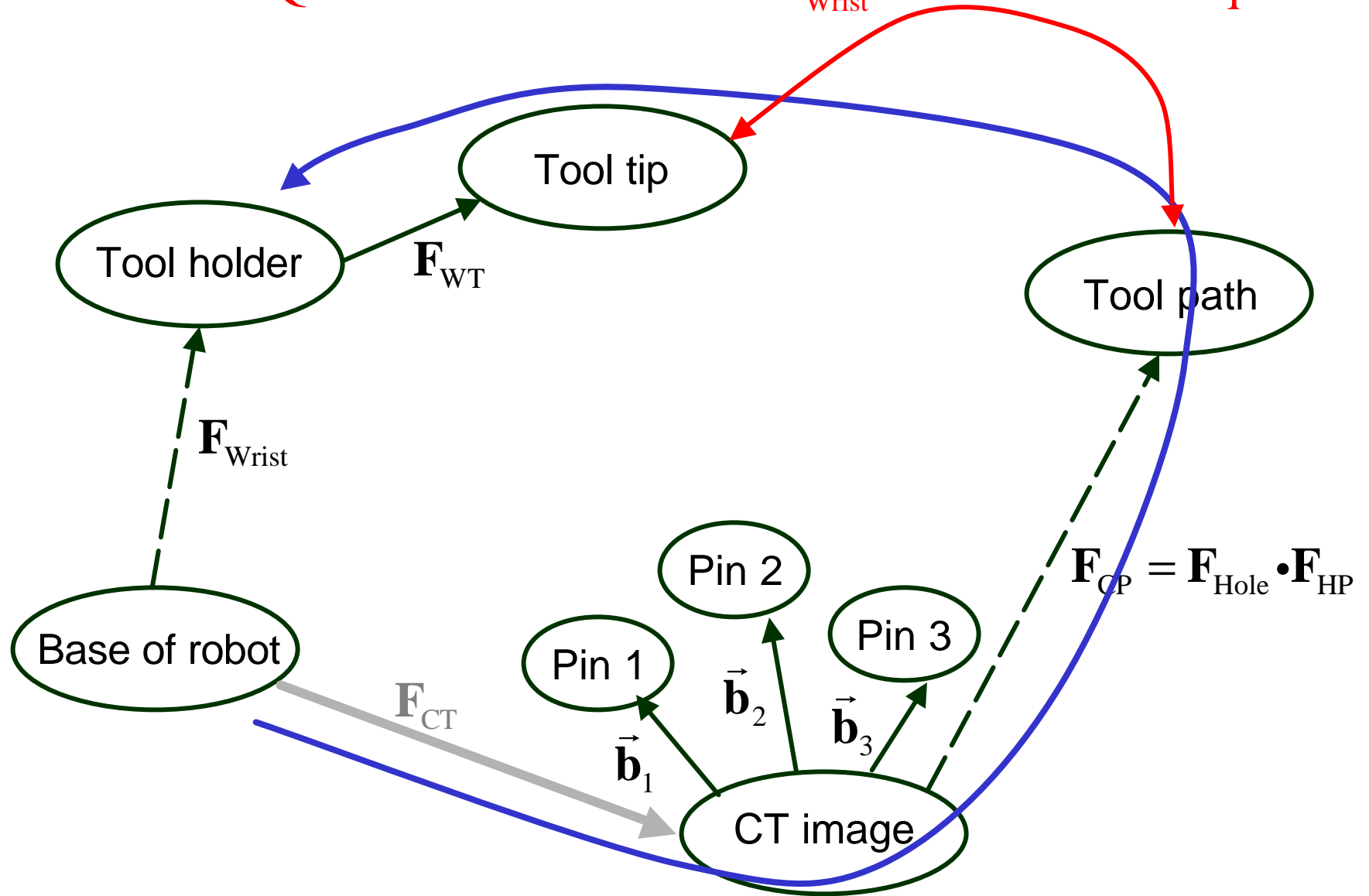
Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$?

Answer:

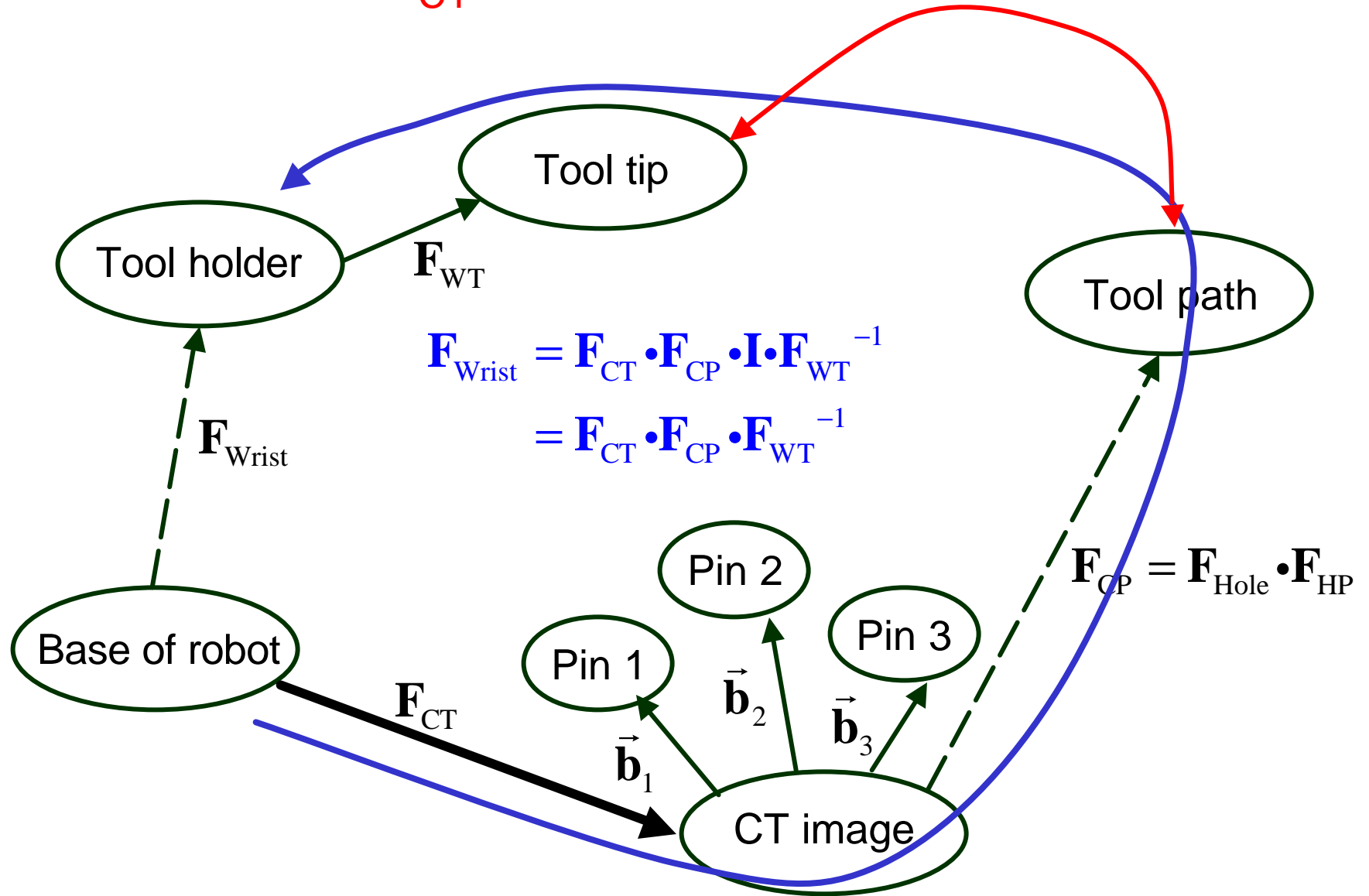
$$\begin{aligned} \mathbf{F}_{\text{Wrist}} &= \mathbf{F}_{\text{Target}} \cdot \mathbf{I} \cdot \mathbf{F}_{\text{WT}}^{-1} \\ &= \mathbf{F}_{\text{Target}} \cdot \mathbf{F}_{\text{WT}}^{-1} \end{aligned}$$



Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make these equal?

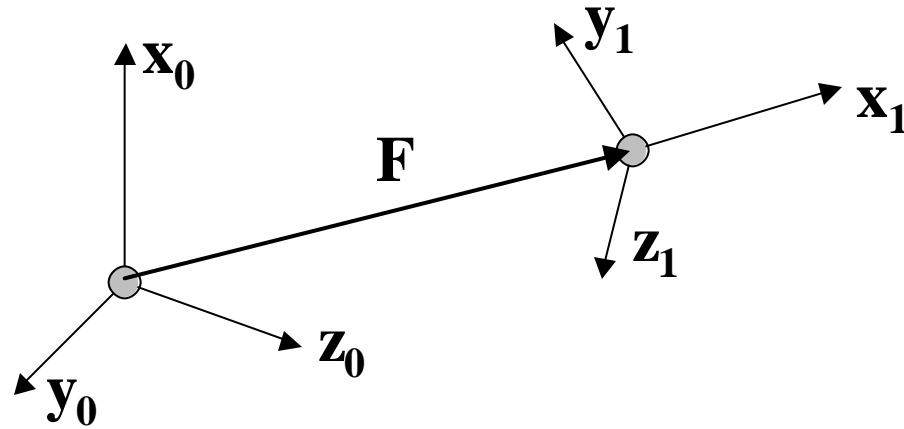


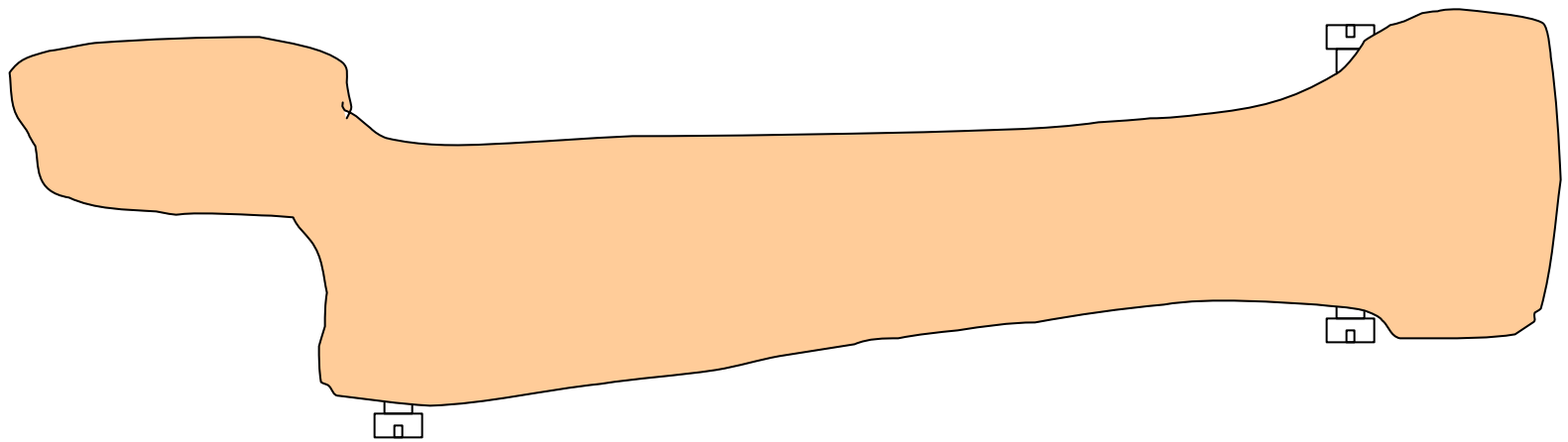
But: We must find F_{CT} ... Let's review some math

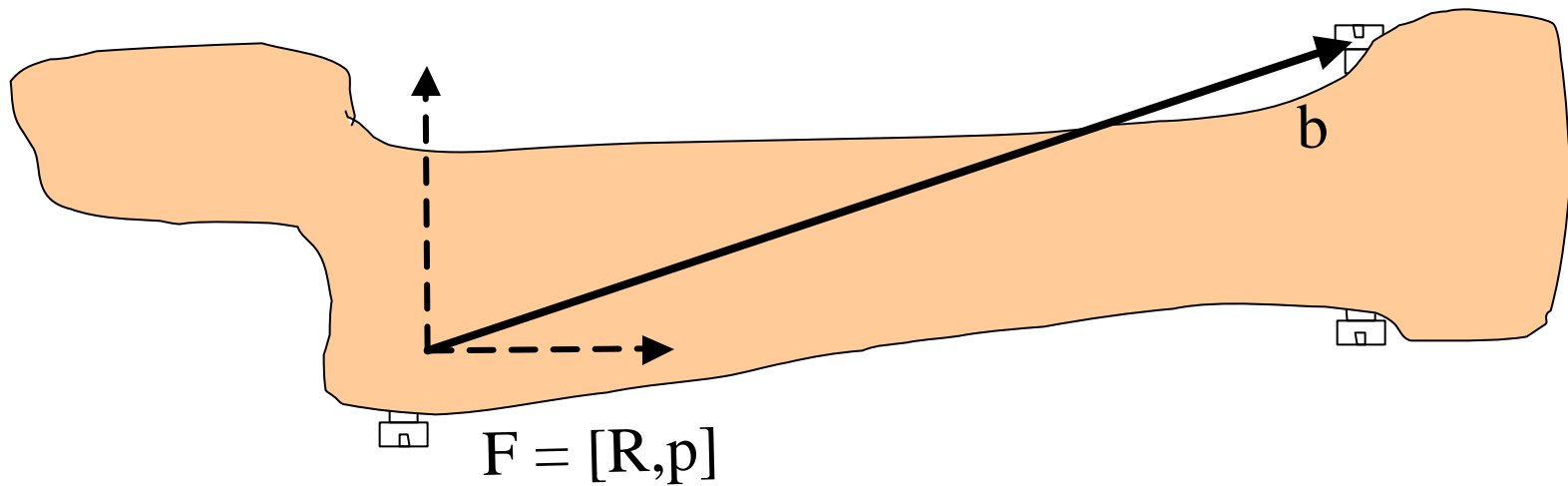


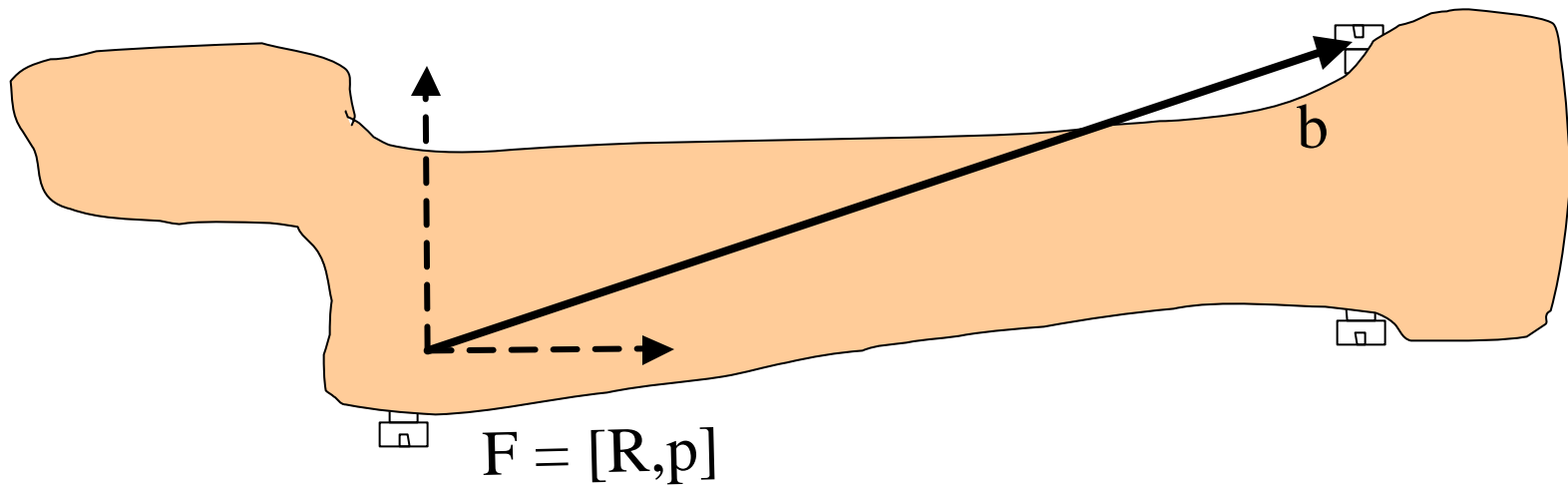
Coordinate Frame Transformation

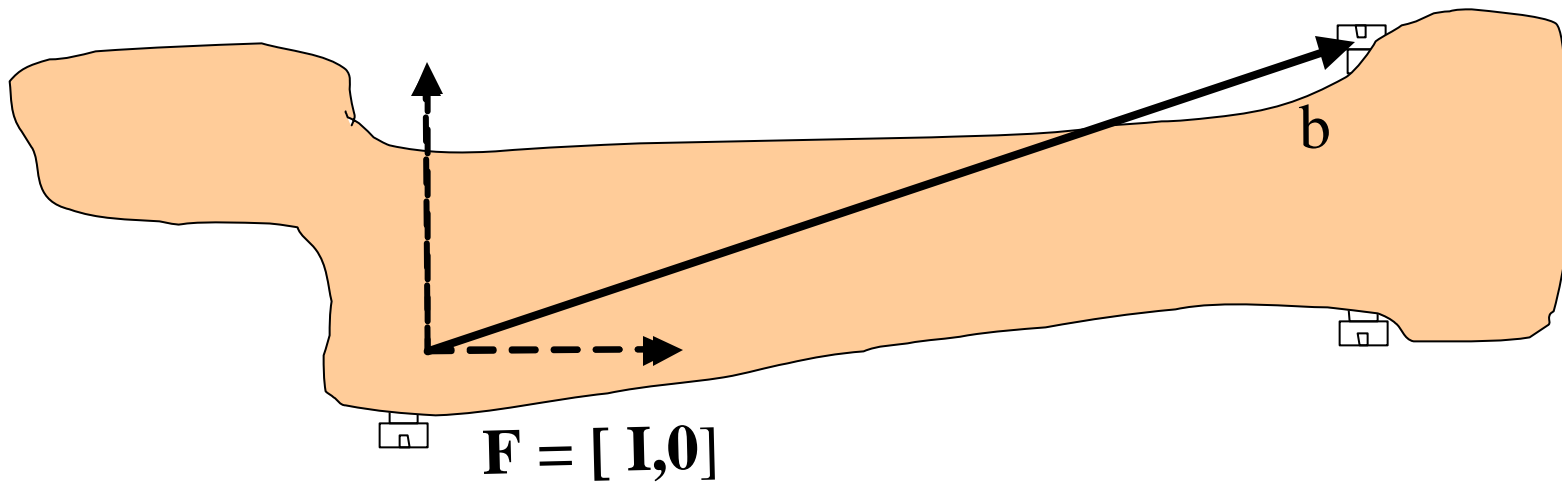
$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$

