Scaling Theorems for Zero Crossings

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Abstract—We characterize some properties of the zero crossings of the Laplacian of signals—in particular images—filtered with linear filters, as a function of the scale of the filter (extending recent work by Witkin [16]). We prove that in any dimension the only filter that does not create generic zero crossings as the scale increases is the Gaussian. This result can be generalized to apply to level crossings of any linear differential operator: it applies in particular to ridges and ravines in the image intensity. In the case of the second derivative along the gradient, there is no filter that avoids creation of zero crossings, unless the filtering is performed *after* the derivative is applied.

Index Terms—Gaussian filters, scale space, zero crossing.

I. Introduction

In most physical phenomena, changes in spatial or temporal structure occur over a wide range of scales. Images are no exception: changes in light intensity reflect the many spatial scales at which visible surfaces are organized. It seems intuitive that a great deal of information can be gained by an analysis of the changes in a signal at different scales. For instance, graphs of one-dimensional functions are a very effective tool for describing complex systems. An important reason is that they allow direct visual access to important properties of the data, chiefly to their changes over different scales.

The idea of scale is critical for a symbolic description of the significant changes in images or other types of signals. Changes must be detected at different levels of detail and over different extents. In general, different physical processes may be associated with a characteristic behavior across different scales. In an image, changes of intensity take place at many spatial scales depending on their physical origin. A multiscale analysis, tracing the behavior of some feature of the signal across scales, can reveal precious information about the nature of the underlying physical process. In images, for instance, spatial coincidence at all scales of zero crossings in the Laplacian of the intensity values filtered with a Gaussian mask may signal a physical "edge" distinct from surface markings or shadows. Not only is it necessary to detect and describe changes in a signal at different scales, but in addition, much useful information can be obtained by combining descriptions across scales.

The importance of this idea has been clearly realized in the field of vision. One of the main contributions of visual psychophysics in the last 10 years was indeed to show that visual information is processed in parallel by a number (perhaps a continuum) of spatial-frequency-tuned channels [3]. The bulk of the data demonstrates that the visual system analyzes the image at different resolutions. Physiological experiments are consistent with the psychophysics. They suggest that, in the visual pathway, spatial filters of different sizes operate at the same location. Furthermore, psychophysics, physiology, and anatomy all show that the spatial grain of analysis continuously changes from foveal to peripheral locations. Receptive and dendritic field sizes of both retinal and cortical neurons increases monotonically with eccentricity, in agreement with the dependency on eccentricity of the psychophysical channels.

In the field of computer vision, Rosenfeld was one of the first to explicitly propose an edge detection scheme based on multiscale analysis performed with filters of different sizes [13]. A similar algorithm was suggested by Marr [8] although with different goals and motivations. More recently, he has strongly advocated the use of derivatives of Gaussian-shaped filters of different sizes with the goal of detecting changes in intensity at different scales [9]. The idea was first proposed in the context of a theory of stereomatching [11]. In that scheme, analysis at the different scales was effectively kept separate. Later, Marr and Hildreth [10] proposed some heuristical rules to combine information from the different channels. However, the important problem of how to combine effectively the different scales of analysis at this early level has remained open, although recent work by Terzopoulos [15] has successfully applied multilevel algorithms to the problem of reconstructing visual surfaces (see also the work by Richards *et al.* [12], Crowley [6], and by Canny [4] on edge detection).

Recently a new way of describing zero crossings across scale was suggested by Witkin [16]. A one-dimensional (1-D) signal is smoothed by convolution with a small (large) Gaussian filter and the zeros of the second derivative are localized and followed as the size of the filter increases (decreases). This procedure originates a plot of the zero contours in the x–σ plane (where σ measures the size of the Gaussian filter).[1] In this way, Witkin was able to classify and label zero crossings achieving an effective description of a signal for purposes of recognition and registration. This is possible mainly because the geometry of the zero-crossings contours is surprisingly simple. Zero

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[1]Stansfield first described the idea of plotting zero crossings over scale—for analyzing commodities trends [14]—but did not develop it.

crossings contours are either lines from small to very large scale or closed, bowl-like shapes. Zero crossings are never created as the scale increases. Babaud *et al.* [1] obtained the striking result that the Gaussian filter is the only filter with this remarkable property in 1-D.

This property of the Gaussian filter is important for two reasons: first, it allows *coarse-to-fine tracking* of zero crossings in scale space and, second, it ensures that the scale-space diagram contains, in some sense, a minimal number of zero crossings (for $\sigma = 0$ the number of zero crossing is determined by the signal; see condition 3) in Section II).

We have independently succeeded in obtaining a proof of this result and extended it to two dimensions (and in fact any number of dimensions). We have also obtained related results for zero and level crossings of other differential operators, in particular for ridges and ravines in the image intensity. The work described here was reported in [17].[2]

The 2-D result is important because it 1) lays the necessary mathematical foundation for using multiresolution labels for classifying zero crossings for a symbolic description of intensity changes, and 2) justifies the use of Gaussian filters and an associated linear derivative because of their "nice" properties under changes in scale.

In this paper, we will first state and prove the 1-D result. We will then show that only a specific 2-D extension is valid. Zero crossings of linear derivatives have the "nice scaling behavior" if and only if the image is filtered by a 2-D rotationally symmetric Gaussian. In particular, the Laplacian of a Gaussian filter suggested by Marr and Hildreth [10] has nice scaling behavior. The second directional derivative along the gradient, however, does not: no filter exists that can ensure a nice scaling behavior of the zeros of this derivative. We have then the following results:

1) for linear derivative operations—in particular, for the Laplacian—the Gaussian is the only filter with a nice scaling behavior, and

2) for the nonlinear directional derivative, no filter will give nice scaling behavior.

II. Assumptions and Results

We will consider filtering the image I with a suitable filter F and then consider the behavior of the zero crossings as we change the scale of the filter. We make five assumptions about the filter, and impose them as conditions.

1) Filtering is shift invariant and, hence, a convolution. We write this as

$$F * I(x) = \int F(x - \zeta)\, I(\zeta)\, d.$$

[2]In an interesting manuscript, that came only recently to our attention, Koenderink, Huys, and Toet (preprint, 1983) discuss multiscale resolution of images using the Gaussian filter and the diffusion equation. Koenderink (personal communication, 1984) has obtained results similar to ours by exploiting properties of the diffusion equation.

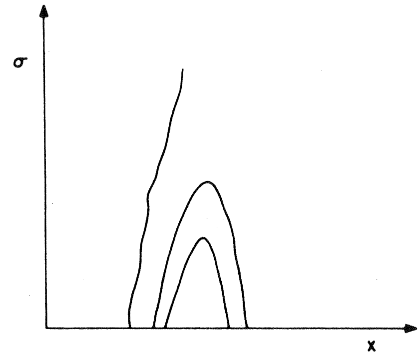

Fig. 1. See text.

