Reprinted from Journal of the Optical Society of America A, Vol. 2, page 683, May 1985

Fingerprints theorems for zero crossings

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Received April 5, 1984; accepted December 18, 1984

We prove that the scale map of the zero crossings of almost all signals filtered by a Gaussian filter of variable size determines the signal uniquely, up to a constant scaling. The proof assumes that the filtered signal can be represented as a polynomial of finite, albeit possibly high, order. The result applies to zero and level crossings of linear differential operators of Gaussian filters. In this case the signal is determined uniquely, modulus the null space of the linear operator. The theorem can be extended to two-dimensional functions. These results are reminiscent of Logan's theorem [Bell Syst. Tech. J. **56**, 487 (1977)]. They imply that extrema of derivatives at different scales are a complete representation of a signal. They are especially relevant for computational vision in the case of the Laplacian operator acting on image intensities, and they suggest rigorous foundations for the primal sketch.

1. INTRODUCTION

Images are often described in terms of edges, which are usually associated with the zeros of some differential operator. For instance, zero crossings in images convolved with the Laplacian operator of a Gaussian filter have been extensively used as the basis representation for later processes, such as stereopsis and motion.[1] In a similar way, sophisticated processing of one-dimensional (1-D) signals requires that a symbolic description first be obtained, in terms of changes in the signal. These descriptions must be concise, and, at the same time, they must capture the meaningful information contained in the signal. It is clearly important, therefore, to characterize in which sense the information in an image or a signal is captured by extrema or zeros of derivatives.

Ideally, one would like to establish a unique correspondence between the changes of intensity in the image and the physical surfaces and edges that generate them through the imaging process. This goal is extremely difficult to achieve in general, although it remains one of the primary objectives of a comprehensive theory of early visual processing.

A more restricted class of results, which does not exploit the constraints dictated by the signal- or image-generation process, has been suggested by work on zero crossings of images filtered with the Laplacian of a Gaussian. Logan[2] had shown that the zero crossings of a 1-D bandpass signal with a bandwidth of less than an octave determine uniquely the filtered signal (up to scaling). The theorem has been extended—only in the special case of oriented bandpass filters—to two-dimensional (2-D) images,[3,4] but it cannot be used for Gaussian filtered signals or images, since they are not ideally bandpass. Nevertheless, Marr *et al.*[4] conjectured that the zero-crossing maps, obtained by filtering the image with the second derivative of Gaussians of variable size, are rich in information about the signal itself.[1,5–8] Extensive physiological and psychophysical data also suggest that the human visual system uses multiple resolutions in the early processing of images. The work on spatial-frequency channels, started by Campbell and Robson,[9] has shown that detection of, and adaptation to, gratings is mediated by independent mechanisms with different spatial resolutions. A subset of retinal ganglion cells is thought to be the first step in filtering the image at multiple scales. Their center-surround receptive field can be approximated by a difference of Gaussian (DOG) function.[6,10]

In their paper on edge-detection Marr and Hildreth proposed a few heuristic rules to classify various edges by exploiting the behavior of zero crossings across four scales. This was part of a premature attempt to construct a primal sketch of the image. We expect that the results presented in this paper will provide the necessary foundations for a more rigorous approach to the problem.

More recently, Witkin[11] and Stansfield[12] introduced a scale-space description of zero crossings, which gives the position of the zero crossing across a continuum of scales, i.e., sizes of the Gaussian filter (parameterized by the standard deviation σ of the Gaussian). The signal—or the result of applying to the signal a linear (differential) operator—is convolved with a Gaussian filter over a continuum of sizes of the filter. Zero or level crossings of the filtered signal are contours on the x–σ plane (and surfaces in the x, y, σ space). The appearance of the scale map of the zero crossing—an example is shown in Fig. 1—is suggestive of a fingerprint. Witkin has proposed that this concise map can be effectively used to obtain a rich and qualitative description of the signal. Furthermore, it has been proved in one dimension[13,14] and two-dimensions[14–16] that the Gaussian filter is the only filter with a "nice" scaling behavior. More precisely, zero crossings are not created as the scale of the filter increases. In this paper, we prove a stronger completeness property: the map of the zero crossing across scales determines the signal uniquely for almost all signals (in the absence of noise). The scale maps obtained by Gaussian filters are true fingerprints of the signal. Our proof is constructive. It shows how the original signal can be reconstructed by information from the zero-crossing contours across scales. It is important to emphasize that our result applies to level crossings of any arbitrary linear (differential) operator of the Gaussian, since it applies to functions that obey the diffusion equation. These results are originally reported by Yuille and Poggio.[15]