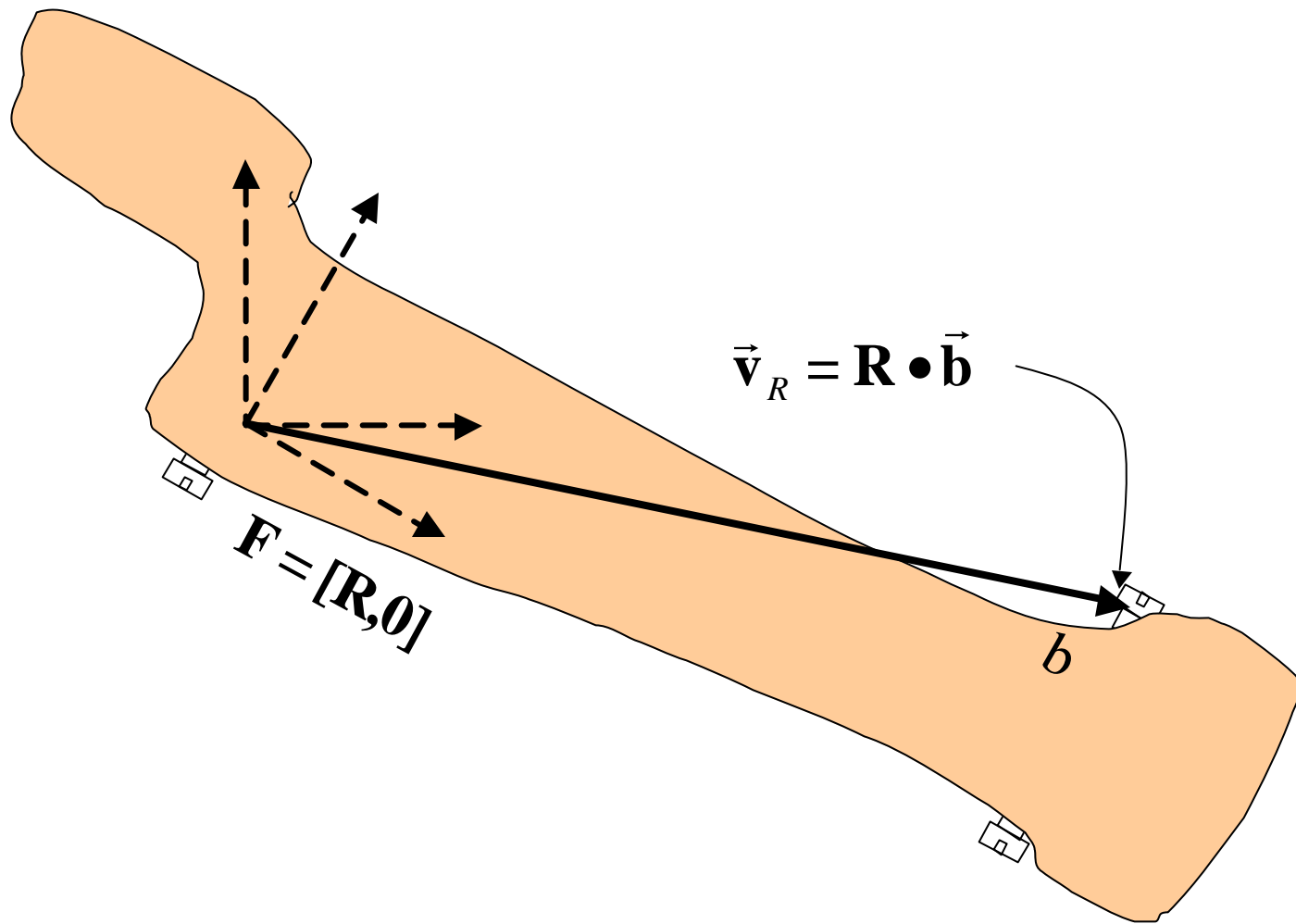


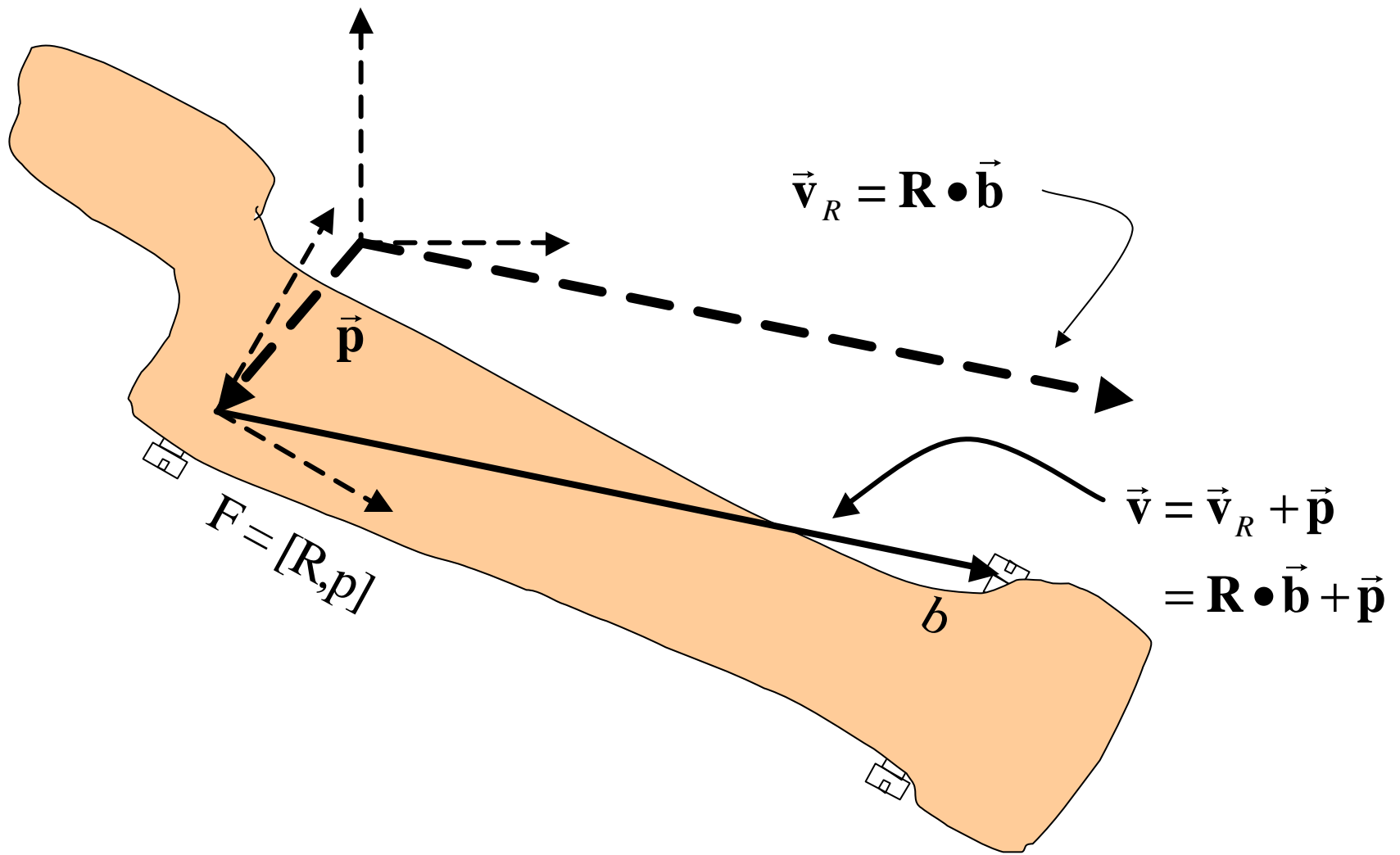




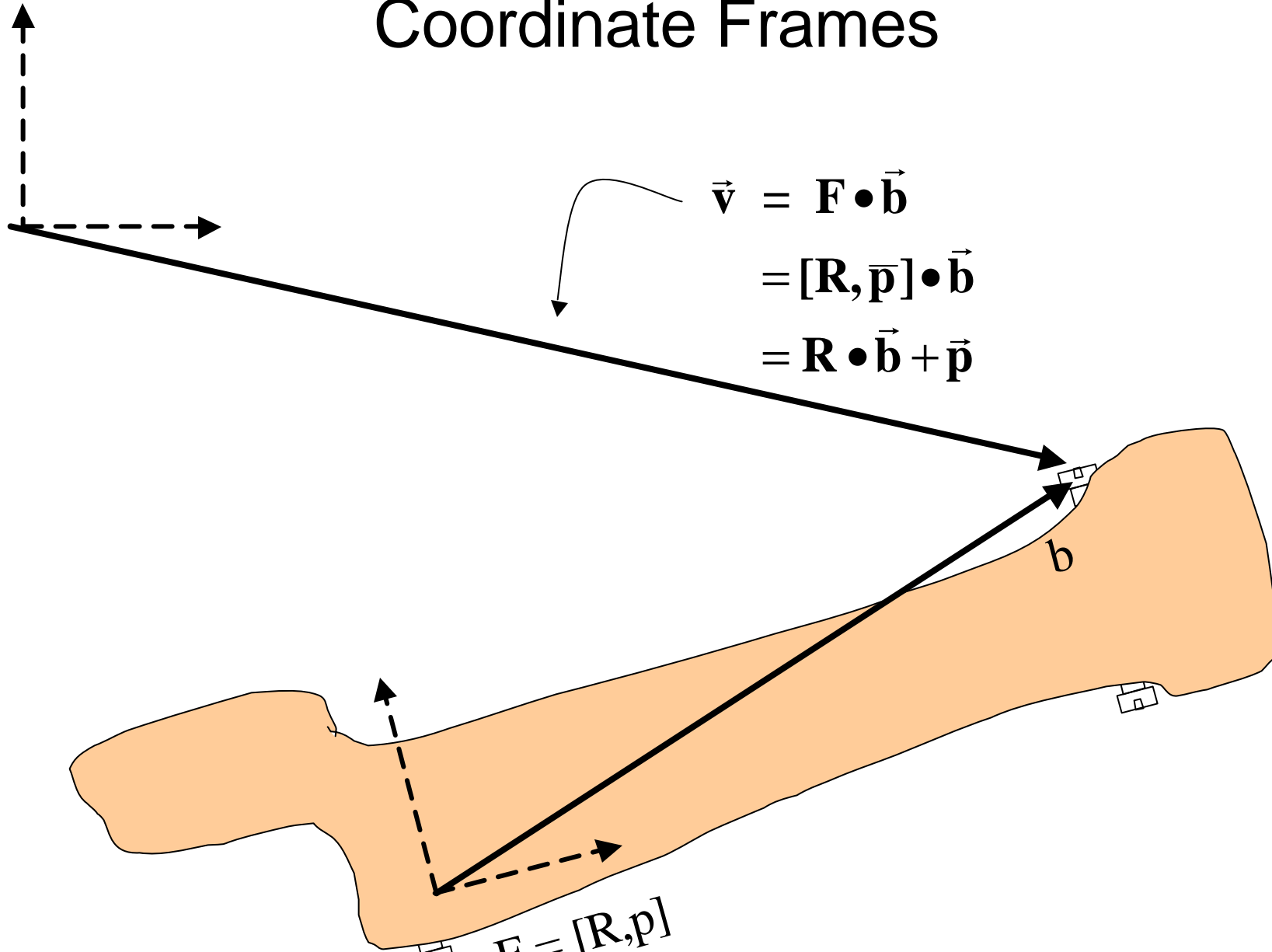




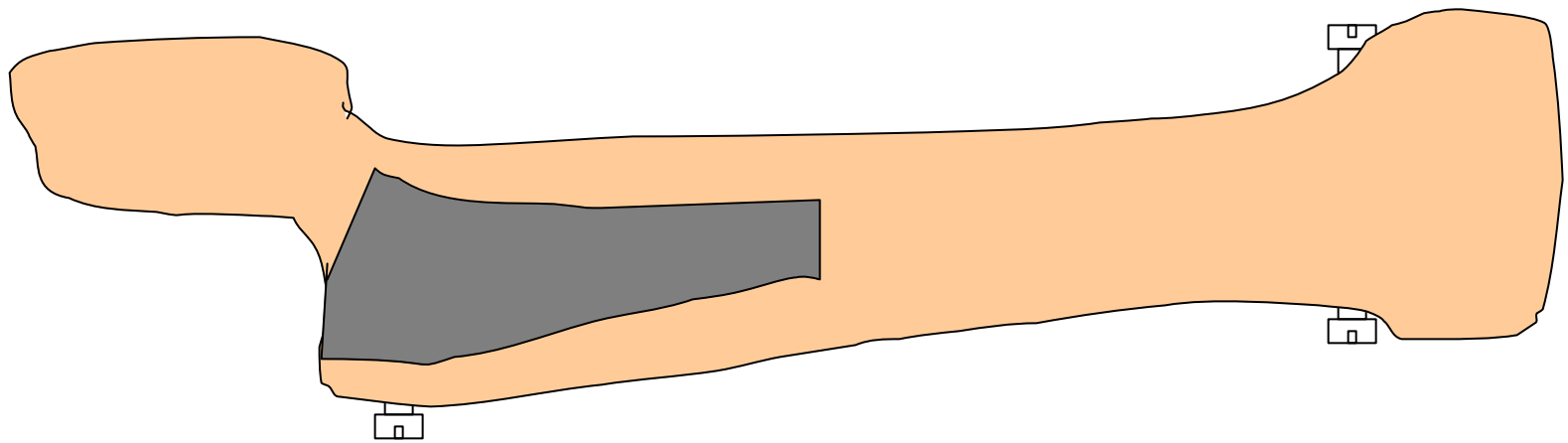




Coordinate Frames



$$\begin{aligned}\vec{v} &= \mathbf{F} \bullet \vec{b} \\ &= [\mathbf{R}, \vec{p}] \bullet \vec{b} \\ &= \mathbf{R} \bullet \vec{b} + \vec{p}\end{aligned}$$



Forward and Inverse Frame Transformations

Forward

$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$

$$\begin{aligned}\mathbf{v} &= \mathbf{F} \bullet \mathbf{b} \\ &= [\mathbf{R}, \mathbf{p}] \bullet \mathbf{b} \\ &= \mathbf{R} \bullet \mathbf{b} + \mathbf{p}\end{aligned}$$

Inverse

$$\mathbf{F}^{-1} \mathbf{v} = \mathbf{b}$$

$$\begin{aligned}\mathbf{b} &= \mathbf{R}^{-1} \bullet (\mathbf{v} - \mathbf{p}) \\ &= \mathbf{R}^{-1} \bullet \mathbf{v} - \mathbf{R}^{-1} \bullet \mathbf{p}\end{aligned}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \mathbf{p}]$$

Composition

Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1]$, $\mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

Then

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\ &= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\ &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\ &= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}\end{aligned}$$

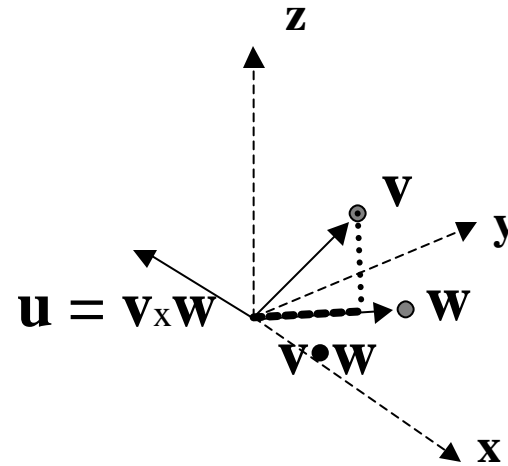
So

$$\begin{aligned}\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\ &= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]\end{aligned}$$

Vectors

$$\mathbf{v}_{col} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \end{bmatrix}$$

$$\mathbf{v}_{row} = \begin{bmatrix} \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \end{bmatrix}$$