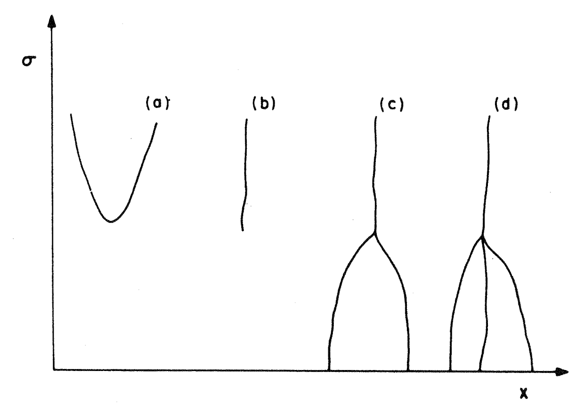

Fig. 2. See text.

2) The filter has no preferred scale. In two dimensions, standard results of dimensional analysis [2] give $F(x, \sigma) = (1/\sigma^2) f(x/\sigma)$ where σ is the scale of the filter. The factor $1/\sigma^2$ ensures that the filter is properly normalized at all scales.

3) The filter recovers the whole image at sufficiently small scales. This is expressed by $\lim_{\sigma \to 0} F(x, \sigma) = \delta(x)$ where $\delta(x)$ denotes the Dirac delta function.

4) The position of the center of the filter is independent of σ. Otherwise, zero crossings of a step edge would change their position with change of scale.

5) The filter goes to zero as $|x| \to \infty$ and as $\sigma \to \infty$.

As will become apparent, our results are independent of scaling the x axis. We usually require that we scale this axis so that the filter is radially symmetric, and state theorems for radially symmetric filters. However, we can relax this requirement by rescaling the axes.

Fig. 1 shows the typical scaling behavior of zero crossings in one dimension observed by Witkin. Fig. 2 shows possible behavior of zero crossings which is never empirically observed when the filter is a Gaussian. The generic properties of the zero-crossings curve in the (x, σ) plane can be derived from the implicit function theorem [5]. To yield a C^r curve (i.e., with continuous derivatives up to the rth order), the theorem requires that the Laplacian of the filtered image is C^r. Therefore, the filter must be reasonably smooth. Observe that filtering with a Gaussian will ensure a C^∞ output for all images, because solutions of the heat equation are entire functions and the Gaussian kernel is the Green functon of the heat equation. The implicit function theorem may break down at degenerate critical points when all first derivatives of the filtered image vanish together with the Hessian.[3] These points are nongeneric in the sense that a small perturbation in the signal will destroy them. Observe that "true" zero crossings (i.e., "simple" zeros, see [7]) can only disappear in pairs in the x, σ plane. Only trivial zeros that do not cross zero can disappear by themselves. They are, however, nongeneric. In this paper we only consider generic zero and level crossings.

In one dimension, the zero crossings in the second derivative of f obey

$$0 = \int_{-\infty}^{\infty} f''\left(\frac{x-\zeta}{\sigma}\right) I(\zeta)\, d\zeta \tag{2.1}$$

where f'' is the second derivative of f.

Equation (2.1) gives x as an implicit function of σ, i.e., $x = x(\sigma)$. If we vary x and σ so that (2.1) is still satisfied, we obtain

$$\frac{dx}{d\sigma} = \frac{\displaystyle\int_{-\infty}^{\infty} \left(\frac{x-\zeta}{\sigma}\right) f'''\left(\frac{x-\zeta}{\sigma}\right) I(\zeta)\, d\zeta}{\displaystyle\int_{-\infty}^{\infty} f'''\left(\frac{x-\zeta}{\sigma}\right) I(\zeta)\, d\zeta}. \tag{2.2}$$

So the tangent to the curve is uniquely defined at a point, as are all the higher order derivatives. This prevents the behavior shown in Figs. 2(b) and (c) (Fig. 2(d) is meant to trigger some thoughts in our readers) with the possible exception of the nongeneric cases, when the implicit function theorem breaks down.

The curve in Fig. 2(a) is more interesting because it corresponds to a pair of zero crossings being "created" as the scale (i.e., σ) increases. The implicit function theorem does not rule out this case. It therefore seems natural to require a filter such that this never occurs. In the following three sections, we will prove some theorems showing that such a filter can only be a Gaussian and, moreover, that not all differential zero-crossings operators can have this property. More precisely, we prove the following theorems.

Theorem 1: In one dimension, with the second derivative, the Gaussian is the only filter obeying our five conditions which never creates zero crossings as the scale increases.

Theorem 2: In two dimensions, with the Laplacian operator, the Gaussian is the only filter obeying the conditions which never create zero crossings as the scale increases.

Theorem 3: In two dimensions, with the directional derivative along the gradient, there is no filter obeying the conditions which never creates zero crossing as the scale increases.

In Section V, we show that results similar to theorems 1 and 2 can be extended to all linear differential operators (in particular, directional derivatives) and, therefore, to other features of the image, such as ravines and ridges (but not peaks) in the image intensity function. These theorems can be extended to any dimension, but we will not give these extensions here.

It should be emphasized that, although zero crossings can only annihilate themselves in pairs as σ increases, the intensity change corresponding to a zero crossing could become arbitrarily smaller with increasing sigma. The zero crossing would then become so weak that for practical purposes the curve terminates.

[3]Zeros of the Hessian correspond to zeros of the Gaussian curvature.

III. The 1-D Case

Let the image be I and the filter be F. We consider the zero crossings in the filtered image.

$$F * I(x) = \int_{-\infty}^{\infty} F(x-\zeta)\, I(\zeta)\, d\zeta. \tag{3.1}$$

Denote $(d^2/dx^2)(F * I)$ by E. Hence, the zero crossings are the solutions of

$$E(x) = 0. \tag{3.2}$$

These form curves in the x–σ plane. The condition that zero crossings are not created at larger scales is that for all such curves $\sigma(x)$ the extrema of $\sigma(x)$ are not minima. Hence, for all points x_o such that $\sigma'(x_o) = 0$, we require that $\sigma''(x_o) < 0$.

Let t be a parameter along a curve in σ–x space. Then

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial \sigma}\frac{d\sigma}{dt}. \tag{3.3}$$

On a curve of zero crossings, $E = 0$, and so $dE/dt = 0$ along the curve. We can choose the parameter t to be x. Then, using the implicit function theorem, we obtain

$$\frac{d\sigma}{dx} = \frac{-E_x}{E_\sigma}. \tag{3.4}$$

This derivative vanishes at x_o if and only if

$$E_x(x_o) = 0 \tag{3.5}$$

and we calculate, at places where (3.5) holds

$$\frac{d^2\sigma(x_o)}{dx^2} = \frac{-E_{xx}(x_o)}{E_\sigma(x_o)}. \tag{3.6}$$

Thus, our filter must be such that if

$$E(x_o) = E_x(x_o) = 0 \tag{3.7}$$

then

$$\frac{E_{xx}(x_o)}{E_\sigma(x_o)} > 0. \tag{3.8}$$

The heat equation can be written as

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{\sigma}\frac{\partial E}{\partial \sigma}. \tag{3.9}$$

Note that by the substitution $t = \sigma^2/2$, we obtain the standard heat equation. If the filter F is a Gaussian

$$F(x) = \frac{1}{\sigma} \exp\left\{\frac{-x^2}{2\sigma^2}\right\} \tag{3.10}$$

then it will obey the heat equation of which it is the Green function [(3.1) is the Green superposition integral associated with the heat equation] and hence $E(x)$ will also obey the equation. Thus, $E_{xx}/E_\sigma = 1/\sigma$ and so a Gaussian filter will always satisfy conditions (3.7) and (3.8).