Our fingerprint theorems can be regarded as an extension of Logan's result to Gaussian-filtered, nonbandpass signals and 2-D images (the 2-D proof, which is an extension of the 1-D proof, is given in full by Yuille and Poggio.[15] There are, however, some important differences between Logan's theorem and the fingerprints theorems. Logan uses a bandpass

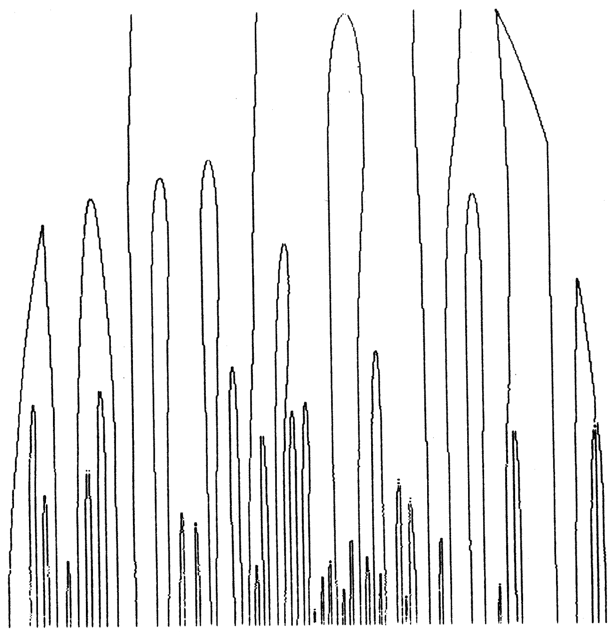

Fig. 1. The scale map of the zero crossings of the second derivative of a signal (a 1-D slice of a natural image). The x axis is the abscissa; the scale, i.e., σ, increases from the bottom to the top. Our theorem states that this map is a true fingerprint since it uniquely determines the signal (modulus the null space of the operator).

filter, at one scale only, and shows that the zero crossings determine the filtered signal. His proof is nonconstructive and applies only in one dimension (2-D generalizations exist,[3] but none is fully satisfactory). The fingerprints theorems determine the original signal from the zero crossings of the signal filtered at different scales. The proof is constructive and applies in both one and two dimensions. Reconstruction of the signal is of course not the goal of early signal processing. Symbolic primitives must be extracted from the signals and used for later processing. Our results imply that scale-space fingerprints are complete primitives, which capture the whole information in the signal and characterize it uniquely. Subsequent processes can therefore work on this more compact representation instead of on the original signal.

Our results have theoretical interest in that they answer the question of what information is conveyed by the zero and level crossings of multiscale Gaussian filtered signals. From a point of view of applications, the results in themselves do not justify the use of the fingerprint representation. Completeness of a representation (connected with Nishihara's sensitivity) is not sufficient.[17] A good representation must, in addition, be robust (i.e., stable in Nishihara's terms) against photometric and geometric distortions (the general-point-of-view argument). It should also, if possible, be compact for the given class of signals. Most importantly, it should make explicit the information that is required by later processes. Fingerprints of images may have these additional properties.[18] Fingerprints may make relevant information explicit. This is still an open question, although recent work by Asada and Brady[19] is encouraging. They consider the fingerprints of the orientation of the tangent vector of a 2-D shape. They show that several robust shape primitives—such as dents, corners, cranks, ends and smooth joins—can be identified. We expect that these results can be extended to 2-D images by using primitives of the primal sketch type.[1] We should emphasize that this use of fingerprints is quite different from Witkin's proposal.