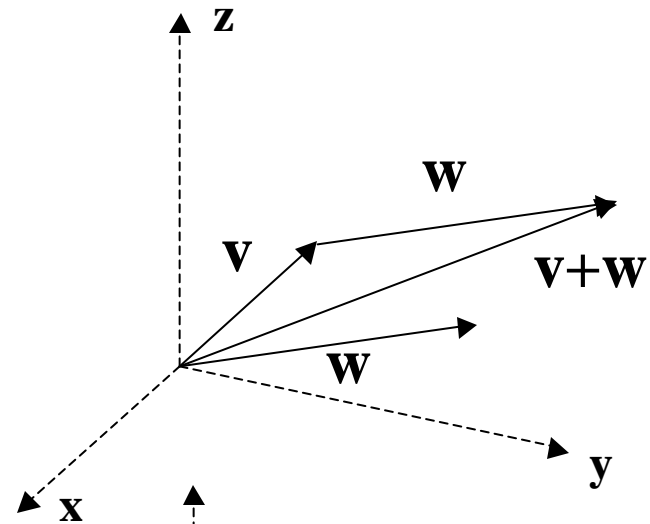
$$\text{length : } \|\mathbf{v}\| = \sqrt{\mathbf{v}_x^2 + \mathbf{v}_y^2 + \mathbf{v}_z^2}$$

$$\text{dot product : } a = \mathbf{v} \cdot \mathbf{w} = (\mathbf{v}_x \mathbf{w}_x + \mathbf{v}_y \mathbf{w}_y + \mathbf{v}_z \mathbf{w}_z) = \|\mathbf{v}\| \|\mathbf{w}\| \cos q$$

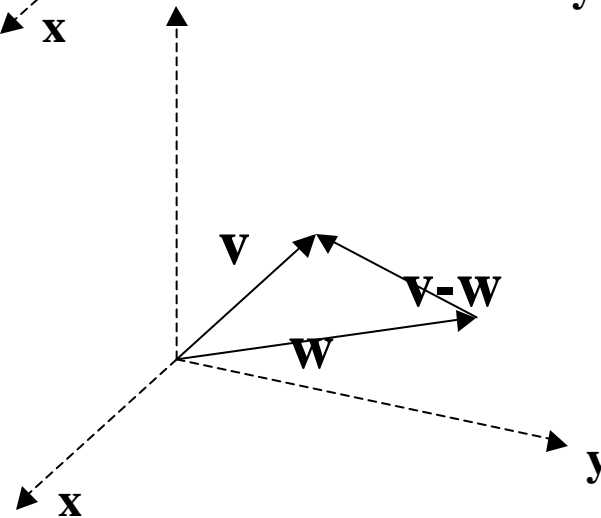
$$\text{cross product : } \mathbf{u} = \mathbf{v} \times \mathbf{w} = \begin{bmatrix} \mathbf{v}_y \mathbf{w}_z - \mathbf{v}_z \mathbf{w}_y \\ \mathbf{v}_z \mathbf{w}_x - \mathbf{v}_x \mathbf{w}_z \\ \mathbf{v}_x \mathbf{w}_y - \mathbf{v}_y \mathbf{w}_x \end{bmatrix}, \|\mathbf{u}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin q$$

Vectors as Displacements

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} \mathbf{v}_x + \mathbf{w}_x \\ \mathbf{v}_y + \mathbf{w}_y \\ \mathbf{v}_z + \mathbf{w}_z \end{bmatrix}$$

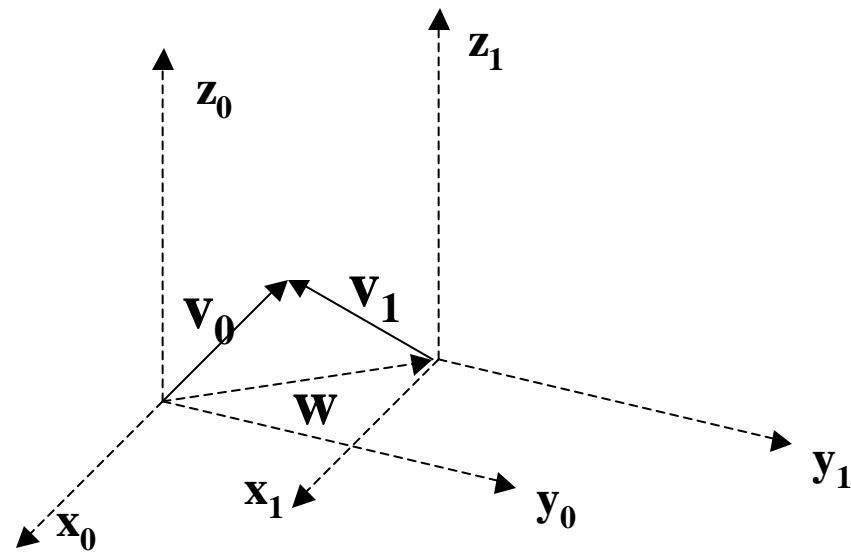


$$\mathbf{v} - \mathbf{w} = \begin{bmatrix} \mathbf{v}_x - \mathbf{w}_x \\ \mathbf{v}_y - \mathbf{w}_y \\ \mathbf{v}_z - \mathbf{w}_z \end{bmatrix}$$



Vectors as Displacements Between Parallel Frames

$$\mathbf{V}_1 = \mathbf{V}_0 - \mathbf{W}$$



Rotations: Some Notation

$Rot(\vec{\mathbf{a}}, \mathbf{a}) \triangleq$ Rotation by angle \mathbf{a} about axis $\vec{\mathbf{a}}$

$\mathbf{R}_{\vec{\mathbf{a}}}(\mathbf{a}) \triangleq$ Rotation by angle \mathbf{a} about axis $\vec{\mathbf{a}}$

$\mathbf{R}(\vec{\mathbf{a}}) \triangleq Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|)$

$\mathbf{R}_{xyz}(\mathbf{a}, \mathbf{b}, \mathbf{g}) \triangleq \mathbf{R}(\vec{\mathbf{x}}, \mathbf{a}) \bullet \mathbf{R}(\vec{\mathbf{y}}, \mathbf{b}) \bullet \mathbf{R}(\vec{\mathbf{z}}, \mathbf{g})$

$\mathbf{R}_{zyz}(\mathbf{a}, \mathbf{b}, \mathbf{g}) \triangleq \mathbf{R}(\vec{\mathbf{z}}, \mathbf{a}) \bullet \mathbf{R}(\vec{\mathbf{y}}, \mathbf{b}) \bullet \mathbf{R}(\vec{\mathbf{z}}, \mathbf{g})$

Rotations: A few useful facts

$$Rot(s\vec{\mathbf{a}}, \mathbf{a}) \bullet \vec{\mathbf{a}} = \vec{\mathbf{a}} \quad \text{and} \quad \|Rot(\vec{\mathbf{a}}, \mathbf{a}) \bullet \vec{\mathbf{b}}\| = \|\vec{\mathbf{b}}\|$$

$$Rot(\vec{\mathbf{a}}, \mathbf{a}) = Rot(\hat{\mathbf{a}}, \mathbf{a}) \quad \text{where} \quad \hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}$$

$$Rot(\hat{\mathbf{a}}, \mathbf{a}) \bullet Rot(\hat{\mathbf{a}}, \mathbf{b}) = Rot(\hat{\mathbf{a}}, \mathbf{a} + \mathbf{b})$$

$$Rot(\hat{\mathbf{a}}, \mathbf{a})^{-1} = Rot(\hat{\mathbf{a}}, -\mathbf{a})$$

$$Rot(\vec{\mathbf{a}}, 0) \bullet \vec{\mathbf{b}} = \vec{\mathbf{b}} \quad \text{i.e.,} \quad Rot(\vec{\mathbf{a}}, 0) = \mathbf{I}_{Rot} = \text{the identity rotation}$$

$$Rot(\hat{\mathbf{a}}, \mathbf{a}) \bullet \vec{\mathbf{b}} = (\hat{\mathbf{a}} \bullet \vec{\mathbf{b}}) \hat{\mathbf{a}} + Rot(\hat{\mathbf{a}}, \mathbf{a}) \bullet (\vec{\mathbf{b}} - (\hat{\mathbf{a}} \bullet \vec{\mathbf{b}}) \hat{\mathbf{a}})$$

$$Rot(\hat{\mathbf{a}}, \mathbf{a}) \bullet Rot(\hat{\mathbf{b}}, \mathbf{b}) = Rot(\hat{\mathbf{b}}, \mathbf{b}) \bullet Rot(Rot(\hat{\mathbf{b}}, -\mathbf{b}) \bullet \hat{\mathbf{a}}, \mathbf{a})$$

Rotations: more facts

If $\vec{v} = [v_x, v_y, v_z]^T$ then a rotation $\mathbf{R} \bullet \vec{v}$ may be described in terms of the effects of \mathbf{R} on orthogonal unit vectors, $\hat{\mathbf{x}} = [1, 0, 0]^T$, $\hat{\mathbf{y}} = [0, 1, 0]^T$, $\hat{\mathbf{z}} = [0, 0, 1]^T$

$$\mathbf{R} \bullet \vec{v} = v_x \vec{\mathbf{r}}_x + v_y \vec{\mathbf{r}}_y + v_z \vec{\mathbf{r}}_z$$

where

$$\vec{\mathbf{r}}_x = \mathbf{R} \bullet \hat{\mathbf{x}}$$

$$\vec{\mathbf{r}}_y = \mathbf{R} \bullet \hat{\mathbf{y}}$$

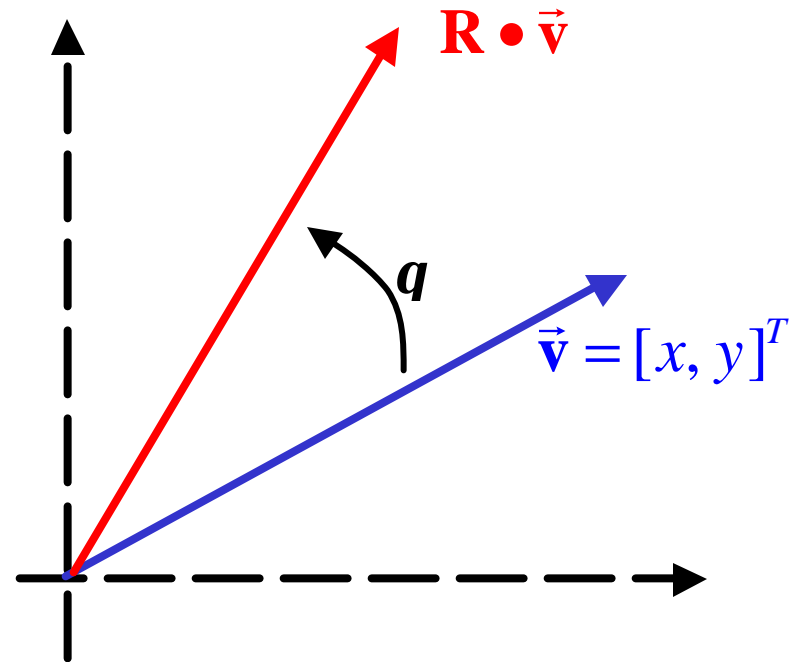
$$\vec{\mathbf{r}}_z = \mathbf{R} \bullet \hat{\mathbf{z}}$$

Thus

$$(\mathbf{R} \bullet \vec{\mathbf{b}}) \bullet (\mathbf{R} \bullet \vec{\mathbf{c}}) = \vec{\mathbf{b}} \bullet \vec{\mathbf{c}}$$

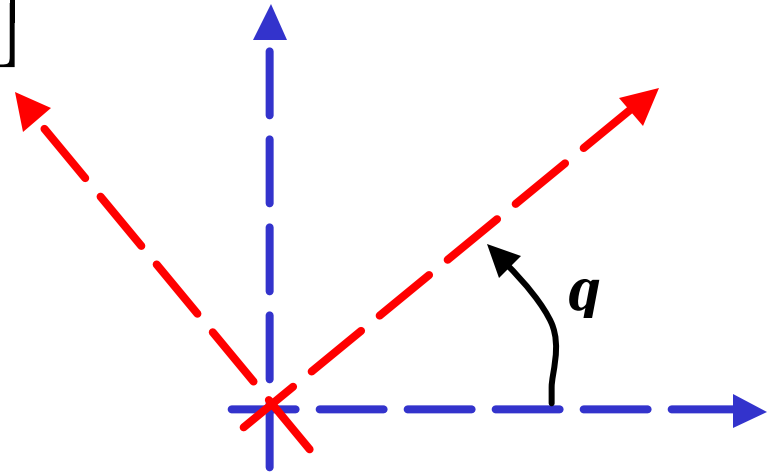
Rotations in the plane

$$\begin{aligned} \mathbf{R} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x \cos \mathbf{q} - y \sin \mathbf{q} \\ x \sin \mathbf{q} + y \cos \mathbf{q} \end{bmatrix} \\ &= \begin{bmatrix} \cos \mathbf{q} & -\sin \mathbf{q} \\ \sin \mathbf{q} & \cos \mathbf{q} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$



Rotations in the plane

$$\begin{aligned} \mathbf{R} \bullet [\hat{\mathbf{x}} \quad \hat{\mathbf{y}}] &= \begin{bmatrix} \cos \mathbf{q} & -\sin \mathbf{q} \\ \sin \mathbf{q} & \cos \mathbf{q} \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= [\mathbf{R} \bullet \hat{\mathbf{x}} \quad \mathbf{R} \bullet \hat{\mathbf{y}}] \end{aligned}$$



3D Rotation Matrices

$$\mathbf{R} \bullet [\hat{\mathbf{x}} \quad \hat{\mathbf{y}} \quad \hat{\mathbf{z}}] = [\mathbf{R} \bullet \hat{\mathbf{x}} \quad \mathbf{R} \bullet \hat{\mathbf{y}} \quad \mathbf{R} \bullet \hat{\mathbf{z}}] \\ = [\hat{\mathbf{r}}_x \quad \hat{\mathbf{r}}_y \quad \hat{\mathbf{r}}_z]$$

$$\mathbf{R}^T \bullet \mathbf{R} = \begin{bmatrix} \hat{\mathbf{r}}_x^T \\ \hat{\mathbf{r}}_y^T \\ \hat{\mathbf{r}}_z^T \end{bmatrix} \bullet [\hat{\mathbf{r}}_x \quad \hat{\mathbf{r}}_y \quad \hat{\mathbf{r}}_z] \\ = \begin{bmatrix} \hat{\mathbf{r}}_x^T \bullet \hat{\mathbf{r}}_x & \hat{\mathbf{r}}_x^T \bullet \hat{\mathbf{r}}_y & \hat{\mathbf{r}}_x^T \bullet \hat{\mathbf{r}}_z \\ \hat{\mathbf{r}}_y^T \bullet \hat{\mathbf{r}}_x & \hat{\mathbf{r}}_y^T \bullet \hat{\mathbf{r}}_y & \hat{\mathbf{r}}_y^T \bullet \hat{\mathbf{r}}_z \\ \hat{\mathbf{r}}_z^T \bullet \hat{\mathbf{r}}_x & \hat{\mathbf{r}}_z^T \bullet \hat{\mathbf{r}}_y & \hat{\mathbf{r}}_z^T \bullet \hat{\mathbf{r}}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of Rotation Matrices

Inverse of a Rotation Matrix equals its transpose:

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

$$\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}$$

The Determinant of a Rotation matrix is equal to +1:

$$\det(\mathbf{R}) = +1$$

Any Rotation can be described by consecutive rotations about the three primary axes, x, y, and z:

$$\mathbf{R} = \mathbf{R}_{z,q} \mathbf{R}_{y,f} \mathbf{R}_{x,y}$$

Canonical 3D Rotation Matrices

Note: Right-Handed Coordinate System

$$\mathbf{R}_{\vec{x}}(\mathbf{q}) = Rot(\vec{x}, \mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(?) & -\sin(?) \\ 0 & \sin(?) & \cos(?) \end{bmatrix}$$

$$\mathbf{R}_{\vec{y}}(\mathbf{q}) = Rot(\vec{y}, \mathbf{q}) = \begin{bmatrix} \cos(?) & 0 & \sin(?) \\ 0 & 1 & 0 \\ -\sin(?) & 0 & \cos(?) \end{bmatrix}$$

$$\mathbf{R}_{\vec{z}}(\mathbf{q}) = Rot(\vec{z}, \mathbf{q}) = \begin{bmatrix} \cos(?) & -\sin(?) & 0 \\ \sin(?) & \cos(?) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Coordinates

- Widely used in graphics, geometric calculations
- Represent 3D vector as 4D quantity
- For our purposes, we will keep the “scale” $s = 1$

$$\vec{\mathbf{V}} = \begin{bmatrix} x / s \\ y / s \\ z / s \\ s \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