We now show that Gaussian is the *only* filter which satisfies the conditions and obeys the conditions specified in Section I.

Consider an image which is the sum of delta functions

$$I(\zeta) = \sum_{i=1}^{n} A_i \delta(\zeta - \zeta_i). \tag{3.11}$$

It is possible to generate any image in this way by taking the limit as $n \to \infty$. Set

$$T(x) = F_{xx}(x). \tag{3.12}$$

Equations (3.7)and (3.8) yield

$$\sum_{i=1}^{n} A_i T(x_o - \zeta_i) = 0 \tag{3.13}$$

$$\sum_{i=1}^{n} A_i T_x(x_o - \zeta_i) = 0 \tag{3.14}$$

and

$$\frac{\Sigma_{i=1}^{n} A_i T_{xx}(x_o - \zeta_i)}{\Sigma_{i=1}^{n} A_i T\sigma(x_o - \zeta_i)} > 0. \tag{3.15}$$

We can construct a counter example if we can solve the simultaneous equations for any $x_o, \zeta_1, \cdots, \zeta_n$ and any positive l^2

$$\sum_{i=1}^{n} A_i T(x_o - \zeta_i) = 0 \tag{3.16}$$

$$\sum_{i=1}^{n} A_i T_x(x_o - \zeta_i) = 0 \tag{3.17}$$

$$\sum_{i=1}^{n} A_i T_{xx}(x_o - \zeta_i) = -l^2 \tag{3.18}$$

$$\sum_{i=1}^{n} A_i T\sigma(x_o - \zeta_i) = 1 \tag{3.19}$$

We can write these as a matrix equation

$$\begin{pmatrix} T(x_o - \zeta_1) & \cdots & T(x_o - \zeta_1) \\ T_x(x_o - \zeta_1) & \cdots & T_x(x_o - \zeta_1) \\ T_{xx}(x_o - \zeta_1) & \cdots & T_{xx}(x_o - \zeta_1) \\ T_\sigma(x_o - \zeta_1) & \cdots & T_\sigma(x_o - \zeta_1) \end{pmatrix} \begin{pmatrix} A_1 \\ \cdot \\ \cdot \\ A_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -l^2 \\ 1 \end{pmatrix}. \tag{3.20}$$

Using Appendix A, a necessary and sufficient condition for it to be impossible to solve these equations for any values of $x_o, \zeta_1 \cdots \zeta_n$ is that there exists a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ independent of x such that,

$$\lambda_1 T(x) + \lambda_2 T_x(x) + \lambda_3 T_{xx}(x) + \lambda_4 T_\sigma(x) = 0 \tag{3.21}$$

and

$$-\lambda_3 l^2 + \lambda_4 \neq 0. \tag{3.22}$$

Equation (3.22) will be satisfied for all positive l^2 if and only if ($\lambda_3 = 0$ and $\lambda_4 = 0$ can be ruled out because of the conditions)

$$\lambda_3 \lambda_4 < 0. \tag{3.23}$$

Our condition 2) means that $F(x)$, and hence $T(x)$, cannot depend on any scale length. The $\boldsymbol{\lambda}$'s are independent of x and so to make (3.21) dimensionally correct [2] we set

$$\lambda_1 = \frac{a}{\sigma^2}, \quad \lambda_2 = \frac{b}{\sigma}, \quad \lambda_3 = c, \quad \lambda_4 = \frac{-d}{\sigma} \tag{3.24}$$

and rewrite it as

$$\frac{aT}{\sigma^2} + \frac{bT_x}{\sigma} + cT_{xx} = \frac{d}{\sigma} T_\sigma. \tag{3.25}$$

Condition (3.23) implies that d/c is positive.

Now $T = d^2F/dx^2$ so F will also satisfy (3.25). Although it is possible to add a term ϕ to F where $d^2\phi/dx^2 = 0$, according to condition 5) ϕ can only be the zero function.

Thus, we have shown that we can always construct a counter example *unless* our filter F obeys to the equation

$$\frac{aF}{\sigma^2} + \frac{b}{\sigma} F_x + cF_{xx} = \frac{d}{\sigma} F_\sigma \tag{3.26}$$

with d/c positive. It is shown in Appendix B that the only solution of this equation obeying conditions is the Gaussian, thus proving theorem 1.

IV. The 2-D Case

We now consider the two-dimensional case when the zero-crossing operator is the Laplacian ∇^2 and the image depends on $\boldsymbol{x} = (x, y)$. Again, we consider the filtered image

$$F * I(\boldsymbol{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\boldsymbol{x} - \boldsymbol{\zeta})\, I(\boldsymbol{\zeta})\, d\boldsymbol{\zeta}. \tag{4.1}$$

We set

$$E(x) = \nabla^2 \{F * I(\boldsymbol{x})\}. \tag{4.2}$$

The zero crossings are solutions of $E(\boldsymbol{x}) = 0$ and form surfaces in the three-dimensional $(\boldsymbol{x}, \sigma)$ space. Our requirements that zero crossings are not created at larger scales is satisfied if the extrema of these zero crossing surfaces are either maxima or saddle points. Minima are forbidden. Thus, if we have a surface $\sigma(x, y)$ and there is a point (x_o, y_o) with

$$\sigma_x(x_o, y_o) = \sigma_y(x_o, y_o) = 0 \tag{4.3}$$

we cannot have $\sigma_{xy} = 0$ and both

$$\sigma_{xx} > 0, \quad \sigma_{yy} > 0. \tag{4.4}$$

The axes are chosen to be along the lines of curvature at the extrema: thus, σ_{xx}, σ_{yy} are eigenvalues of the Hessian.

Let t be a parameter of a curve of the surface $E(x) = 0$. Then,

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial \sigma}\frac{d\sigma}{dt}. \tag{4.5}$$

Since we are on the zero crossing surface, we have $dE/dt = 0$ and setting $t = x$ and then $t = y$, we obtain

$$\sigma_x = \frac{-E_x}{E_\sigma} \tag{4.6}$$

$$\sigma_y = \frac{-E_y}{E_\sigma}. \tag{4.7}$$

Suppose we are at an extremum (x_o, y_o). Choose the x and y axes so that they coincide with the directions of principal curvature at (x_o, y_o). Then we calculate

$$\sigma_{xx}(x_o, y_o) = \frac{-E_{xx}(x_o y_o)}{E_\sigma(x_o, y_o)} \tag{4.8}$$

$$\sigma_{yy}(x_o, y_o) = \frac{-E_{yy}(x_o y_o)}{E_\sigma(x_o, y_o)}. \tag{4.9}$$

It should be emphasized that (4.8) and (4.9) are true only at an extremum of $\sigma(x, y)$ and only if the x and y axes are taken along the directions of the lines of curvature (this ensures $\sigma_{xy} = 0$).