2. ASSUMPTIONS AND RESULTS

We consider the zero crossings of a signal $F(x)$, space-scale filtered with a Gaussian, as a function of x, σ. Let F and E be defined by

$$E(x, \sigma) = F(x) * G(x, \sigma),$$

$$E(x, \sigma) = \int F(\zeta) \frac{1}{\sigma} \exp\left[-\frac{(x-\zeta)^2}{2\sigma^2}\right] d\zeta. \qquad (2.1)$$

Notice that $E(x, \sigma)$ obeys the diffusion equation in x and σ:

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{\sigma}\frac{\partial E}{\partial \sigma}. \qquad (2.2)$$

A special case, which is especially interesting for vision, arises when one considers the zero crossings of a second-order differential operator applied to the image. In this case we consider $F(x)$ as the result of applying a second derivative to the image $I(x)$, that is, $F(x) = (d^2/dx^2)I$. We restrict ourselves to images, or signals, of class P such that E can be expressed as a finite Taylor series of arbitrarily high order and such that E is not antisymmetric about all its zeros. Observe that any filtered image can be approximated arbitrarily well in this way, because of the classical Weierstrass approximation thereom. We can make even stronger claims, however. E is not only continuous but also very smooth, even if F is discontinuous. More precisely, it is an entire analytic function for two reasons. First, diffraction-limited optics makes images band limited and therefore entire. Second, convolution of any bounded function with the Gaussian is an entire analytic function. Therefore the restriction to polynomials is justified.

It is easy to show that functions that are antisymmetric around all their zeros—for instance, a sine wave—have zero-crossing contours that go vertically upward at all scales and conversely. Note that, for a finite-order polynomial, functions antisymmetric about all their zeros only have one zero-crossing contour. We show below a simple way to extend our results to this class of functions.

We will show that the local behavior of the zero-crossing curves [defined by $E(x, \sigma) = 0$] on the x–σ plane determines $F(x)$ up to a constant scaling factor. We will also discuss its (obvious) extension to zero and level crossings of linear (differential) operators. More precisely, we will prove the following theorem:

Theorem 1: The derivatives (including the zero-order derivative) of the zero-crossing contours defined by $E(x, \sigma) = 0$, at two distinct points at the same scale, determine uniquely a generic signal of class P up to a constant scaling.

Note that the theorem applies only to signals that have at least two distinct zero-crossing contours. Another remark is relevant here: The Gaussian filter seems critical for our proof, but we cannot show that it is the only filter with this property. Theorem 1 can be extended to the 2-D case without difficulties.[14,15]

Theorem 2: Derivatives of the zero-crossing contours, defined by $E(x, y, \sigma) = 0$, at two distinct points at the same scale, uniquely determine a generic image of class P up to a scaling factor.

If the signal is not a polynomial, a similar, weaker, result can be proved. A best solution can always be found, but it may not be unique. These theorems break down when all the zero-crossing contours are independent of scale (i.e., the contours go straight up in the scale-space fingerprint). This class of functions is discussed in detail in a forthcoming paper,[20] in which it is shown that additional information about the gradient of the function on the zero crossings is sufficient to determine the signal. An even simpler alternative is to consider the level crossings of the signal at any nonzero level: These uniquely determine functions of this class.

The proofs of the theorems are based on the implicit function theorem, which may break down at isolated points in the signal E. For our purposes this does not matter, since we can avoid these points (the theorems need only two points). For a more detailed discussion, see Yuille and Poggio.[20]

The theorems do not directly address the stability of the fingerprint representation. The first question is related to the noise sensitivity of reconstructing the filtered function $E(x, \sigma)$ at the σ where the derivatives are taken. Our proof relies only on two points on the zero-crossing contours. Exploitation of the whole zero-crossing contours should make the information content of zero-crossing contours considerably robust.

A second question is about the stability of the recovery of the unfiltered signal $F(x) = E(x, 0)$ and $E(x, \sigma)$. This is equivalent to inverting the diffusion equation, which is an ill-posed problem, numerically unstable. Reconstruction is, however, possible, with an error depending on the signal-to-noise ratio behavior.[21,22]

For vision we are particularly interested in zero crossings of the Laplacian of a Gaussian.[6] In this case the image is determined up to the null space of the Laplacian, a harmonic function. This indeterminancy is not a problem. It has long been known that the human vision system is rather insensitive to linear illumination gradients.