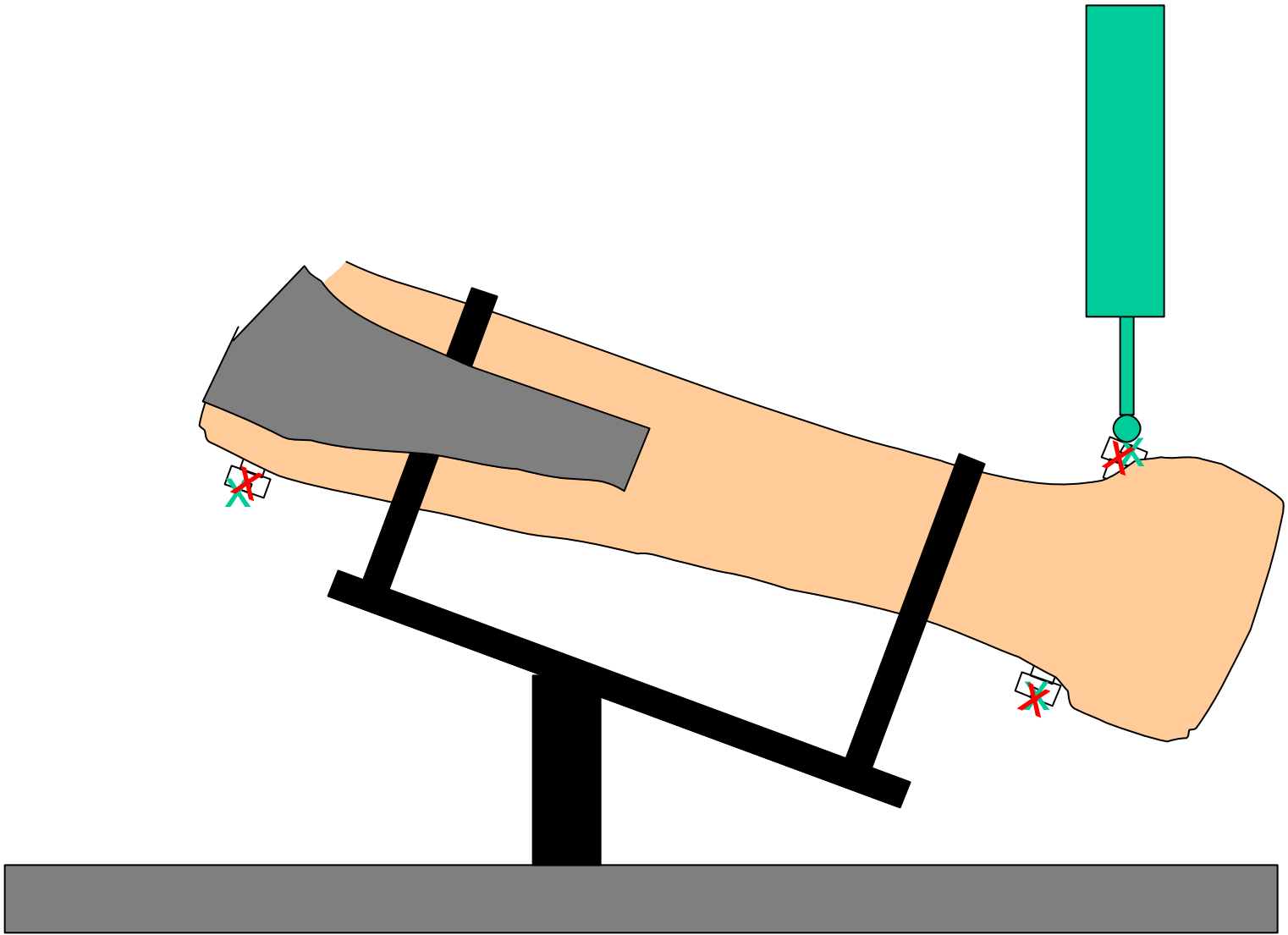
Representing Frame Transformations as Matrices

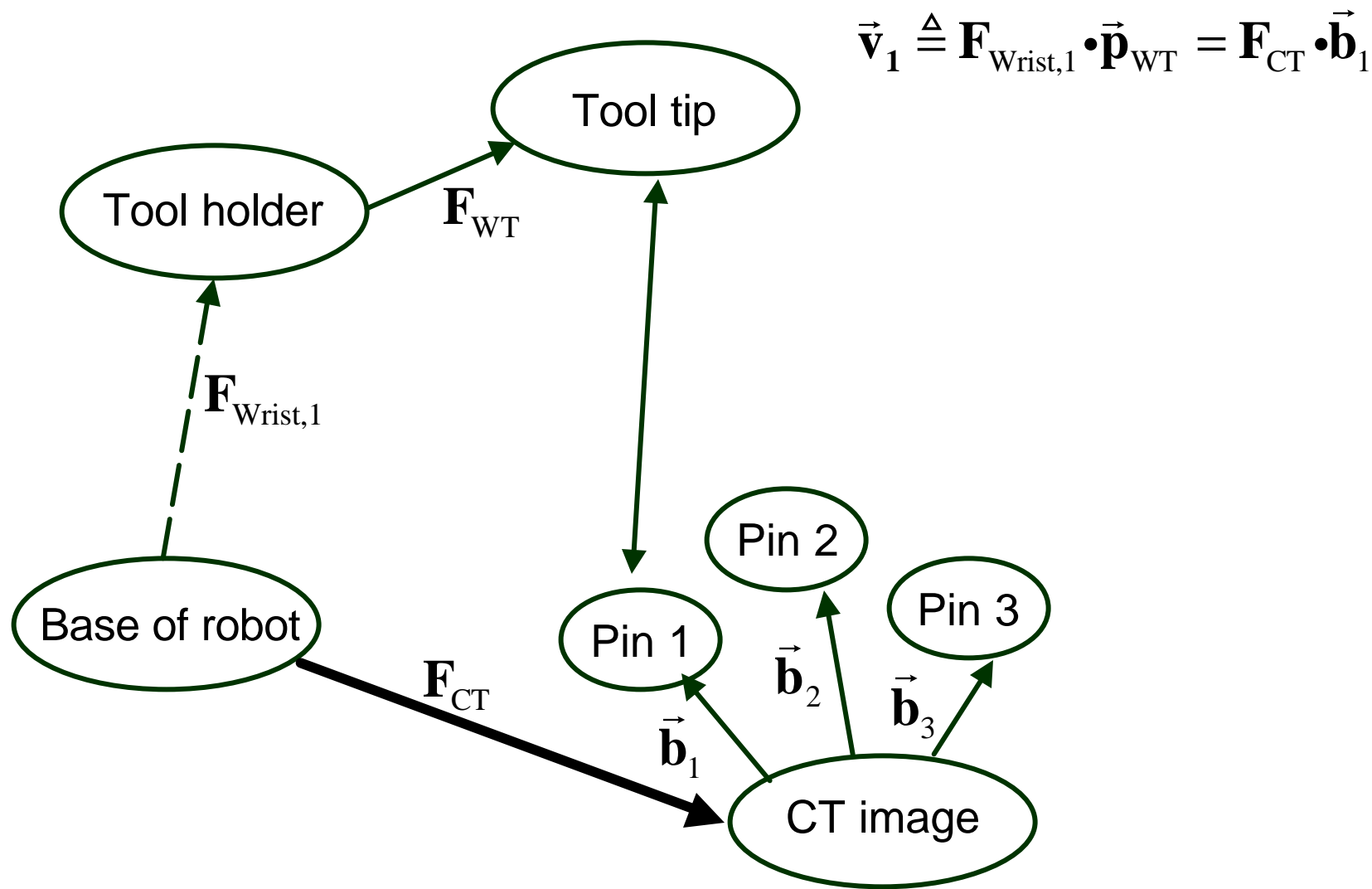
$$\mathbf{v} + \mathbf{p} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \mathbf{p}_x \\ 0 & 1 & 0 & \mathbf{p}_y \\ 0 & 0 & 1 & \mathbf{p}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \\ 1 \end{bmatrix} = \mathbf{P} \bullet \mathbf{v}$$

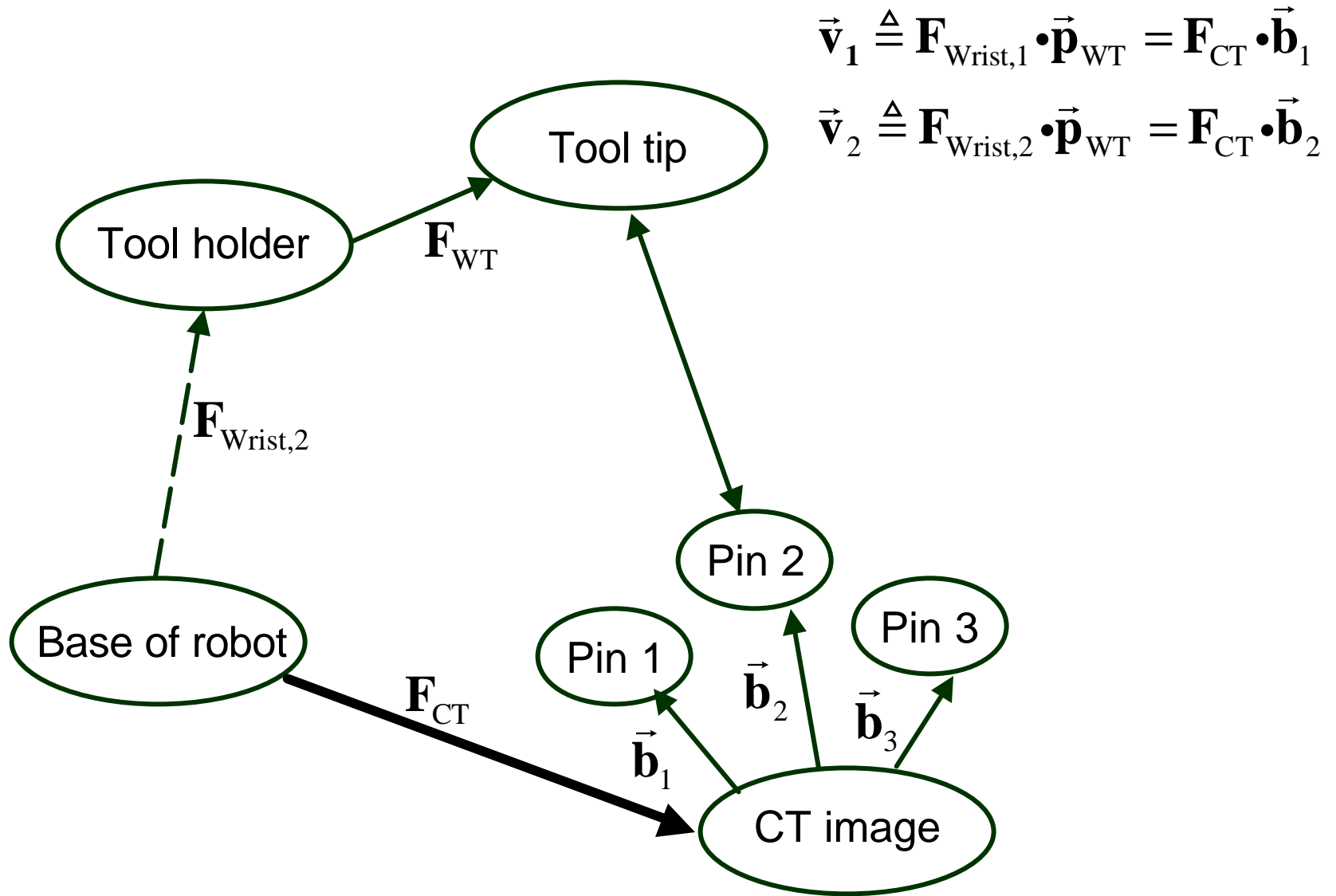
$$\mathbf{R} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$$

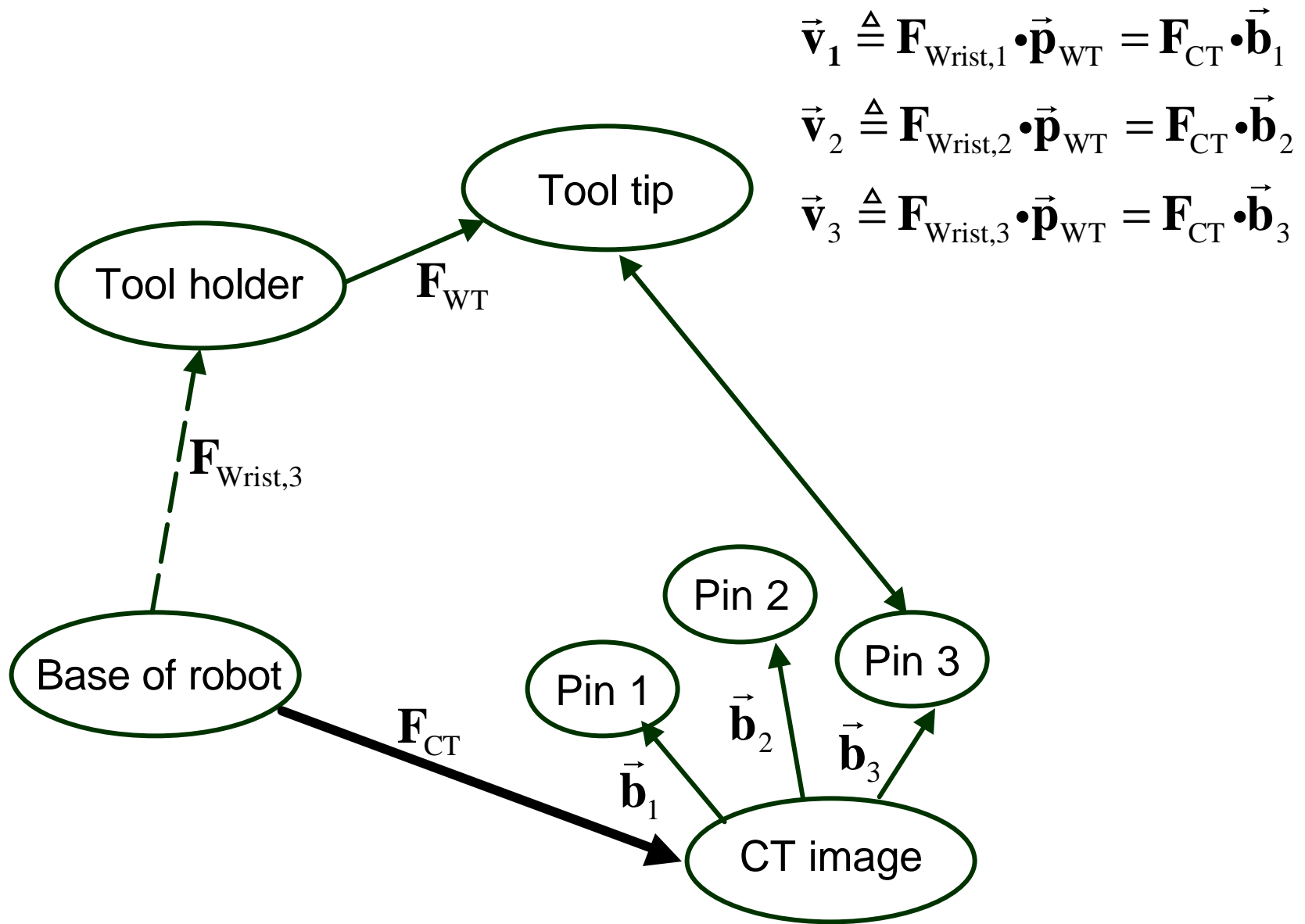
$$\mathbf{P} \bullet \mathbf{R} \rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \bullet \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = [\mathbf{R}, \mathbf{p}] = \mathbf{F}$$

$$\mathbf{F} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} (\mathbf{R} \bullet \mathbf{v}) + \mathbf{p} \\ 1 \end{bmatrix}$$