It follows, as in the 1-D case, that the conditions 3) and 4) will always be satisfied if E obeys the heat equation. If $\sigma_{xx}(x_o, y_o)$ and $\sigma_{yy}(x_o, y_o)$ are both positive, (4.8) and (4.9) imply that $E_{xx}(x_o y_o)/E_\sigma(x_o, y_o)$ and $E_{yy}(x_o y_o)/E_\sigma(x_o, y_o)$ and $\sigma_{xx}(x_o, y_o)$ are both negative. Thus, a Gaussian filter will always obey our condition.

We now show that if the filter is not a Gaussian, we can conduct a counterexample. The argument is a generalization of the proof of Theorem 1. Let

$$I(\zeta) = \sum_{i=1}^{n} A_i \delta(\zeta - \zeta_i) \tag{4.10}$$

set

$$T(x) = \nabla^2 F(x). \tag{4.11}$$

We can conduct a counterexample if we can solve the matrix equation for any x_o, $\zeta_1 \cdots, \zeta_n$ and any positive l_1^2 and l_2^2

$$\begin{bmatrix} T(x_o - \zeta_1) & \cdots & T(x_o - \zeta_n) \\ T_x(x_o - \zeta_1) & \cdots & T_x(x_o - \zeta_n) \\ T_y(x_o - \zeta_1) & \cdots & T_y(x_o - \zeta_n) \\ T_{xx}(x_o - \zeta_1) & \cdots & T_{xx}(x_o - \zeta_n) \\ T_{xy}(x_o - \zeta_1) & \cdots & T_{xy}(x_o - \zeta_n) \\ T_{yy}(x_o - \zeta_1) & \cdots & T_{yy}(x_o - \zeta_n) \\ T_\sigma(x_o - \zeta_1) & \cdots & T_\sigma(x_o - \zeta_n) \end{bmatrix} \begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -l_1^2 \\ 0 \\ -l_2^2 \\ 1 \end{bmatrix}. \tag{4.12}$$

Using Appendix A, a necessary and sufficient condition for no solution to exist for all x_o, $\zeta_1, \cdots, \zeta_n$ is that we can find $= (\lambda_1, \cdots, \lambda_5)$ such that

$$\lambda_1(x) + \lambda_2 T_x(x) + \lambda_3 T_y(x) + \lambda_4 T_{xx}(x) + \lambda_7 T_{xy}(x) + \lambda_5 T_{yy} + \lambda_6 T_\sigma(x) = 0 \tag{4.13}$$

and

$$-l_1^2\lambda_4 - l_2^2\lambda_5 + \lambda_6 \neq 0. \tag{4.14}$$

Equation (4.14) can be satisfied for all positive l_1^2 and l_2^2 if and only if (because of the conditions)

$$\lambda_4\lambda_5 > 0, \qquad \lambda_4\lambda_6 < 0. \tag{4.15}$$

Again, condition 2) implies the λ's are of form

$$\lambda_1 = \frac{a}{\sigma^2}, \quad \lambda_2 = \frac{b_1}{\sigma}, \quad \lambda_3 = \frac{b_2}{\sigma}, \quad \lambda_4 = c_1$$
$$\lambda_5 = c_2, \quad \lambda_6 = \frac{-d}{\sigma}, \quad \lambda_7 = c_3 \tag{4.16}$$

and T satisfies

$$\frac{aT}{\sigma^2} + \frac{b_1}{\sigma}T_x + \frac{b_2}{\sigma}T_y + c_1 T_{xx} + c_2 T_{yy} + c_3 T_{xy} = \frac{d}{\sigma}T_\sigma. \tag{4.17}$$

with $c_1c_2 > 0$ and $c_1 d > 0$[4]. e can obtain restrictions on c_3 by requiring that the curvature at the extreme points is negative. This means that elliptic operators—and hence a skewed Gaussian filter—will also have the desired scaling properties. We are not interested in these since we require the filter to be symmetric (see Section II).

F will satisfy (4.17) up to a term ψ with $\nabla^2\psi = 0$, but because of condition 5), ψ can be taken to be zero.

It is shown in Appendix B that the only solution of (4.17) which obeys our conditions is the product of two 2-D Gaussians. If we make the additional assumption of symmetry, we obtain a 2-D symmetric Gaussian. Hence, the Gaussian is the only filter which satisfied our condition, and we have proven theorem 2.

There is an additional property of Gaussian filters: allowed zero-crossing surfaces in (x, y, σ) space cannot have saddle points with positive mean curvature H because $H = (\sigma_{xy} + \sigma_{yy})/2$. The result of this section forbids the existence of upside-down mountains or pits [in the (x, y, σ) plane] and also of upside-down volcanos. Sections of the zero-crossings surfaces normal to the (x, y) plane may appear as suggesting that lines of zero crossings are created. In fact, because of saddle points of the surface, zeros can be traced *continuously* along the zero-crossing surface to smaller and smaller scales.

V. Further Results

It is clear that the methods of proof we have developed do not only apply on zero crossings. For example, consider

[4]W.

the 1-D case and look for solutions of

$$\frac{d}{dx}(F * I) = 0. \quad (5.1)$$

These correspond to maxima and minima of the filtered signal which we call peaks and troughs. If we set $E = (d/dx)(F * I)$ and duplicate the arguments of Section II, we find that having a Gaussian filter is a necessary and sufficient condition for peaks and troughs to not be created.

More generally, if $L(\boldsymbol{x})$ is a differential operator in any dimension that commutes with the diffusion equation, then solutions of

$$L(F * I) = \text{const} \quad (5.2)$$

will not be created if and only if the filter is Gaussian. Zeros of all linear differential operators can be encompassed by theorem 1.

In particular, in two dimensions, surfaces obeying $(d/dx)(F * T) = 0$ can only be created by a non-Gaussian filter. Thus, ridges and ravines whose creation necessarily involves creation of zeros along some direction, can only be created, as the scale increases, by a non-Gaussian filter. The argument, however, does not apply to extremum points (nondegenerate critical points, such as peaks and pits, where all derivatives vanish simultaneously).