OUTLINE OF THE ONE-DIMENSIONAL PROOF

We summarize here the 1-D proof from a slightly different point of view that clarifies its bare structure.

The proof starts by taking derivatives along the zero-crossing contours at a certain point. Such derivatives split into combinations of x and t derivatives (where $t = \sigma^2/2$). Because the filter is assumed to be Gaussian, however, derivatives can be expressed in terms of x derivatives. This is a key point: Since the filtered signal $E(x, t)$ satisfies the diffusion equation, the t derivatives can be expressed in terms of the x derivatives simply by $E_t = E_{xx}$. The next stage is to find the x derivatives of $E(x, t)$ up to an arbitrary degree n from such derivatives along the zero-crossing contours in the x–t plane. We show that this can be done by using two points on two contours. (It is possible that one point is sufficient, but we are as yet unable to prove this.) Since $E(x, t)$ is entire analytic, because of the Gaussian filtering, it can be represented by a Taylor series expansion in x. Since we know the values of the n derivatives of $E(x, t)$ with respect to x, we know its Taylor series expansion and hence $E(x, t)$. The unfiltered signal $F(x)$, $[E(x, t) = F(x) * G(x, t)]$ can then be recovered in the ideal noiseless case by deblurring the Gaussian. A particularly simple way of doing this is provided by a property of the function ϕ_n in which we will expand the function F: The coefficients of an expansion of $F(x)$ in terms of ϕ_n are equal to the coefficients of the Taylor series expansion of $E(x, t)$. In the presence of noise, the recovery of $F(x)$ from $E(x, t)$ is obviously unstable. It is limited by to signal-to-noise ratio since high spatial frequencies in the signal are masked by the noise for increasing t. [For instance, if $F(x) = \Sigma a_\mu \exp(i\mu x)$, the filtered signal is $E(x, t) = \Sigma a_\mu \exp(i\mu x) \exp(-\mu^2 t)$.] Note that since the zero-crossing contours are available at all scales a reconstruction scheme that exploits more than two points will be significantly more robust. As one would expect, the reconstruction of the unfiltered signal is therefore affected by noise. The reconstruction of the filtered signal $E(x, t)$ is likely to be considerably more robust. We plan to study theoretically and with computer simulations the noise sensitivity of the reconstruction scheme.

3. PROOF OF THE THEOREM IN ONE DIMENSIONAL

We divide our proof into two main steps. In the first, Subsection 3.A, we show that derivatives at a point on a zero-crossing contour put strong constraints on the coefficients of the Taylor series expansion of $E(x, \sigma)$ [see Eq. (3.A.4)]. Appendix 1 relates the coefficients of the Taylor series expansion to the coefficients of the expansion of $F(x) = E(x, 0)$ in functions related to the Hermite polynomial. In Subsection 3.B we show that the coefficients can be uniquely determined by the derivatives on a second point of a different zero-crossing contour.

A. Coefficients of the Signal Are Constrained by the Zero-Crossing Contours

Let the Fourier transform of the signal $I(x)$ be $\tilde{I}(\omega)$ and the Gaussian filter be $G(x, \sigma) = (1/\sigma)\exp(-x^2/2\sigma^2)$ with Fourier transform $\tilde{G}(\omega) = \exp(-\sigma^2\omega^2/2)$.

The zero crossings are given by solutions of $E(x, t) = 0$. $E(x, t)$ is an analytic function in x:

$$E(x, t) = \sum_{n=0}^{\infty} C_n \frac{(x - x_0)^n}{n!}, \tag{3.A1}$$

where

$$C_n = \frac{\partial^n E(x, t)}{\partial x^n} \tag{3.A2}$$

evaluated at $x = x_0$. The position of x_0 does not matter. The implicit function theorem gives curves $x(t)$, which are C^∞ (this is a property of the Gaussian filter and of the diffusion equation; see Yuille and Poggio[14]). Let ζ be a parameter of the zero-crossing curve. Then

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} = \frac{\mathrm{d}x}{\mathrm{d}\zeta}\frac{\partial}{\partial x} + \frac{\mathrm{d}t}{\mathrm{d}\zeta}\frac{\partial}{\partial t}. \tag{3.A3}$$

On the zero-crossing surface, $E = 0$ and $(\mathrm{d}^n/\mathrm{d}\zeta^n)E = 0$ for all integers n. Knowledge of the zero-crossing curve is equivalent to knowledge of all the derivatives of x and t with respect to ζ.