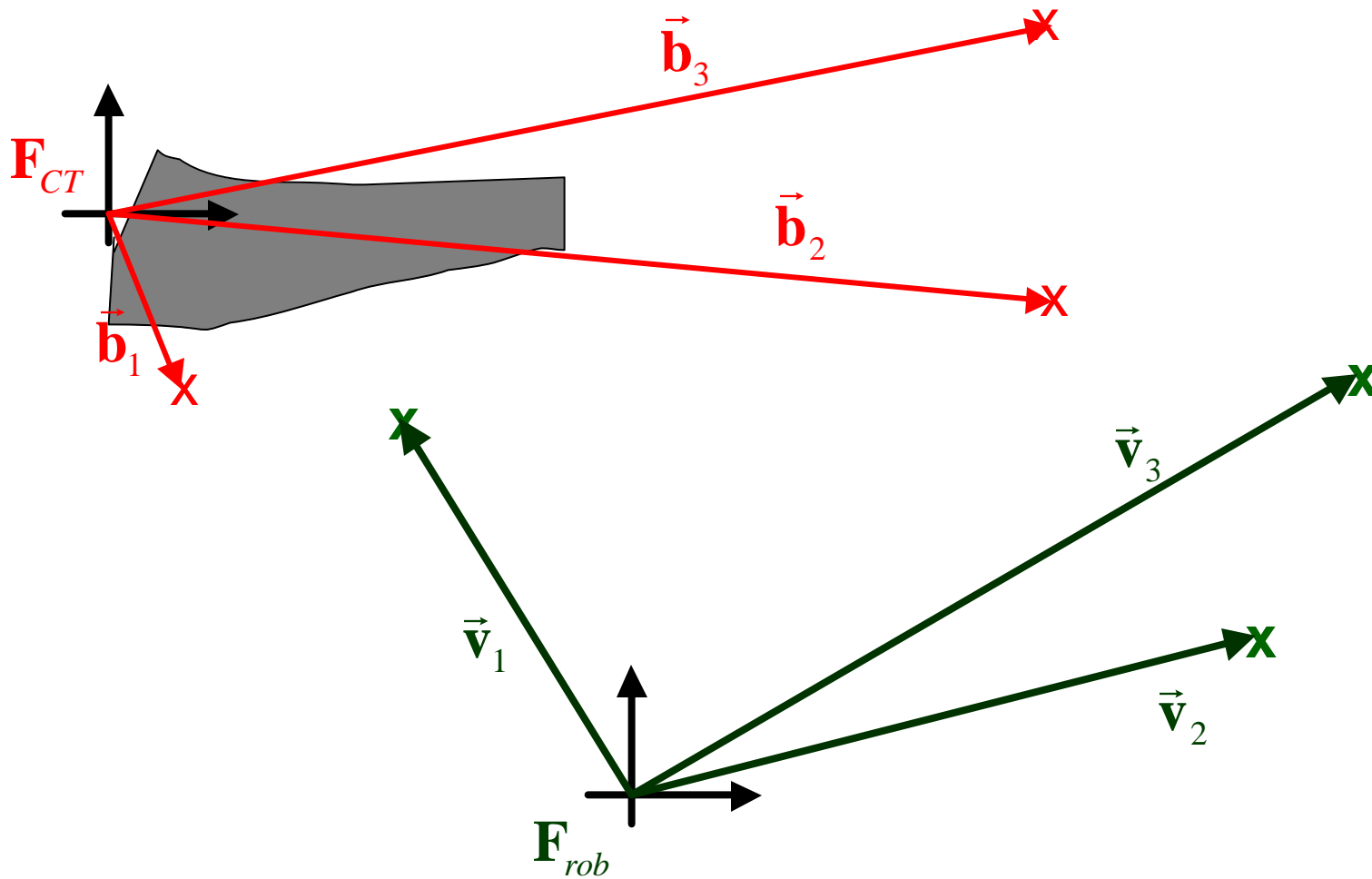




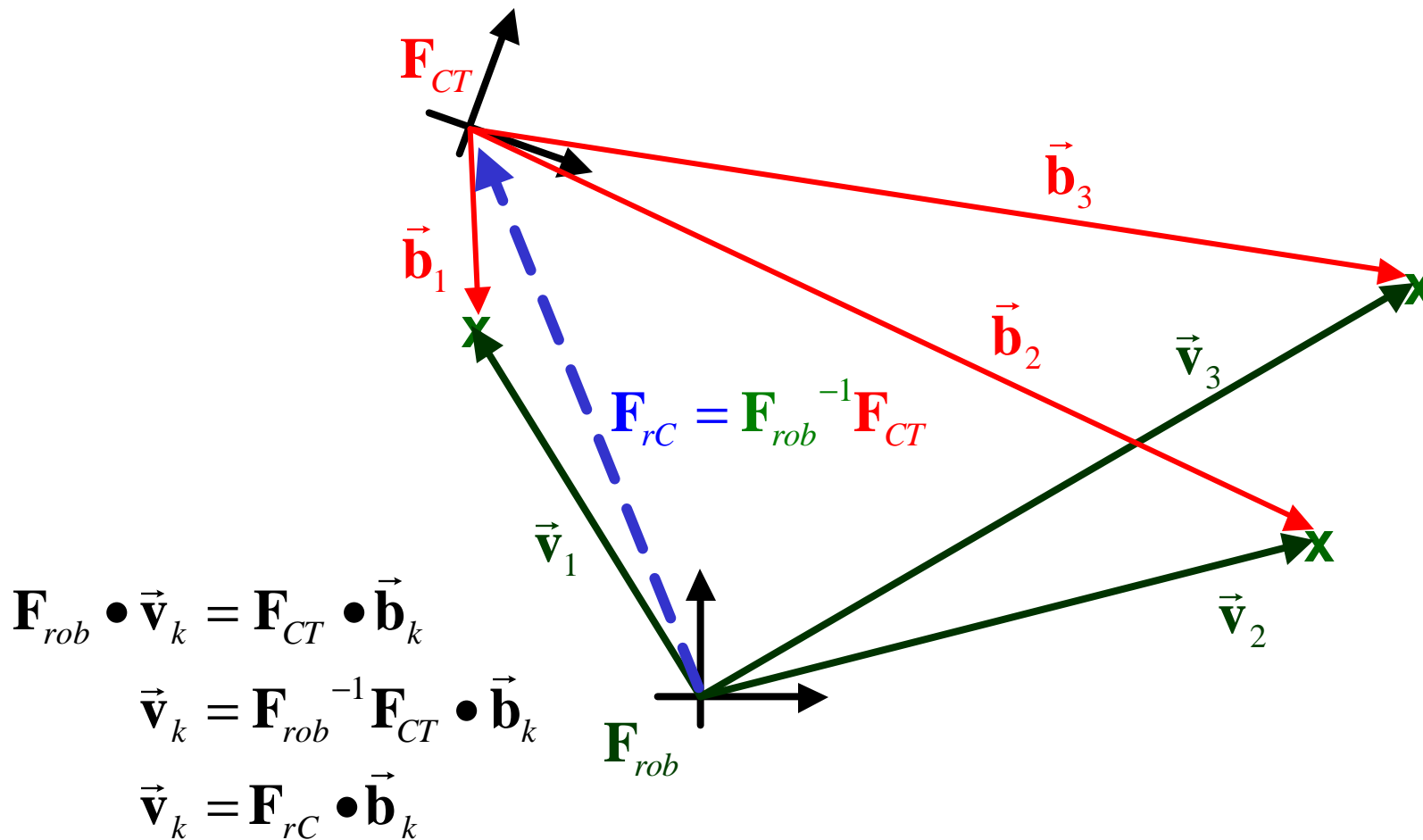




Frame transformation from 3 point pairs



Frame transformation from 3 point pairs



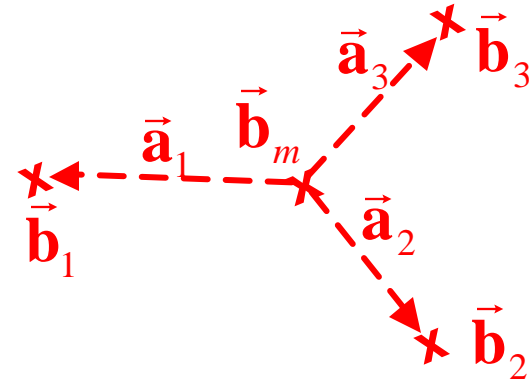
Frame transformation from 3 point pairs

$$\vec{v}_k = \mathbf{F}_{rC} \vec{b}_k = \mathbf{R}_{rC} \vec{b}_k + \vec{p}_{rC}$$

Define

$$\vec{v}_m = \frac{1}{3} \sum_1^3 \vec{v}_k \quad \vec{b}_m = \frac{1}{3} \sum_1^3 \vec{b}_k$$

$$\vec{u}_k = \vec{v}_k - \vec{v}_m \quad \vec{a}_k = \vec{b}_k - \vec{b}_m$$



$$\mathbf{F}_{rC} \vec{a}_k = \mathbf{R}_{rC} \vec{a}_k + \vec{p}_{rC}$$

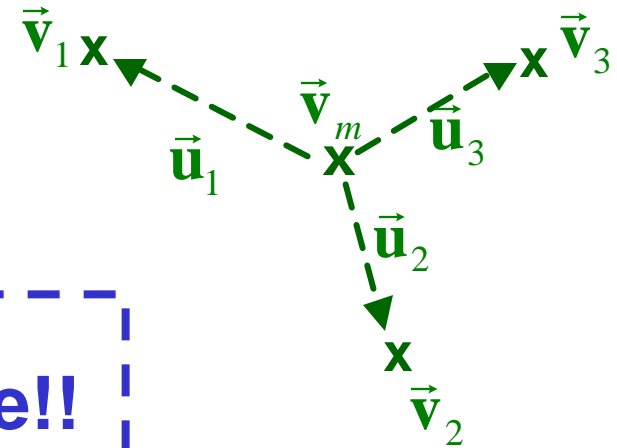
$$\mathbf{R}_{rC} \vec{a}_k + \vec{p}_{rC} = \mathbf{R}_{rC} (\vec{b}_k - \vec{b}_m) + \vec{p}_{rC}$$

$$\mathbf{R}_{rC} \vec{a}_k = \mathbf{R}_{rC} \vec{b}_k + \vec{p}_{rC} - \mathbf{R}_{rC} \vec{b}_m - \vec{p}_{rC}$$

$$\mathbf{R}_{rC} \vec{a}_k = \vec{v}_k - \vec{v}_m = \vec{u}_k$$

$$\vec{p}_{rC} = \vec{u}_m - \mathbf{R}_{rC} \vec{b}_m$$

Solve These!!



Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{\mathbf{a}}_k = \vec{\mathbf{u}}_k$ for $k = 1, \dots, n$ the problem is to estimate \mathbf{R} . This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = [\vec{\mathbf{u}}_1 \ \dots \ \vec{\mathbf{u}}_n]$ and $\mathbf{A} = [\vec{\mathbf{a}}_1 \ \dots \ \vec{\mathbf{a}}_n]$

Step 2: Solve the system $\mathbf{R}\mathbf{A} = \mathbf{U}$ for \mathbf{R} . E.g., by $\mathbf{R} = \mathbf{U}\mathbf{A}^{-1}$

Step 3: Renormalize \mathbf{R} to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = \left[\vec{\mathbf{r}}_x \mid \vec{\mathbf{r}}_y \mid \vec{\mathbf{r}}_z \right]$, modify it so $\mathbf{R}^T \mathbf{R} = \mathbf{I}$.

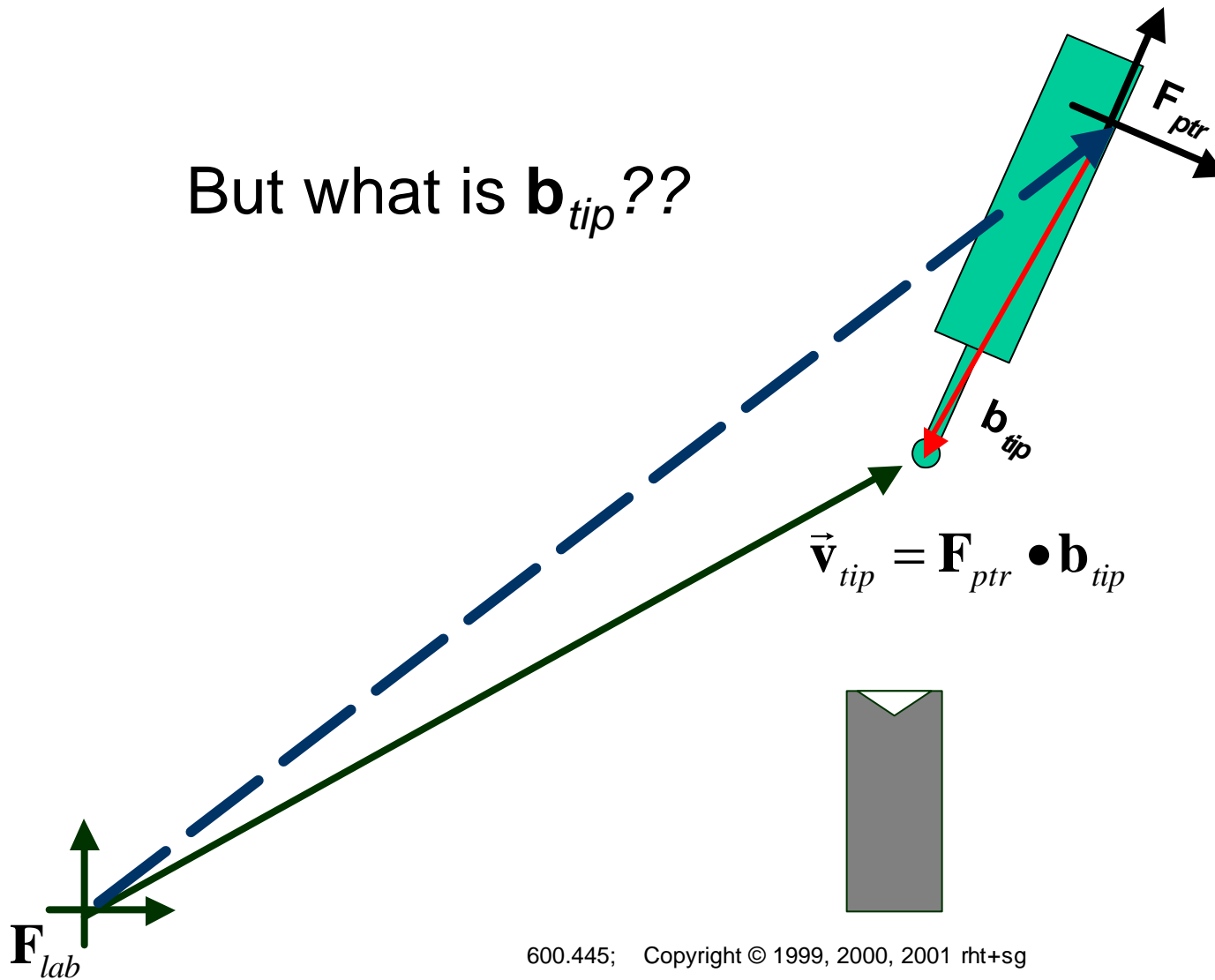
Step 1: $\vec{\mathbf{a}} = \vec{\mathbf{r}}_y \times \vec{\mathbf{r}}_z$

Step 2: $\vec{\mathbf{b}} = \vec{\mathbf{r}}_z \times \vec{\mathbf{a}}$

Step 3: $\mathbf{R}_{normalized} = \left[\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \mid \frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \mid \frac{\vec{\mathbf{r}}_z}{\|\vec{\mathbf{r}}_z\|} \right]$

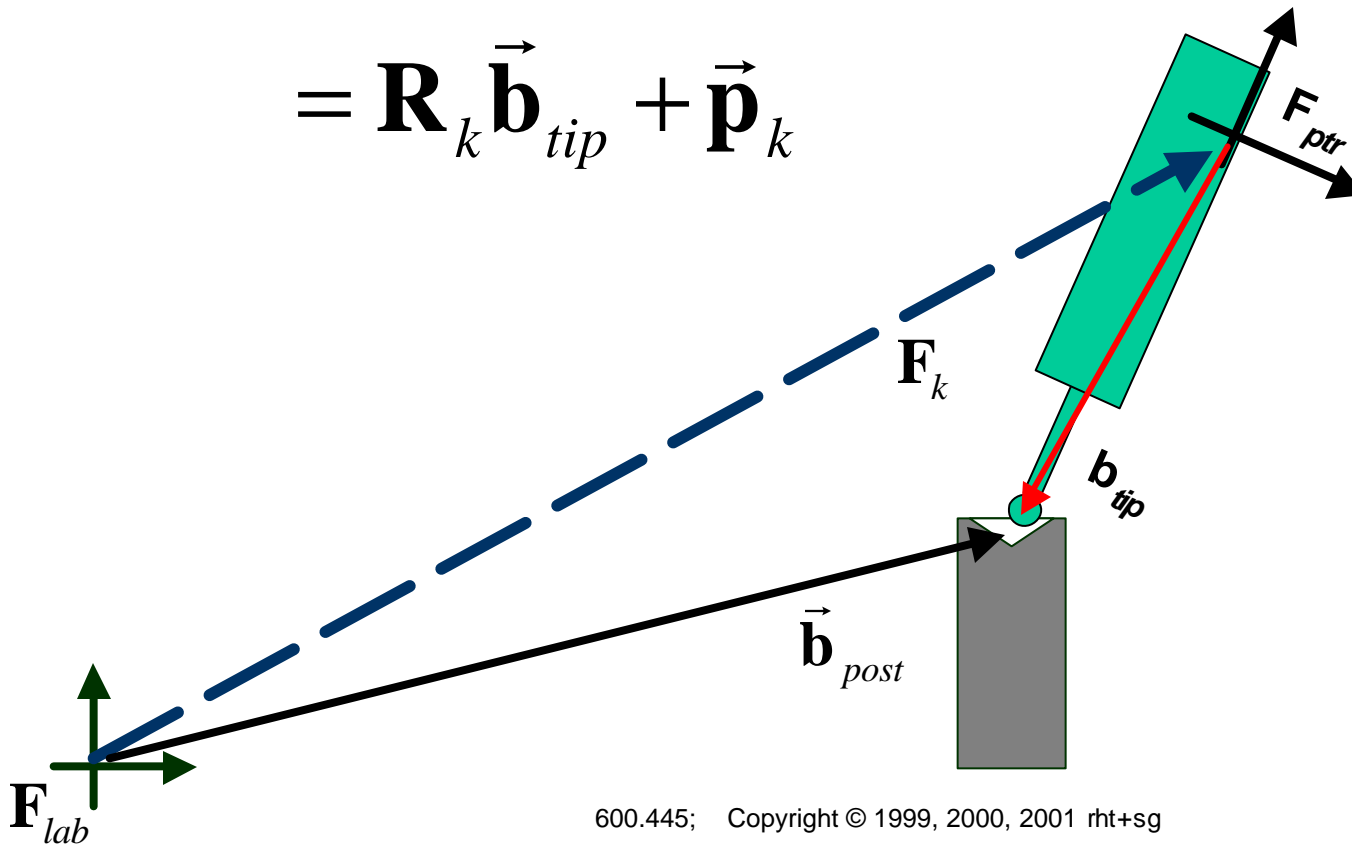
Calibrating a pointer

But what is \mathbf{b}_{tip} ??