VI. Directional Operator

We have considered the 2-D case when our operator is the second directional derivative along the direction of the gradient in the filtered image. Let

$$H(\boldsymbol{x}) = \int\int F(\boldsymbol{x} - \boldsymbol{\zeta})\, I(\boldsymbol{\zeta})\, d\boldsymbol{\zeta}. \quad (6.1)$$

The directional operator is

$$\frac{d}{dt} = \frac{1}{\left|\frac{\partial H}{\partial x_i}\right|}\frac{\partial H}{\partial x_j}\cdot\frac{\partial}{\partial x_j} \quad (6.2)$$

using the standard summation convention on the j indexes. The second directional derviative along the gradient is then

$$\frac{d^2H}{dt^2} = \frac{H_iH_jH_{ij}}{H_kH_k} \quad (6.3)$$

where $H_i = \partial H/\partial x_i$, $H_{ij} = \partial^2 H/\partial x_i \partial x_j$ and we use the summation convention over repeated indexes. We set

$$E(\boldsymbol{x}) = H_i(\boldsymbol{x})H_j(\boldsymbol{x})H_{ij}(\boldsymbol{x}). \quad (6.4)$$

The zero crossings lie on the surface $\sigma(x, y)$ where $E(\boldsymbol{x}) = 0$. Our condition is that if we have a point (x_o, y_o) where

$$\sigma_x(x_o, y_o) = \sigma_y(x_o, y_o) = 0 \quad (6.5)$$

and the x and y axes are along the direction of the lines of curvature of the $\sigma(x, y)$ surface at that point (so that $\sigma_{xy} = 0$), then it is impossible for both σ_{xx} and σ_{yy} to be positive, i.e.,

$$\sigma_{xx}(x_o, y_o) > 0, \qquad \sigma_{yy}(x_o, y_o) > 0. \quad (6.6)$$

We use the implicit function theorem to obtain

$$\sigma_x = \frac{-E_x}{E_\sigma} \quad (6.7)$$

$$\sigma_y = \frac{-E_y}{E_\sigma} \quad (6.8)$$

and we calculate, at places where (6.5) holds,

$$\sigma_{xx}(x_o, y_o) = \frac{-E_{xx}(x_o, y_o)}{E_\sigma(x_o, y)} \quad (6.9)$$

$$\sigma_{yy}(x_o, y_o) = \frac{-E_{yy}(x_o, y_o)}{E_\sigma(x_o, y_o)}. \quad (6.10)$$

Again, note that if E obeys the diffusion equation, then the conditions (6.5) and (6.6) cannot be satisfied. However, E is no longer a linear function of the filter, and so we cannot directly obtain a condition the filter must satisfy. Now set

$$I(x) = \sum_{\alpha=1}^{n} A_\alpha \delta(x - \zeta_\alpha). \quad (6.11)$$

We find

$$H_iH_jH_{ij} = A_\alpha A_\beta A_\gamma F_i(\alpha)\, F_j(\beta)\, F_{ij}(\gamma) \quad (6.12)$$

where the summation convention applies to α, β, γ as well as to i, j. We define

$$T(\alpha\beta\gamma) = \tfrac{1}{6}\{F_i(\alpha)F_j(\beta)F_{ij}(\gamma) + F_i(\beta)F_j(\alpha)F_{ij}(\gamma) + F_i(\alpha)F_j(\gamma)F_{ij}(\beta) + F_i(\beta)F_j(\gamma)F_{ij}(\alpha) + F_i(\gamma)F_j(\alpha)F_{ij}(\beta) + F_i(\gamma)F_j(\beta)F_{ij}(\alpha)\} \quad (6.13)$$

and write (6.12) as

$$H_iH_jH_{ij} = T(\alpha\beta\gamma)\, A_\alpha A_\beta A_\gamma. \quad (6.14)$$

We can produce a counterexample if we can satisfy

$$\begin{bmatrix} T(\alpha\beta\gamma) & \cdots \\ T_x(\alpha\beta\gamma) & \cdots \\ T_y(\alpha\beta\gamma) & \cdots \\ T_{xx}(\alpha\beta\gamma) & \cdots \\ T_{xy}(\alpha\beta\gamma) & \cdots \\ T_{yy}(\alpha\beta\gamma) & \cdots \\ T_s(\alpha\beta\gamma) & \cdots \end{bmatrix} \begin{bmatrix} A_\alpha A_\beta A_\gamma \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -l_1^2 \\ 0 \\ -l_2^2 \\ 1 \end{bmatrix}. \quad (6.15)$$

It follows from Appendix A that no solution exists if there is $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)$ such that

$$_1T(\alpha\beta\gamma) + \lambda_2 T_x(\alpha\beta\gamma) + \lambda_3 T_y(\alpha\beta\gamma) + \lambda_4 T_{xx}(\alpha\beta\gamma) + \lambda_7 T_{xy}(\alpha\beta\gamma)\, \lambda_5 T_{yy}(\alpha\beta\gamma) + \lambda_6 T_\sigma(\alpha\beta\gamma) = 0 \quad (6.16)$$

but

$$-l_1^2\lambda_4 - l_2^2\lambda_5 + \lambda_6 \neq 0. \tag{6.17}$$

As in Section III, we can use dimensional arguments to show this means that $T(\alpha\beta\gamma)$ satisfies the generalized diffusion equation. As in Appendix B, we set $\lambda_7 = 0$ to preserve symmetry.

However, since we require solutions to (6.15) of specific form $A_\alpha A_\beta A_\gamma$, it is possible that there are no solutions of (6.15) even if $T(\alpha\beta\gamma)$ does not obey the generalized diffusion equation. To rule this out, we must show that it is possible to find a solution of form $A_\alpha A_\beta A_\gamma$. From Appendix B it is possible to get a solution $B_{\alpha\beta\gamma}$ of

$$\begin{bmatrix} T(\alpha\beta\gamma) \\ \cdot \\ \cdot \\ \cdot \\ T_\sigma(\alpha\beta\gamma) \end{bmatrix} \begin{bmatrix} B_{\alpha\beta\gamma} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -l_1^2 \\ -l_2^2 \\ 1 \end{bmatrix} \tag{6.18}$$

if and only if the vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -l_1^2 \\ -l_2^2 \\ 1 \end{bmatrix}$$

lies in the spaced spanned by the

$$\begin{bmatrix} T(\alpha\beta\gamma) \\ \cdot \\ \cdot \\ \cdot \\ T_\sigma(\alpha\beta\gamma) \end{bmatrix}$$

as α, β, γ vary. Denote

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -l_1^2 \\ -l_2^2 \\ 1 \end{bmatrix}$$

by l^i and

$$\begin{bmatrix} T_{\alpha\beta\gamma} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ T_\sigma(\alpha\beta\gamma) \end{bmatrix}$$

by $T^i_{\alpha\beta\gamma}$ where $i = 1$–6.