Calibrating a pointer

$$\begin{aligned}\vec{\mathbf{b}}_{post} &= \mathbf{F}_k \vec{\mathbf{b}}_{tip} \\ &= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k\end{aligned}$$



Calibrating a pointer

For each measurement k , we have

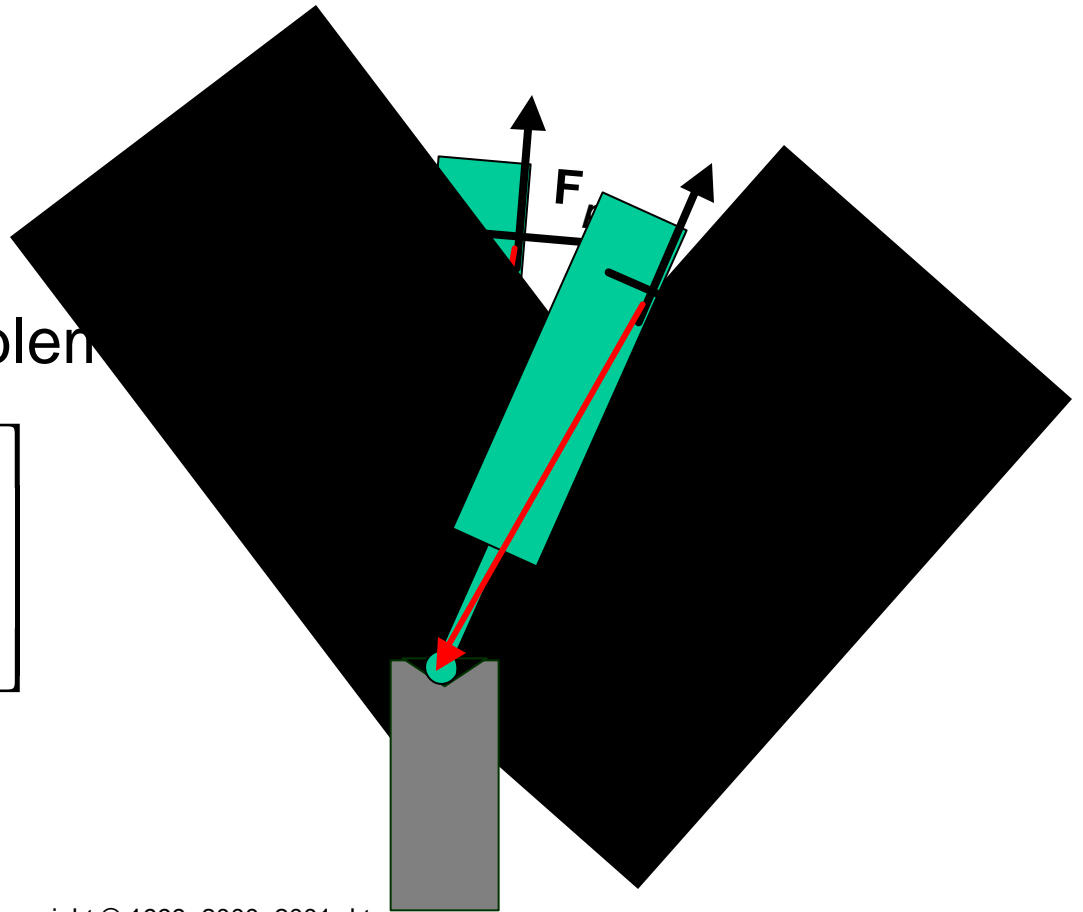
$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

i. e.,

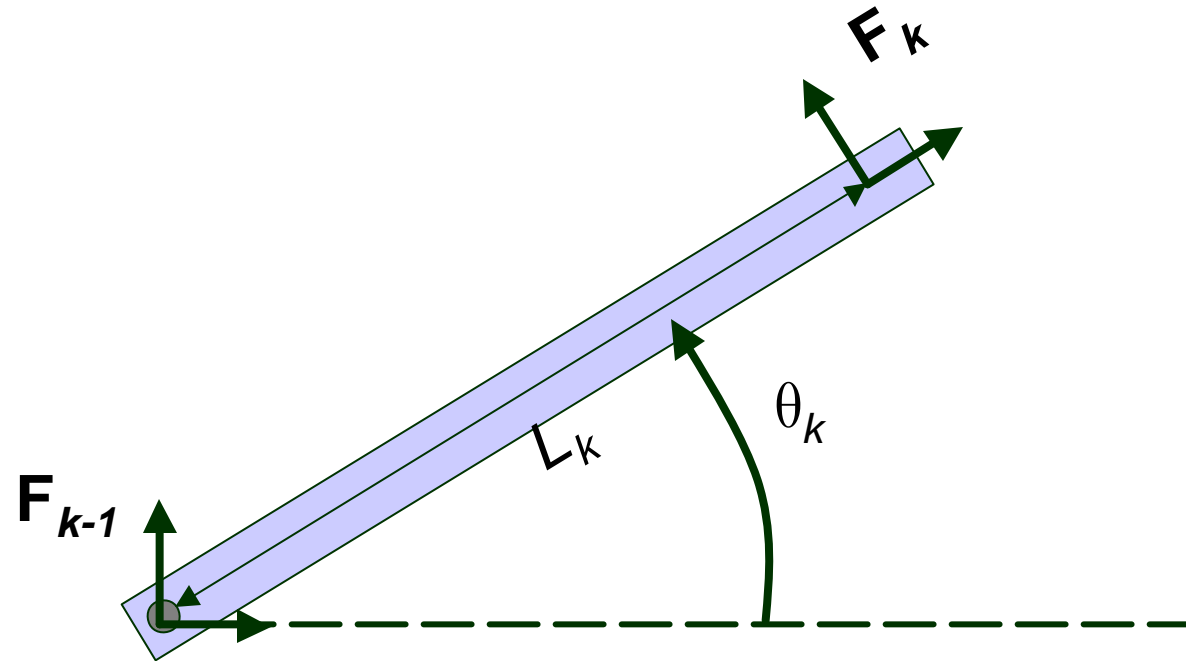
$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\begin{bmatrix} \vdots & \vdots \\ \hline \mathbf{R}_k & -\mathbf{I} \\ \hline \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{tip} \\ \vec{\mathbf{b}}_{post} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{bmatrix}$$



Kinematic Links

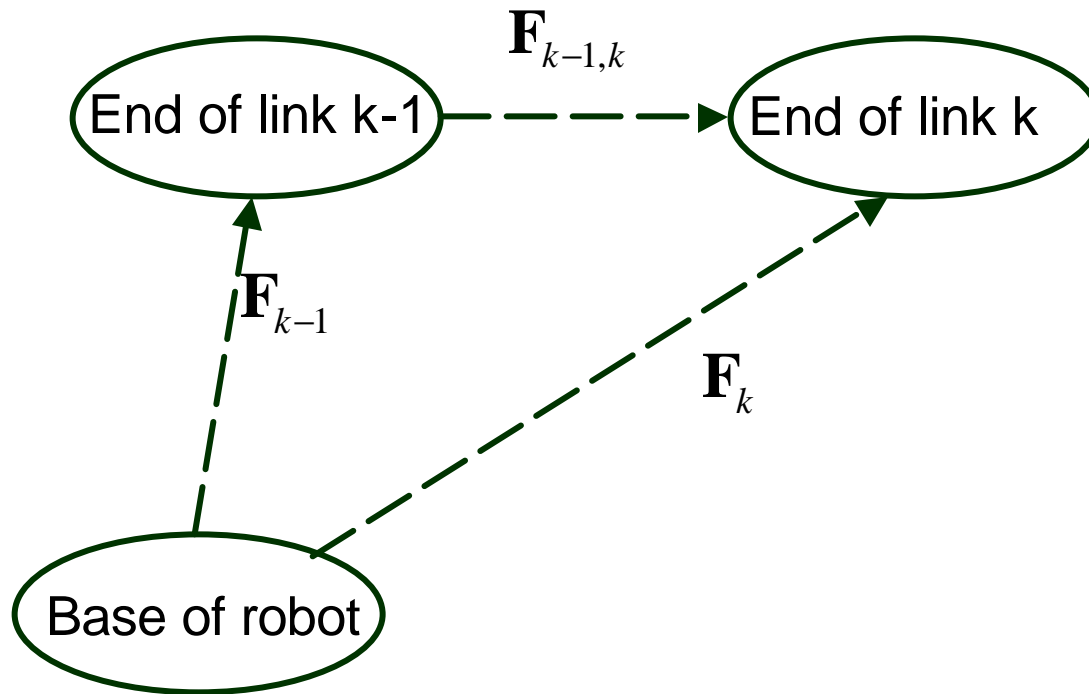


$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$[\mathbf{R}_k, \vec{\mathbf{p}}_k] = [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}]$$

$$= [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [Rot(\vec{\mathbf{r}}_k, \mathbf{q}_k), L_k Rot(\vec{\mathbf{r}}_k, \mathbf{q}_k) \bullet \vec{\mathbf{x}}]$$

Kinematic Links



$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$[\mathbf{R}_k, \vec{\mathbf{p}}_k] = [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}]$$

$$= [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [Rot(\vec{\mathbf{r}}_k, \mathbf{q}_k), L_k Rot(\vec{\mathbf{r}}_k, \mathbf{q}_k) \bullet \vec{\mathbf{x}}]$$

Kinematic Chains

$$\mathbf{F}_0 = [\mathbf{I}, \vec{\mathbf{0}}]$$

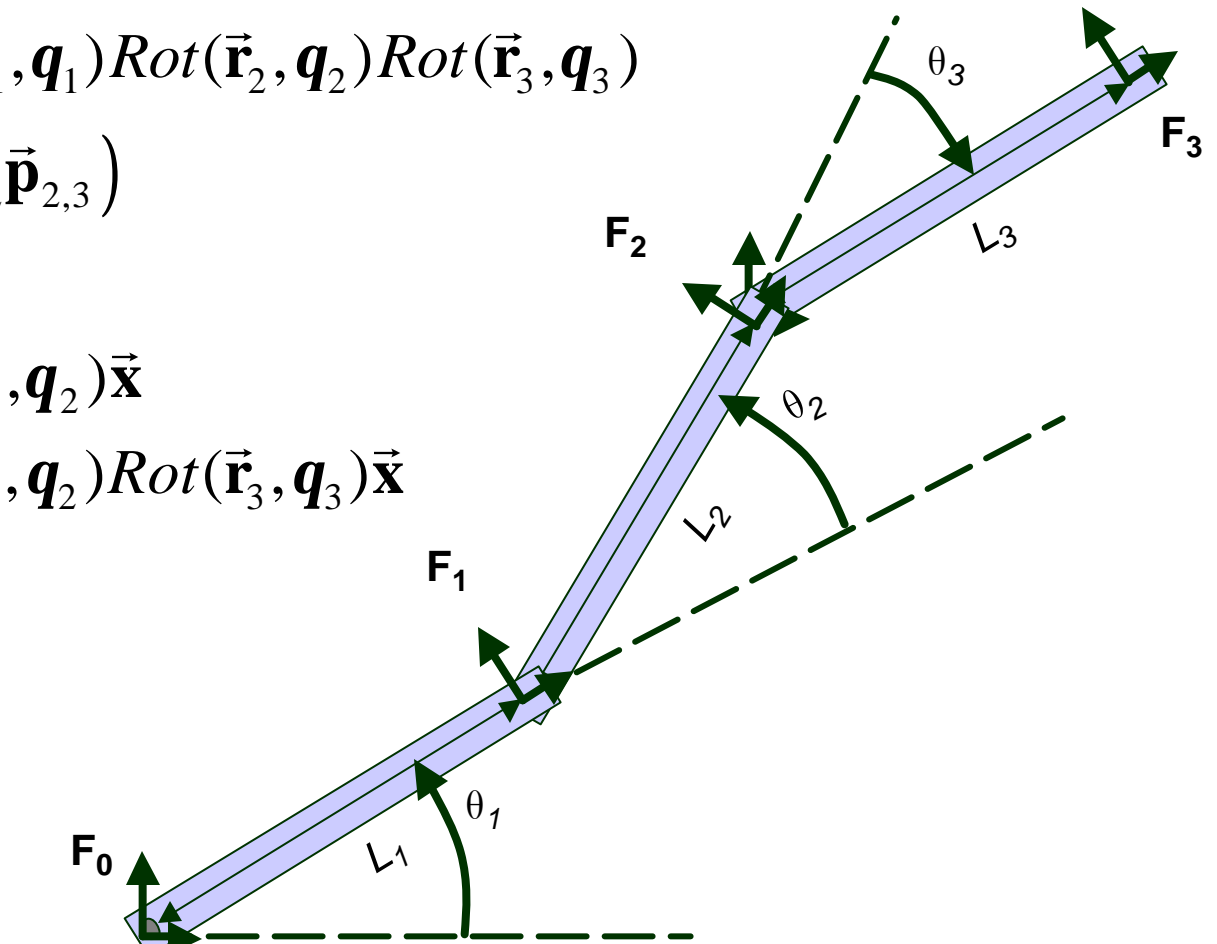
$$\mathbf{R}_3 = \mathbf{R}_{0,1} \mathbf{R}_{1,2} \mathbf{R}_{2,3} = \text{Rot}(\vec{\mathbf{r}}_1, \mathbf{q}_1) \text{Rot}(\vec{\mathbf{r}}_2, \mathbf{q}_2) \text{Rot}(\vec{\mathbf{r}}_3, \mathbf{q}_3)$$

$$\vec{\mathbf{p}}_3 = \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1} (\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2} \vec{\mathbf{p}}_{2,3})$$

$$= L_1 \text{Rot}(\vec{\mathbf{r}}_1, \mathbf{q}_1) \vec{\mathbf{x}}$$

$$+ L_2 \text{Rot}(\vec{\mathbf{r}}_1, \mathbf{q}_1) \text{Rot}(\vec{\mathbf{r}}_2, \mathbf{q}_2) \vec{\mathbf{x}}$$

$$+ L_3 \text{Rot}(\vec{\mathbf{r}}_1, \mathbf{q}_1) \text{Rot}(\vec{\mathbf{r}}_2, \mathbf{q}_2) \text{Rot}(\vec{\mathbf{r}}_3, \mathbf{q}_3) \vec{\mathbf{x}}$$



Kinematic Chains

$$\text{If } \vec{r}_1 = \vec{r}_2 = \vec{r}_3 = \vec{z},$$

$$\mathbf{R}_3 = \text{Rot}(\vec{z}, \mathbf{q}_1) \text{Rot}(\vec{z}, \mathbf{q}_2) \text{Rot}(\vec{z}, \mathbf{q}_3)$$

$$= \text{Rot}(\vec{z}, \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)$$

$$\vec{p}_3 = \vec{p}_{0,1} + \mathbf{R}_{0,1} (\vec{p}_{1,2} + \mathbf{R}_{1,2} \vec{p}_{2,3})$$

$$= L_1 \text{Rot}(\vec{z}, \mathbf{q}_1) \vec{x}$$

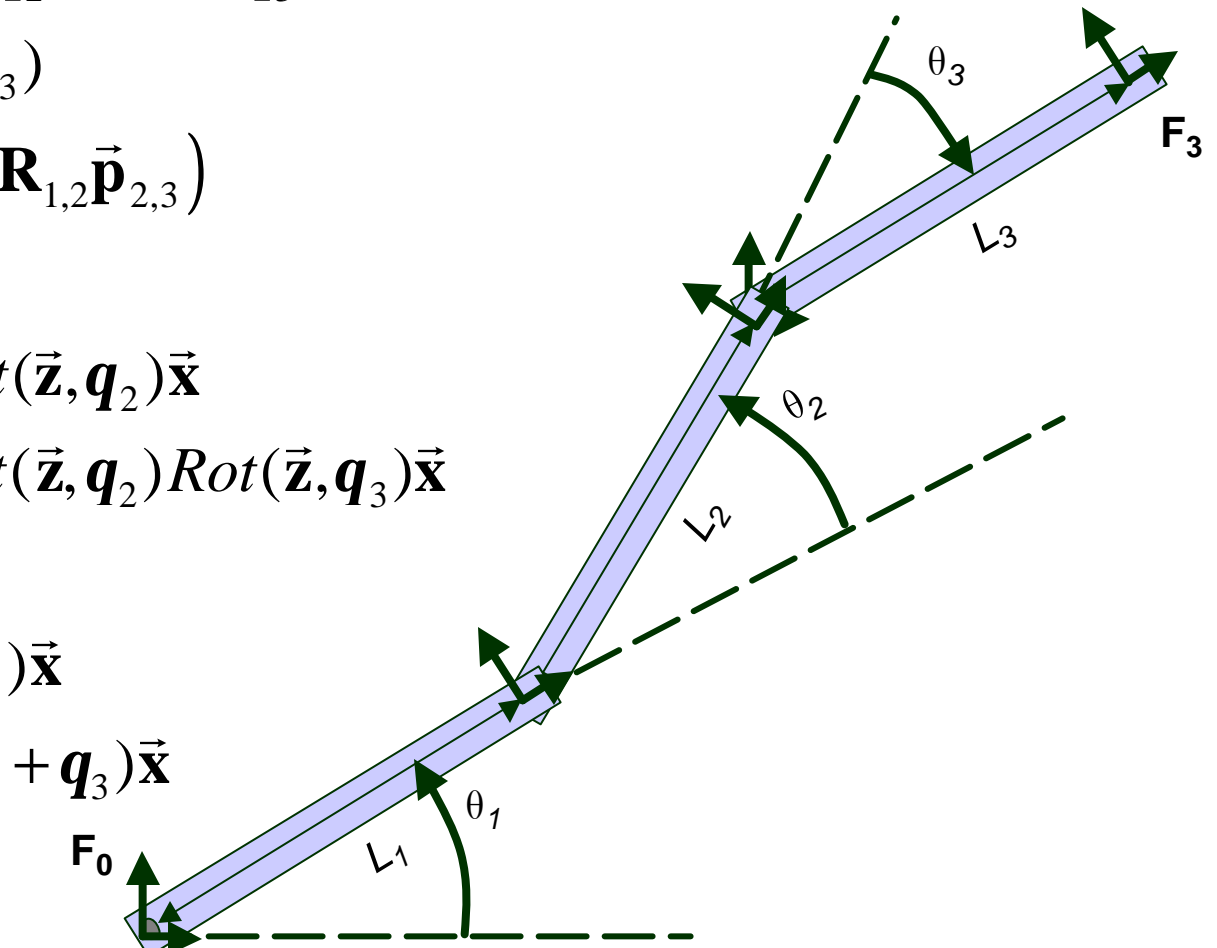
$$+ L_2 \text{Rot}(\vec{z}, \mathbf{q}_1) \text{Rot}(\vec{z}, \mathbf{q}_2) \vec{x}$$

$$+ L_3 \text{Rot}(\vec{z}, \mathbf{q}_1) \text{Rot}(\vec{z}, \mathbf{q}_2) \text{Rot}(\vec{z}, \mathbf{q}_3) \vec{x}$$

$$= L_1 \text{Rot}(\vec{z}, \mathbf{q}_1) \vec{x}$$

$$+ L_2 \text{Rot}(\vec{z}, \mathbf{q}_1 + \mathbf{q}_2) \vec{x}$$

$$+ L_3 \text{Rot}(\vec{z}, \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \vec{x}$$

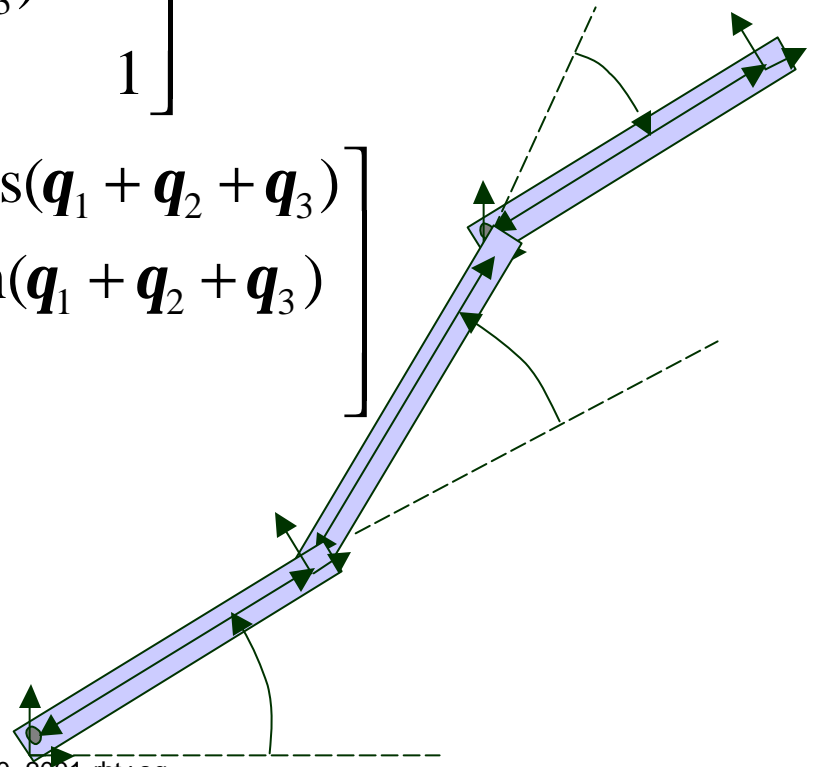


Kinematic Chains

If $\vec{r}_1 = \vec{r}_2 = \vec{r}_3 = \vec{z}$,

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) & -\sin(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) & 0 \\ \sin(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) & \cos(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\vec{p}_3 = \begin{bmatrix} L_1 \cos(\mathbf{q}_1) + L_2 \cos(\mathbf{q}_1 + \mathbf{q}_2) + L_3 \cos(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \\ L_1 \sin(\mathbf{q}_1) + L_2 \sin(\mathbf{q}_1 + \mathbf{q}_2) + L_3 \sin(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \\ 0 \end{bmatrix}$$



“Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

Small Rotations

$\Delta\mathbf{R} \triangleq$ a small rotation

$\mathbf{R}_{\vec{a}}(\Delta\mathbf{a}) \triangleq$ a rotation by a small angle $\Delta\mathbf{a}$ about axis \vec{a}

$\text{Rot}(\vec{a}, \|\vec{a}\|) \bullet \vec{b} \approx \vec{a} \times \vec{b} + \vec{b}$ for $\|\vec{a}\|$ sufficiently small

$\Delta\mathbf{R}(\vec{a}) \triangleq$ a rotation that is small enough for this approximation

$\Delta\mathbf{R}(l\vec{a}) \bullet \Delta\mathbf{R}(m\vec{b}) \cong \Delta\mathbf{R}(l\vec{a} + m\vec{b})$ (Linearity for small rotations)

Exercise: Work out the linearity proposition by substitution

Approximations to “Small” Frames

$$\Delta\mathbf{F}(\vec{\mathbf{a}}, \Delta\vec{\mathbf{p}}) \triangleq [\Delta\mathbf{R}(\vec{\mathbf{a}}), \Delta\vec{\mathbf{p}}]$$

$$\begin{aligned}\Delta\mathbf{F}(\vec{\mathbf{a}}, \Delta\vec{\mathbf{p}}) \bullet \vec{\mathbf{v}} &= \Delta\mathbf{R}(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}} \\ &\approx \vec{\mathbf{v}} + \vec{\mathbf{a}} \times \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}\end{aligned}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = \mathit{skew}(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

$$\triangleq \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \bullet \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}}) \approx \mathbf{I} + \mathit{skew}(\vec{\mathbf{a}})$$

$$\mathit{skew}(\vec{\mathbf{a}}) \bullet \vec{\mathbf{a}} = \vec{\mathbf{0}}$$

Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

$$\mathbf{F}_{\text{actual}} = \mathbf{F}_{\text{nominal}} \bullet \Delta\mathbf{F}$$

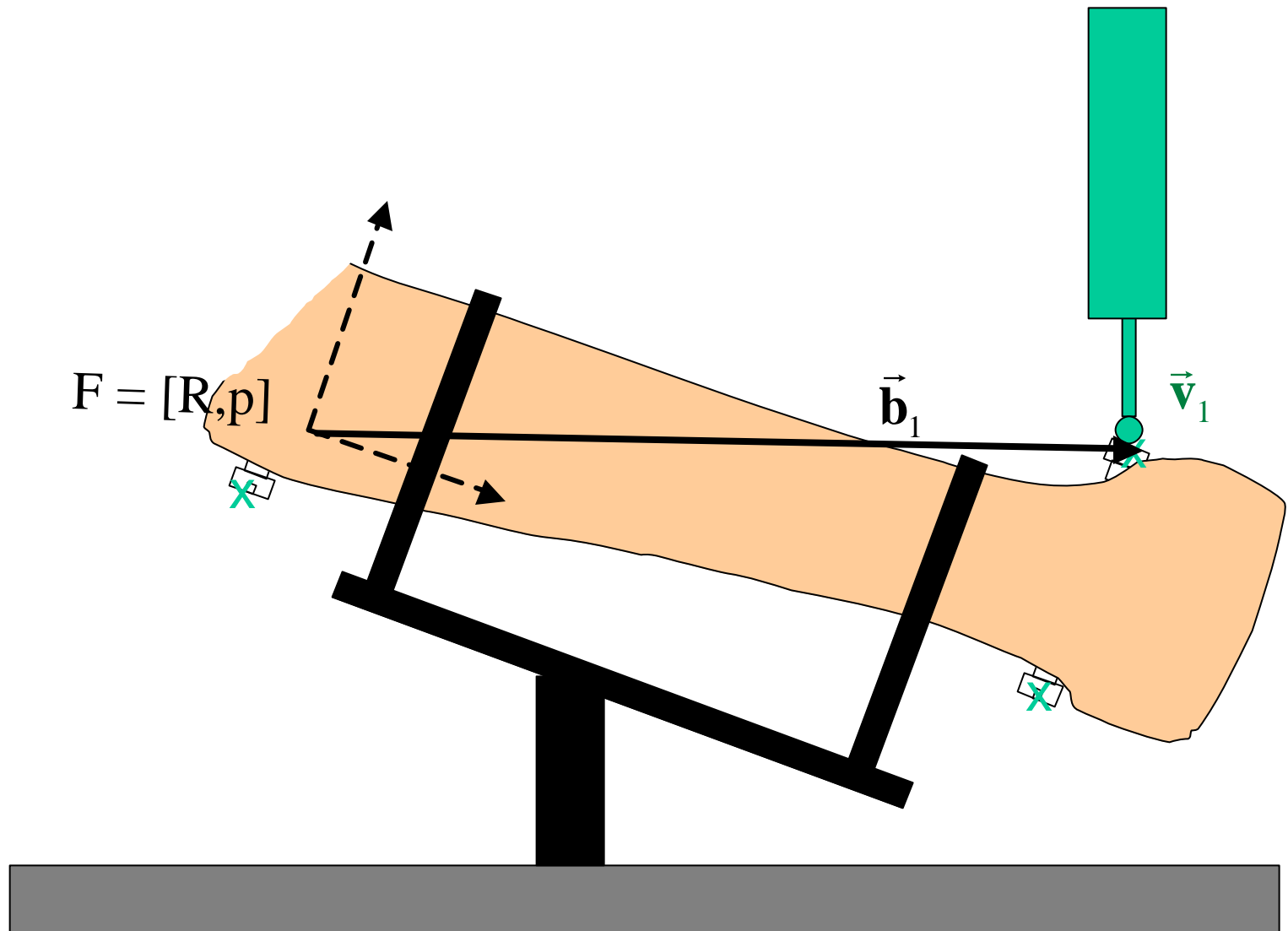
Often, we will use the notation \mathbf{F}^* for $\mathbf{F}_{\text{actual}}$ and will just use \mathbf{F} for $\mathbf{F}_{\text{nominal}}$. Thus we may write something like

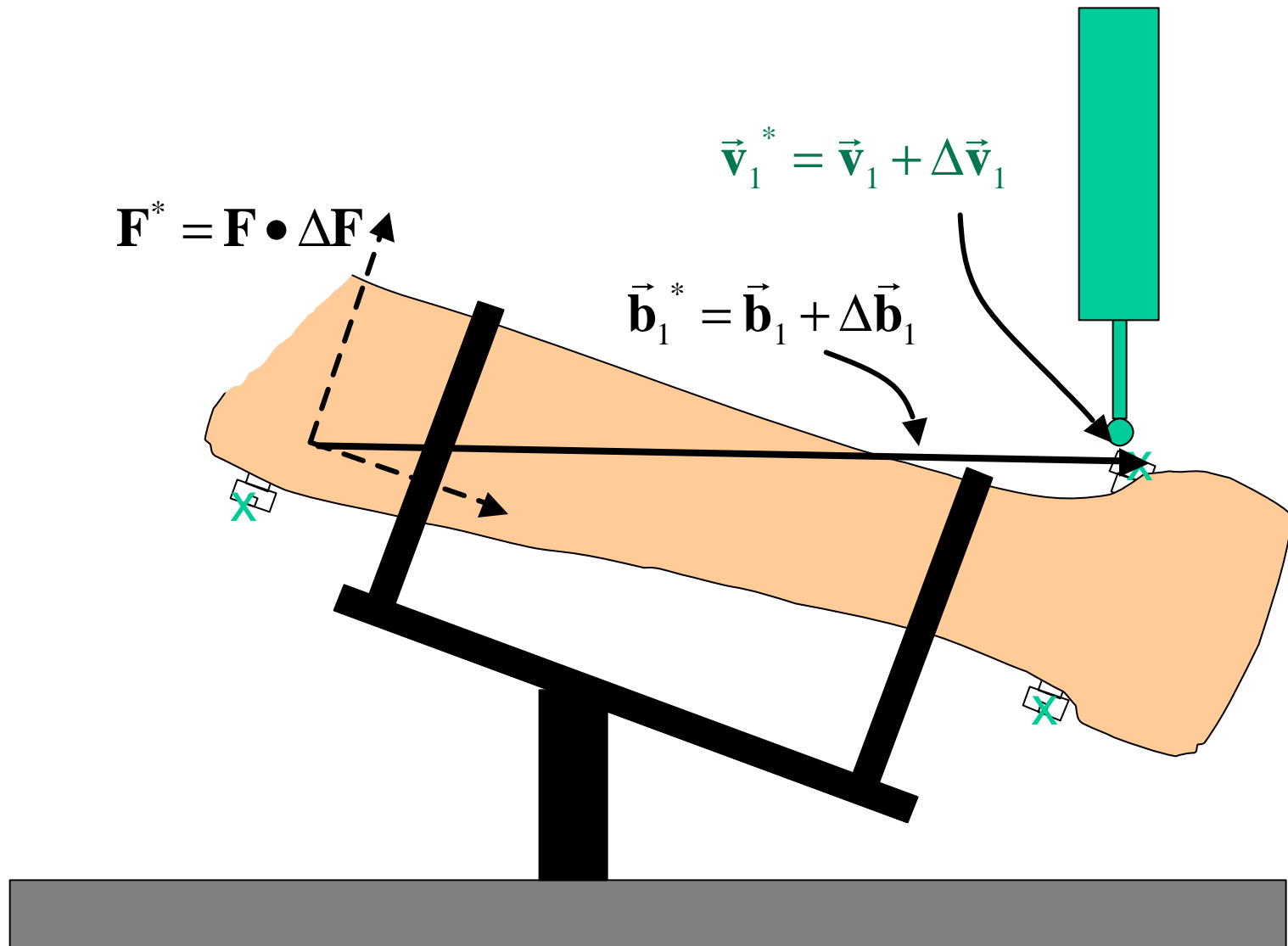
$$\mathbf{F}^* = \mathbf{F} \bullet \Delta\mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta\mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}$, *etc.*

Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\begin{aligned}\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\ &= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}})\end{aligned}$$



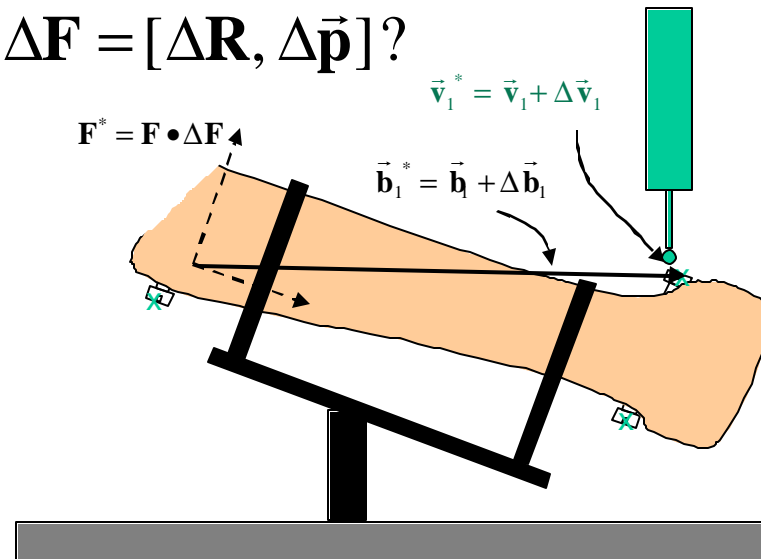


Errors & Sensitivity

Suppose that we know nominal values for \mathbf{F} , $\bar{\mathbf{b}}$, and $\bar{\mathbf{v}}$ and that

$$[-\mathbf{e}, -\mathbf{e}, -\mathbf{e}]^T \leq \Delta\bar{\mathbf{v}}_1 \leq [\mathbf{e}, \mathbf{e}, \mathbf{e}]^T$$

What does this tell us about $\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\bar{\mathbf{p}}]$?