Each $T^i(\alpha\beta\gamma)$ is symmetric in all indexes α, β and γ, and so there are $N = n(n + 1)(n + 2)/6$ such vectors. They have only six components each and so they are not linearly independent. There will be at least N-6 linearly independent vectors $\zeta^{(p)}_{\alpha\beta\gamma}$ such that

$$\sum_{\alpha\beta\gamma} T^{(i)}_{(\alpha\beta\gamma)} \zeta^{(p)}_{\alpha\beta\gamma} = 0, \qquad p = 1 \text{ to } N = 6. \tag{6.19}$$

If $T_{(\alpha\beta\gamma)}$ does not obey the generalized diffusion equation, there will be at least one solution $B_{\alpha\beta\gamma}$ to (6.18). The general solution is of the form

$$B_{\alpha\beta\gamma} + \sum_{p=1}^{N-6} \mu_p \zeta^p_{\alpha\beta\gamma} \tag{6.20}$$

where μ is arbitrary. We now ask under what conditions can we find A_α and μ which satisfy

$$B_{\alpha\beta\gamma} + \sum_{p=1}^{N-6} \mu_p \zeta^p_{\alpha\beta\gamma} = A_\alpha A_\beta A_\gamma. \tag{6.21}$$

From the form of (6.15) it is clear that scaling the A's will not affect the counterexample. Hence, satisfying (6.21) is equivalent to finding an A_α such that $A_\alpha A_\beta A_\gamma$ lies in the N-5-dimensional vector space spanned by $B_{\alpha\beta\gamma}$, $\zeta^1_{\alpha\beta\gamma}, \cdots, \zeta^{N-6}_{\alpha\beta\gamma}$. A necessary and sufficient condition is that $A_\alpha A_\beta A_\gamma$ is perpendicular to the five vectors which span the complement of this N-dimensional space in the full N-dimensional space.

Let the five vectors be $P_{\alpha\beta\gamma}$, $Q_{\alpha\beta\gamma}$, $T_{\alpha\beta\gamma}$, and $Y_{\alpha\beta\gamma}$. It will be possible to solve (6.21)'and hence (6.15) if we can satisfy

$$\begin{aligned} P_{\alpha\beta\gamma} A_{\alpha\nu t} A_\beta A_\gamma &= 0 \\ Q_{\alpha\beta\gamma} A_\alpha A_\beta A_\gamma &= 0 \\ T_{\alpha\beta\gamma} A_\alpha A_\beta A_\gamma &= 0 \\ X_{\alpha\beta\gamma} A_\alpha A_\beta A_\gamma &= 0 \\ Y_{\alpha\beta\gamma} A_\alpha A_\beta A_\gamma &= 0. \end{aligned} \tag{6.22}$$

This is a system of five simultaneous cubic equations in n variables. If we take n sufficiently large, it will always be possible to solve them (see [17]).

Thus, unless $T(\alpha\beta\gamma)$ obeys the generalized diffusion equation, it will always be possible to construct a counterexample.

We now show that no reasonable filter will satisfy these requirements. First, suppose we have a Gaussian filter $G(\boldsymbol{x}, \sigma)$

$$G(\boldsymbol{x}, \sigma) = \frac{1}{\sigma^m} \exp \{\frac{-\boldsymbol{x}^2}{2\sigma^2}\} \tag{6.23}$$

where m is an arbitrary number. Then we find

$$G_i(\alpha) = \frac{-(\boldsymbol{x} - \zeta_\alpha)_i}{\sigma^{m+2}} \exp\left\{\frac{-(\boldsymbol{x} - \zeta_\alpha)^2}{2\sigma^2}\right\} \tag{6.24}$$

$$G_{ij}(\alpha) = \frac{-\delta_{ij}}{\sigma^{m+2}} \exp\left\{-\frac{(\boldsymbol{x} - \zeta_\alpha)^2}{2\sigma^2}\right\} + \frac{(\boldsymbol{x} - \zeta_\alpha)_i(\boldsymbol{x} - \zeta_\alpha)_j}{\sigma^{m+4}} \exp\left\{-\frac{(\boldsymbol{x} - \zeta_\alpha)^2}{2\sigma^2}\right\} \tag{6.25}$$

So we obtain

$$\begin{aligned} T(\alpha\beta\gamma) = {} & 2 \exp\left\{-\frac{(\boldsymbol{x} - \zeta_\alpha)^2}{2\sigma^2} - \frac{(\boldsymbol{x} - \zeta_\beta)^2}{2\sigma^2} - \frac{(\boldsymbol{x} - \zeta_\gamma)^2}{2\sigma^2}\right\} \\ & \times \frac{1}{\sigma^{3m+6}} \left\{-(\boldsymbol{x} - \zeta_\alpha) \cdot (\boldsymbol{x} - \zeta_\beta)\right. \\ & - (\boldsymbol{x} - \zeta_\beta) \cdot (\boldsymbol{x} - \zeta_\gamma) - (\boldsymbol{x} - \zeta_\gamma) \cdot (\boldsymbol{x} - \zeta_\alpha) \\ & + \frac{(\boldsymbol{x} - \zeta_\alpha)^2(\boldsymbol{x} - \zeta_\beta)}{\sigma^2} \cdot (\boldsymbol{x} - \zeta_\gamma) \\ & + \frac{(\boldsymbol{x} - \zeta_\gamma)^2(\boldsymbol{x} - \zeta_\alpha)}{\sigma^2} \cdot (\boldsymbol{x} - \zeta_\beta) \\ & \left. + \frac{(\boldsymbol{x} - \zeta_\alpha)^2(\boldsymbol{x} - \zeta_\beta)}{\sigma^2} \cdot (\boldsymbol{x} - \zeta_\gamma)\right\}. \end{aligned} \tag{6.26}$$

As shown in Appendix B, the general diffusion equation can be written

$$\frac{b_1}{\sigma} T_x + \frac{b_2}{\sigma} T_y + c_1 T_{xx} + c_2 T_{yy} = \frac{d}{\sigma} T_\sigma. \tag{6.27}$$

If we substitute (6.26) into (6.27) we see that $c_1 T_{xx} + c_2 T_{yy}$ contains a term

$$Z = -2 \frac{(c_1 + c_2)}{\sigma^{3m+6}} \exp\left\{-\frac{(\boldsymbol{x} - \zeta_\alpha)^2}{2\sigma^2} - \frac{(\boldsymbol{x} - \zeta_\beta)^2}{2\sigma^2} - \frac{(\boldsymbol{x} - \zeta_\gamma)^2}{2\sigma^2}\right\}. \tag{6.28}$$

All other terms in (6.27) will be of this form multiplied by powers of $(\boldsymbol{x} - \zeta_\gamma)$, $(\boldsymbol{x} - \zeta_\beta)$, and $(\boldsymbol{x} - \xi_\gamma)$. From (6.17), c_1 and c_2 have the same sign and so it is impossible for Z to be zero and, hence, (6.27) cannot be satisfied if the filter is a Gaussian.