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Errors & Sensitivity

$$\begin{aligned}\vec{v}^* &= \mathbf{F}^* \bullet \vec{b}^* \\ &= \mathbf{F} \bullet \Delta \mathbf{F} \bullet (\vec{b} + \Delta \vec{b}) \\ &= \mathbf{R} \bullet \left(\Delta \mathbf{R}(\vec{a}) \bullet (\vec{b} + \Delta \vec{b}) + \Delta \vec{p} \right) + \vec{p} \\ &\cong \mathbf{R} \bullet \left(\vec{b} + \Delta \vec{b} + \vec{a} \times \vec{b} + \vec{a} \times \Delta \vec{b} + \Delta \vec{p} \right) + \vec{p} \\ &= \mathbf{R} \bullet \vec{b} + \vec{p} + \mathbf{R} \bullet \left(\Delta \vec{b} + \vec{a} \times \vec{b} + \vec{a} \times \Delta \vec{b} + \Delta \vec{p} \right) \\ &\cong \vec{v} + \mathbf{R} \bullet \left(\Delta \vec{b} + \vec{a} \times \vec{b} + \Delta \vec{p} \right)\end{aligned}$$

if $\|\vec{a} \times \Delta \vec{b}\| \leq \|\vec{a}\| \|\Delta \vec{b}\|$ is negligible (it usually is)

so

$$\Delta \vec{v} = \vec{v}^* - \vec{v} \cong \mathbf{R} \bullet \left(\Delta \vec{b} + \vec{a} \times \vec{b} + \Delta \vec{p} \right) = \mathbf{R} \bullet \Delta \vec{b} + \mathbf{R} \bullet \vec{a} \times \vec{b} + \mathbf{R} \bullet \Delta \vec{p}$$

Digression: “rotation triple product”

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is linear in $\vec{\mathbf{a}}$, but is not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$\begin{aligned}\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\ &= \mathbf{R} \bullet \mathit{skew}(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\ &= \left[\mathbf{R} \bullet \mathit{skew}(\vec{\mathbf{b}})^T \right] \bullet \vec{\mathbf{a}}\end{aligned}$$

Errors & Sensitivity

Previous expression was

$$\Delta \vec{v}_1 \cong \mathbf{R} \bullet (\Delta \vec{b}_1 + \vec{a} \times \vec{b} + \Delta \vec{p}_1)$$

Substituting triple product and rearranging gives

$$\Delta \vec{v}_1 \cong \left[\mathbf{R} \mid \mathbf{R} \mid \mathbf{R} \bullet skew(-\vec{b}) \right] \bullet \begin{bmatrix} \Delta \vec{b}_1 \\ \Delta \vec{p} \\ \vec{a} \end{bmatrix}$$

So

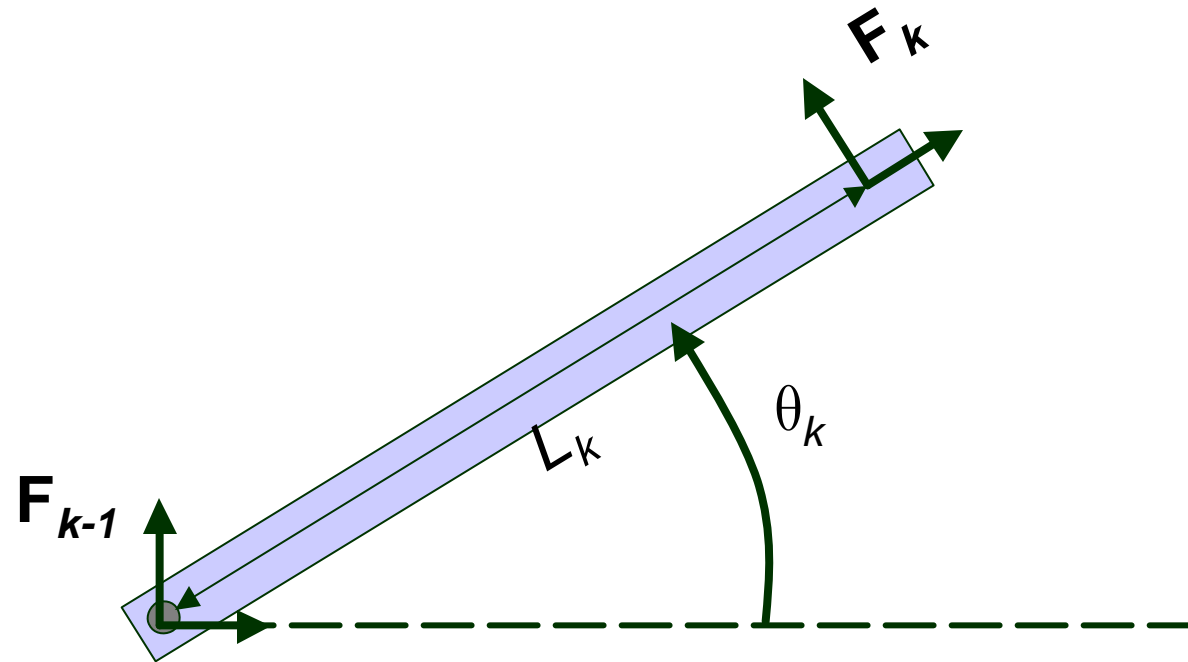
$$\begin{bmatrix} -e \\ -e \\ -e \end{bmatrix} \leq \left[\mathbf{R} \mid \mathbf{R} \mid \mathbf{R} \bullet skew(-\vec{b}) \right] \begin{bmatrix} \Delta \vec{b}_1 \\ \Delta \vec{p} \\ \vec{a} \end{bmatrix} \leq \begin{bmatrix} -e \\ -e \\ -e \end{bmatrix}$$

Errors & Sensitivity

Now, suppose we know that $|\Delta\vec{\mathbf{b}}_1| \leq \mathbf{b}$, this will give us a system of linear constraints

$$\begin{bmatrix} -\mathbf{e} \\ -\mathbf{e} \\ -\mathbf{e} \\ -\mathbf{b} \\ -\mathbf{b} \\ -\mathbf{b} \end{bmatrix} \leq \begin{bmatrix} \mathbf{R} & \mathbf{R} & \mathbf{R} \bullet \mathit{skew}(-\vec{\mathbf{b}}) \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}}_1 \\ \vec{\mathbf{a}} \end{bmatrix} \leq \begin{bmatrix} \mathbf{e} \\ \mathbf{e} \\ \mathbf{e} \\ \mathbf{b} \\ \mathbf{b} \\ \mathbf{b} \end{bmatrix}$$

Error Propagation in Chains



$$\mathbf{F}_k^* = \mathbf{F}_{k-1}^* \bullet \mathbf{F}_{k-1,k}^*$$

$$\mathbf{F}_k \Delta \mathbf{F}_k = \mathbf{F}_{k-1} \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta \mathbf{F}_{k-1,k}$$

$$\Delta \mathbf{F}_k = \left(\mathbf{F}_k^{-1} \mathbf{F}_{k-1} \right) \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \Delta \mathbf{F}_{k-1,k}$$

$$= \left(\mathbf{F}_{k-1,k}^{-1} \Delta \mathbf{F}_{k-1} \mathbf{F}_{k-1,k} \right) \Delta \mathbf{F}_{k-1,k}$$

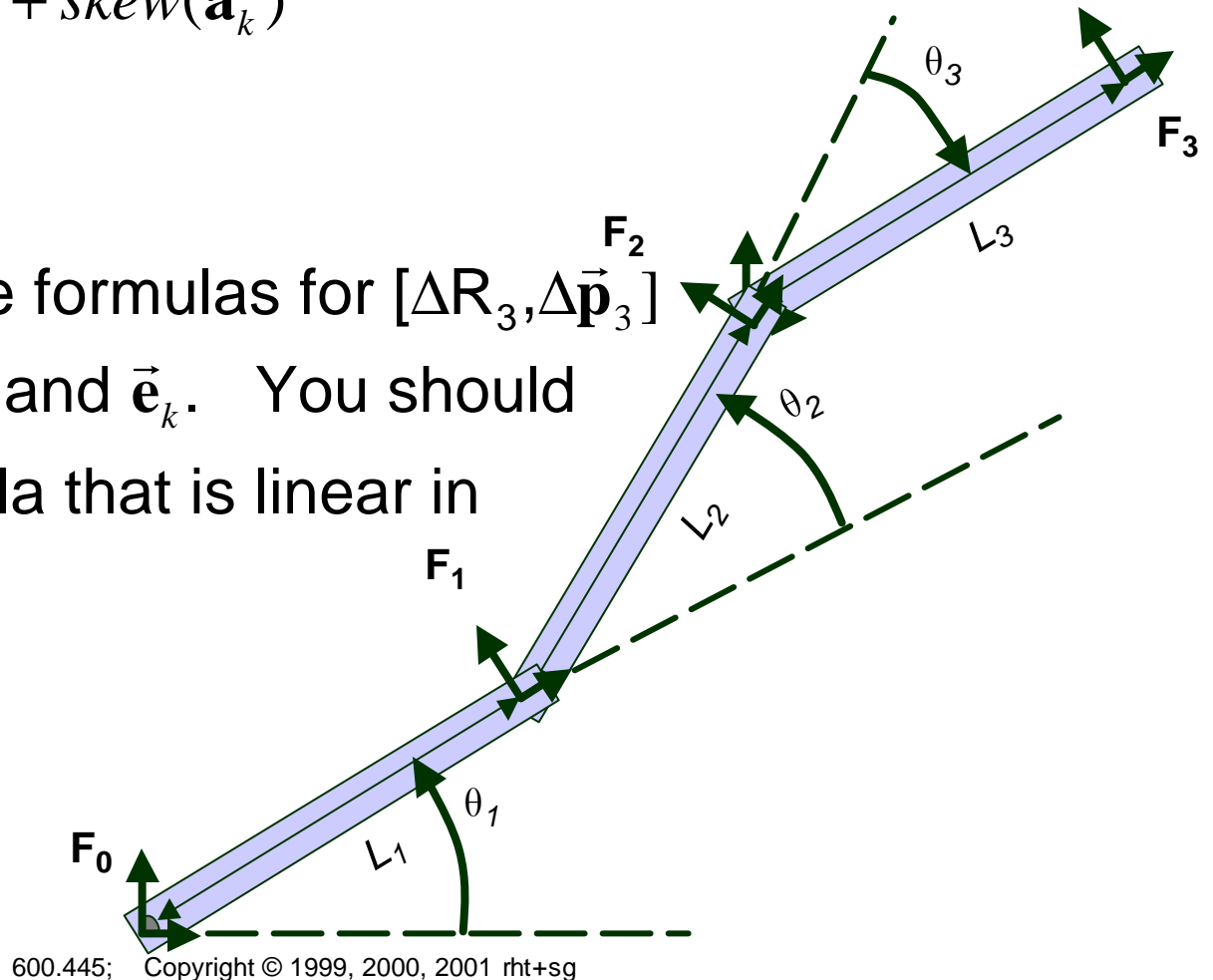
Exercise

Suppose that you have

$$\Delta \mathbf{R}_{k-1,k} = \Delta \mathbf{R}(\vec{\mathbf{a}}_k) \cong \mathbf{I} + \text{skew}(\vec{\mathbf{a}}_k)$$

$$\Delta \vec{\mathbf{p}}_{k-1,k} = \vec{\mathbf{e}}_k$$

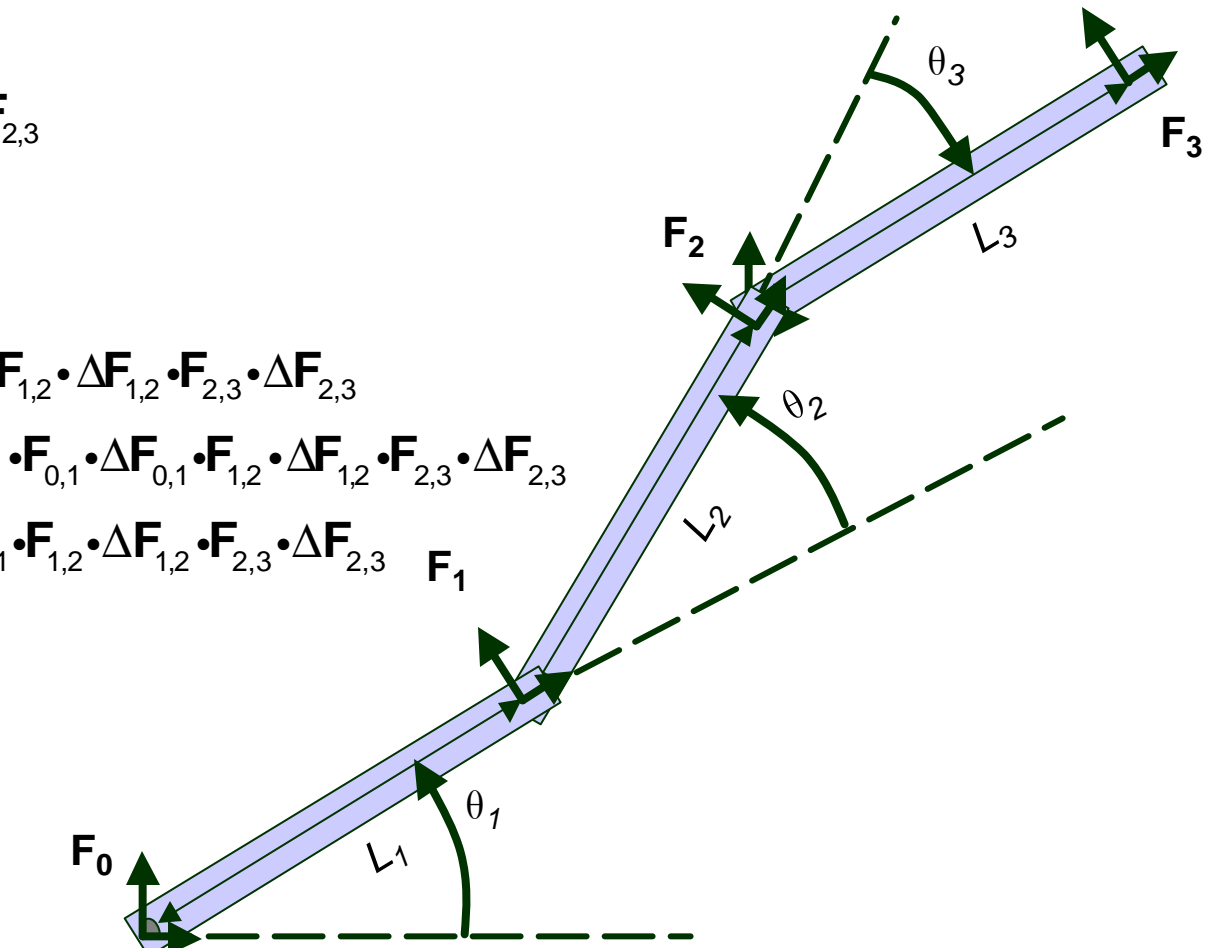
Work out approximate formulas for $[\Delta \mathbf{R}_3, \Delta \vec{\mathbf{p}}_3]$ in terms of $L_k, \vec{\mathbf{r}}_k, \mathbf{q}_k, \vec{\mathbf{a}}_k$ and $\vec{\mathbf{e}}_k$. You should come up with a formula that is linear in $L_k, \vec{\mathbf{a}}_k$, and $\vec{\mathbf{e}}_k$.



Exercise

Suppose we want to know error in $\mathbf{F}_{0,3} = \mathbf{F}_0^{-1} \mathbf{F}_3$

$$\begin{aligned} \mathbf{F}_{0,3} &= \mathbf{F}_0^{-1} \mathbf{F}_0 \cdot \mathbf{F}_{0,1} \cdot \mathbf{F}_{1,2} \cdot \mathbf{F}_{2,3} \\ \mathbf{F}_{0,3}^* &= \mathbf{F}_{0,1}^* \cdot \mathbf{F}_{1,2}^* \cdot \mathbf{F}_{2,3}^* \\ \mathbf{F}_{0,3} \cdot \Delta \mathbf{F}_{0,3} &= \mathbf{F}_{0,1}^* \cdot \mathbf{F}_{1,2}^* \cdot \mathbf{F}_{2,3}^* \\ \Delta \mathbf{F}_3 &= \mathbf{F}_{0,3}^{-1} \cdot \mathbf{F}_{0,1} \cdot \Delta \mathbf{F}_{0,1} \cdot \mathbf{F}_{1,2} \cdot \Delta \mathbf{F}_{1,2} \cdot \mathbf{F}_{2,3} \cdot \Delta \mathbf{F}_{2,3} \\ &= \mathbf{F}_{2,3}^{-1} \cdot \mathbf{F}_{1,2}^{-1} \cdot \mathbf{F}_{0,1}^{-1} \cdot \mathbf{F}_{0,1} \cdot \Delta \mathbf{F}_{0,1} \cdot \mathbf{F}_{1,2} \cdot \Delta \mathbf{F}_{1,2} \cdot \mathbf{F}_{2,3} \cdot \Delta \mathbf{F}_{2,3} \\ &= \mathbf{F}_{2,3}^{-1} \cdot \mathbf{F}_{1,2}^{-1} \cdot \Delta \mathbf{F}_{0,1} \cdot \mathbf{F}_{1,2} \cdot \Delta \mathbf{F}_{1,2} \cdot \mathbf{F}_{2,3} \cdot \Delta \mathbf{F}_{2,3} \end{aligned}$$



Parametric Sensitivity

Suppose you have an explicit formula like

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1 \cos(\mathbf{q}_1) + L_2 \cos(\mathbf{q}_1 + \mathbf{q}_2) + L_3 \cos(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \\ L_1 \sin(\mathbf{q}_1) + L_2 \sin(\mathbf{q}_1 + \mathbf{q}_2) + L_3 \sin(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3) \\ 0 \end{bmatrix}$$

and know that the only variation is in parameters like L_k and \mathbf{q}_k . Then you can estimate the variation in $\vec{\mathbf{p}}_3$ as a function of variation in L_k and \mathbf{q}_k by remembering your calculus.

$$\Delta \vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial \vec{\mathbf{p}}_3}{\partial \vec{L}} & \frac{\partial \vec{\mathbf{p}}_3}{\partial \vec{\mathbf{q}}} \end{bmatrix} \begin{bmatrix} \Delta \vec{L} \\ \Delta \vec{\mathbf{q}} \end{bmatrix}$$

Parametric Sensitivity

Grinding this out gives:

$$\Delta \vec{p}_3 \cong \begin{bmatrix} \frac{\partial \vec{p}_3}{\partial \vec{L}} & \frac{\partial \vec{p}_3}{\partial \vec{q}} \end{bmatrix} \begin{bmatrix} \Delta \vec{L} \\ \Delta \vec{q} \end{bmatrix}$$

where

$$\vec{L} = [L_1, L_2, L_3]^T$$

$$\vec{q} = [q_1, q_2, q_3]^T$$

$$\frac{\partial \vec{p}_3}{\partial \vec{L}} = \begin{bmatrix} \cos(q_1) & \cos(q_1 + q_2) & \cos(q_1 + q_2 + q_3) \\ \sin(q_1) & \sin(q_1 + q_2) & \sin(q_1 + q_2 + q_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \vec{p}_3}{\partial \vec{q}} = \begin{bmatrix} -L_1 \sin(q_1) - L_2 \sin(q_1 + q_2) - L_3 \sin(q_1 + q_2 + q_3) & -L_2 \sin(q_1 + q_2) - L_3 \sin(q_1 + q_2 + q_3) & -L_3 \sin(q_1 + q_2 + q_3) \\ L_1 \cos(q_1) + L_2 \cos(q_1 + q_2) + L_3 \cos(q_1 + q_2 + q_3) & L_2 \cos(q_1 + q_2) + L_3 \cos(q_1 + q_2 + q_3) & L_3 \cos(q_1 + q_2 + q_3) \\ 0 & 0 & 0 \end{bmatrix}$$