Now suppose we have a filter which satisfies this requirement. Set $\zeta_\gamma = \zeta_\alpha + \zeta_\beta$ and integrate $T(\alpha\beta\gamma)$ with respect to ζ_γ and ζ_β. We find

$$\begin{aligned} \int\int & F_i(\boldsymbol{x} - \zeta_\alpha) F_j(\boldsymbol{x} - \zeta_\beta) \\ & F_{ij}(\boldsymbol{x} + (\zeta_\alpha + \zeta_\beta))\, d\zeta_\alpha d\zeta_\beta \\ & = \mathrm{F}_i * \mathrm{F}_j * \mathrm{F}_{ij}(3\boldsymbol{x}). \end{aligned} \tag{6.29}$$

hence, with $\zeta_\gamma = \zeta_\alpha + \zeta_\beta$, we have

$$\int\int T(\alpha\beta\gamma)\, d\zeta_\alpha\, d\zeta_\beta = F_i * F_j * F_{ij}(3\boldsymbol{x}). \tag{6.30}$$

This will satisfy the generalized diffusion equation since $T(\alpha\beta\gamma)$ obeys this equation for all values of ζ_α, ζ_β, and ζ_γ. From Appendix B, the solution to the generalized diffusion equation is $P * f(\boldsymbol{x})$ where f is an arbitrary function and

$$P(\boldsymbol{x}, \sigma) = \frac{1}{\sigma^2} \exp\left\{-\frac{(x + b_1\sigma)}{2\sigma^2} \frac{d}{c_1}\right\} \cdot \exp\left\{\frac{(y + b_2\sigma)^2}{2\sigma^2} \frac{d}{c_2}\right\}. \tag{6.31}$$

We have

$$F_i * F_j * F_{ij}(3\boldsymbol{x}) = P * f(\boldsymbol{x}). \tag{6.32}$$

The condition 4) means that $b_1 = b_2 = 0$ and we can scale the x and y axes to make P a Gaussian. Thus,

$$F_i * F_j * F_{ij}(3\boldsymbol{x}) = G * f(\boldsymbol{x}). \tag{6.33}$$

We Fourier transform this equation and denote the Fourier transform of a function $g(\boldsymbol{x})$ by $\tau g(\boldsymbol{\omega})$.

$$\tau F_i(\boldsymbol{\omega})\, \tau F_j(\boldsymbol{\omega})\, \tau F_{ij}(\boldsymbol{\omega}) = \tau G(\boldsymbol{\omega})\, \tau f(3\boldsymbol{\omega}). \tag{6.34}$$

But we have

$$\tau F_i(\boldsymbol{\omega}) = -i\omega_i \tau F(\boldsymbol{\omega}) \tag{6.35}$$

and

$$\tau G(3\boldsymbol{\omega}) = \exp\left\{\frac{-9\omega^2}{2\sigma^2}\right\}. \tag{6.36}$$

Hence,

$$\omega^4\{\tau F(\boldsymbol{\omega})\}^3 = \exp\left\{\frac{-9\omega^2}{2\sigma^2}\right\} \tau f(3\boldsymbol{\omega}) \tag{6.37}$$

and

$$\tau F(\boldsymbol{\omega}) = \left\{\frac{\tau f(3\boldsymbol{\omega})}{\omega^4}\right\}^{1/3} \exp\left\{\frac{-3\omega^2}{2\sigma^2}\right\}. \tag{6.38}$$

Thus, F is the convolution of a function with a Gaussian and obeys the diffusion equation. But, as shown in Appendix B, the only such filter which satisfies the conditions is a Gaussian.

So a filter which obeys the conditions (6.16) and (6.17) must be a Gaussian, and yet a Gaussian cannot satisfy these conditions. Therefore, for this directional operator, it is impossible to satisfy our requirement. Notice that if the gradient direction does not change rapidly, the second directional derivative along the gradient can be approximated by the second derivative along the x axis where the x axis is chosen in the direction of the gradient. The arguments of Section V then show that no zero crossings are created if, and only if, the filter is Gaussian. If these assumptions are satisfied at one scale, they may break down at larger scales because of the influence of other parts of the image. We therefore expect that, at large scales, zero crossings may be created even for Gaussian filters, unless the image is very simple (for instance an isolated, straight step edge).

VII. Conclusions

The behavior of the zero (or level) crossings is more complex in two dimensions than in one dimension. In the 2-D case, two zero-crossing contours can merge into one closed contour as the scale increases. The zero-crossing surface has a 2-D cross section (for given y, say) that corresponds to an allowed 2-D case. In 2-D, however, the "complementary" situation can also occur: a closed zero-crossing contour can split into two as the scale increases, just as the trunk of a tree may split into two branches. This occurs at saddle points of the zero-crossing surface. This case would correspond in 1-D to the "creation" of a zero crossing (imagine a 1-D section of the zero crossing surface) which is forbidden. In 2-D, however, no new zero crossing is created, since the corresponding surface is continuous down to zero scale. We have constructed 2-D examples of both these two cases, using the Gaussian filters. Both examples would also work for all other filters.

Several other functions have been proposed for the filtering images. We expect that they only give a nice scaling behavior for values of σ for which they approximate the solution of the diffusion equation. The DOG (difference of Gaussians) *does not* satisfy the diffusion equation, but is a good approximation except when σ is very small. One-dimensional real Gabor functions (the product of a Gaussian and a sine or a cosine) approximate the solution of the diffusion equation only for large values of σ. Our conditions are violated even more by the sine function which only satisfies the diffusion equation at best in a weak asymptotic sense. Fig. 3 shows an example of the zero crossings generated by the Gaussian and the sine filter.

It is interesting that our proof implies that the heat equation is the only linear equation that has a nice scaling behavior of its solutions, with suitable boundary conditions.

In summary, we have shown here that the Gaussian is the only filter that guarantees a nice scaling behavior of the zero and level crossings of linear differential operators. (Notice that the Gaussian need not be symmetric: Elongated directional filters, obtained by stretching the axes, also have a nice scaling behavior.) Surprisingly, zero and level crossings of most signals filtered with a Gaussian filter at different scales uniquely characterize the signal [18] up to overall scaling in one and more dimensions, for most functions that can be approximated arbitrarily well by polynomials. In 1-D Logan [7] proved a similar result for bandpass signals of the sequential type. The result cannot be extended to two-dimensional signals. Very recently, Curtis [19] has shown that in two (but not one) dimensions, the result holds even for bandlimited functions. Both Logan's and Curtis' results do not need Gaussian filtering and multiple scales, but assure certain types of smooth signals. Thus, Gaussian filtering across scale originates a "nice" *and* complete representation in terms of zero crossings.

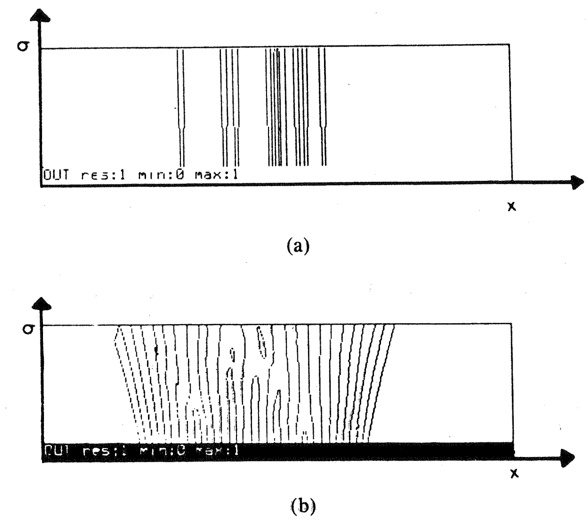

Fig. 3. Examples of the zero crossings of the second derivative of the (a) Gaussian and (b) sine filter for the same input function.

Appendix A

If we have a matrix equation

$$\boldsymbol{Bx} = \boldsymbol{a} \tag{1}$$

the necessary and sufficient condition for the existence of a solution is that

$$\text{rank}\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ b_{m1} & \cdot & b_{mn} \end{pmatrix} = \text{rank}\begin{pmatrix} b_{11} & \cdots & b_{1n} & a_1 \\ \cdot & \cdots & \cdot & \\ b_{m1} & \cdot & b_{mn} & a_m \end{pmatrix} \tag{2}$$

Hence, a necessary and sufficient condition for the nonexistence of a solution is that we can find a vector $= (\lambda_1, \cdots, \lambda_m)$, such that

$$\lambda_1(b_{11}, \cdots, b_{1n}) + \cdots + \lambda_m \cdot (b_{m1}, \cdots, b_{mn}) = 0 \tag{3}$$

but for which

$$\lambda_1 a_1 + \cdots + \lambda_m a_m \neq 0. \tag{4}$$

Appendix B

Suppose we have a generalized diffusion equation of the form

$$a\frac{F}{\sigma^2} + \frac{bF_x}{\sigma} + cF_{xx} = \frac{dF_\sigma}{\sigma}. \tag{1}$$

We can remove the first term by scaling $F \rightarrow \sigma^{-(a/d)}F$. Consider the remaining terms

$$\frac{bF_x}{\sigma} + cF_{xx} = \frac{dF_\sigma}{\sigma}. \quad (2)$$

We write

$$F(x, \sigma) = \frac{1}{\sqrt{2\pi}} \int f(\omega, \sigma)\, e^{i\omega x}\, d\omega \quad (3)$$

where $f(\omega, \sigma)$ is the Fourier transform of $F(x, \sigma)$ with respect to x. Combining (3) and (2) we obtain

$$\frac{b(-i\omega)}{\sigma} f + c(-\omega^2) f = \frac{d}{\sigma}\frac{\partial f}{\partial \sigma}. \quad (4)$$

We integrate and get

$$f(\omega, \sigma) = g(\omega)\{e^{-i\omega b\sigma/d}\, e^{(-c\omega^2/d)(\sigma^2/2)}\} \quad (5)$$

where $g(\omega)$ is a function of integration independent of σ. Hence, substituting (5) into (3) gives us

$$F(x, \sigma) = \frac{1}{\sqrt{2\pi}} \int g(\omega)\{e^{-i\omega b\sigma/d}\, e^{(-c\omega^2/d)(\sigma^2/2)}\}\, e^{-i\omega x}\, d\omega. \quad (6)$$

Note that we are considering equations for which c/d is positive and so the integral is well defined. We now apply the convolution theorem to (6) and get

$$F(x, \sigma) = \frac{1}{\sqrt{2\pi}} \int \lambda(x - \zeta, \sigma)\, \mu(\zeta)\, d\zeta \quad (7)$$

where $\mu(\zeta)$ is the Fourier transform of $g(\omega)$ and $\lambda(x, \sigma)$ is the Fourier transform of $\{e^{-i\omega b\sigma/d}\, e^{-\omega^2\sigma^2/2d}\}$. We calculate

$$\lambda(x, \sigma) = \sqrt{\frac{d}{c}\frac{1}{\sigma}}\, e^{(-d/2c\sigma^2)(x + b\sigma)^2}. \quad (8)$$

Thus, the general solution to (1)is of the form

$$F(x, \sigma) = \frac{1}{\sqrt{2\pi}} \sigma^{(a/d)-1} \sqrt{\frac{d}{c}} \int e^{(-d/2c\sigma^2)(x - \zeta + b\sigma)^2}\, \mu(\zeta)\, d\zeta. \quad (9)$$

We now impose the conditions stated in Section I. First, note that $\lambda(x, \sigma)$ is a Gaussian with center $x = -b\sigma$. The requirement that the center of the filter does not move implies that $b = 0$.

Write

$$F(x, \sigma) = \sigma^{a/d} \frac{1}{\sqrt{2\pi}} \int \sqrt{\frac{d}{c}\frac{1}{\sigma}}\, e^{(-d/2c\sigma^2)(x-\zeta)^2}\, \mu(\zeta)\, d\zeta \quad (10)$$

and consider the limit as σ tends to 0. Now,

$$\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}} \int \sqrt{\frac{d}{c}\frac{1}{\sigma}}\, e^{(-d/2\sigma^2)(x-\zeta)^2} = \delta(x - \zeta) \quad (11)$$

where δ denotes the Dirac delta function. If (a/d) is nonzero, the limits of $F(x, \sigma)$ will either be undefined or zero. Hence, our condition 3) forces $a = 0$. Moreover, substituting into (10) we obtain

$$\lim_{\sigma \to 0} F(x, \sigma) = \mu(x) \quad (12)$$

and condition 3) means that $\mu(x)$ must be the delta function. Hence, on substituting this back into (10), the only solution of (1) which satisfies our condition is the Gaussian

$$G(x, \sigma) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{d}{c}\frac{1}{\sigma}}\, e^{(-d/2c)(x^2/\sigma^2)}. \quad (13)$$

This analysis can be extended to the 2-D generalized diffusion equation

$$\frac{aF}{\sigma^2} + \frac{b_1F_x}{\sigma} + \frac{b_2F_y}{\sigma} + c_1F_{xx} + c_2F_{yy} + c_3F_{xy} = \frac{d}{\sigma} F\sigma. \quad (14)$$

A similar argument shows that the only solution obeying the condition in a 2-D space is, with $c_3 = 0$ because of the symmetry requirements of Section I

$$G(x, y, \sigma) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{d}{c_1}} \sqrt{\frac{d}{c_2}\frac{1}{\sigma^2}}\, e^{(-d/2c_1)(x^2/\sigma^2)}\, e^{(-d/2c_1)(y^2/\sigma^2)}. \quad (15)$$

We again use the symmetry requirement of Section I to set $c_1 = c_2$. Then we obtain

$$G(x, y, \sigma) = \frac{1}{2\pi}\frac{d}{c}\frac{1}{\sigma^2}\, e^{(-d/2c)[(x^2+y^2)/\sigma^2)]}. \quad (16)$$

We can scale the σ axis by $\sqrt{c/d}$ and write (13) and (16) as

$$G(x, \sigma) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\, e^{-x^2/2\sigma^2} \quad (17)$$

and

$$G(x, y, \sigma) = \frac{1}{2\pi}\frac{1}{\sigma^2}\, e^{-(x^2+y^2)/2\sigma^2}, \quad (18)$$

respectively. This ensures that σ is the standard deviation of the function.

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