We compute the derivatives of E with respect to ζ at (x_0, t_0). Since $E(x, t)$ obeys the diffusion equation $E_t = E_{xx}$, we can substitute for partial derivatives with respect to t. The first derivative is

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}E(x,t) = \frac{\mathrm{d}x}{\mathrm{d}\zeta}C_1 + \frac{\mathrm{d}t}{\mathrm{d}\zeta}C_2 \tag{3.A4}$$

and is expressed in terms of the first and second coefficients of the Taylor series expansion of $E(x, t)$.

The second derivative is

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}E(x,t) = \frac{\mathrm{d}^2x}{\mathrm{d}\zeta^2}C_1 + \frac{\mathrm{d}^2t}{\mathrm{d}\zeta^2}C_2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right)^2 C_2 + 2\frac{\mathrm{d}x}{\mathrm{d}\zeta}\frac{\mathrm{d}t}{\mathrm{d}\zeta}C_3 + \left(\frac{\mathrm{d}t}{\mathrm{d}\zeta}\right)^2 C_4. \tag{3.A5}$$

Since the parametric derivatives along the zero-crossing curve are zero, Eq. (3.A4) is a homogeneous linear equation in the first two coefficients. Similarly, Eq. (3.A5) is a homogeneous linear equation in the first four coefficients. In general, the nth equation, $(\mathrm{d}^n/\mathrm{d}\zeta^n)E(x,t) = 0$, is a homogeneous equation in the first $2n$ coefficients. We choose our axes such that $x_0 = 0$. Appendix A shows that the coefficients of the Taylor series expansion of $E(x, t)$ are the coefficients in the expression of the function $F(x)$ in Hermite polynomials. So we have n equations in the first $2n$ coefficients C_n. To determine the C_n uniquely, we need n additional and independent equations, which, as we will show in Subsection 3.B, can be provided by considering a neighboring zero-crossing curve at (x_1, t_0).

B. Combining Information from Two Contours

The derivatives at (x_0, t_0) give us n equations in the first $2n$ coefficients of the Taylor series expansion of $E(x, t)$ about $x = x_0$. We can relate them to the expansion coefficients K_n of $E(x, t)$ about a second point $x = x_1$:

$$E(x,t) = \sum_{n=0}^{\infty} K_n \frac{(x-x_1)^n}{n!}. \tag{3.B1}$$

We have n equations for the $2n$ unknowns K_n. Now observe that

$$\sum_{n=0}^{\infty} K_n \frac{(x-x_1)^n}{n!} = \sum_{n=0}^{\infty} C_n \frac{(x-x_0)^n}{n!}. \tag{3.B2}$$

Without loss of generality, we set $x_0 = 0$. Then we can use Eq. (3.B2) to relate the C_n to the K_n by

$$C_n = \sum_{m=n}^{\infty} \frac{K_m {x_1}^m (-1)^m}{m!} \tag{3.B3}$$

and

$$K_n = \sum_{m=n}^{\infty} \frac{C_m {x_1}^m}{m!}. \tag{3.B4}$$

Thus we express each K_n in terms of C_n's, and then we combine the equations from two points to obtain $2n$ equations for the $2n$ coefficients C_n. Thus with the results of the next two subsections the proof will be complete.

C. Independence of the Equations

We have to show that information from two points yields a unique solution. The first n equations in the $2n$ first moments from a point can be written as

$$\begin{bmatrix} \frac{\mathrm{d}x}{\mathrm{d}\zeta}\ \frac{\mathrm{d}t}{\mathrm{d}\zeta} & 0 & 0 & 0 & \cdots & 0 \\ \frac{\mathrm{d}^2x}{\mathrm{d}\zeta^2}\ \frac{\mathrm{d}^2t}{\mathrm{d}\zeta^2} + \left(\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right)^2 & 2\frac{\mathrm{d}x}{\mathrm{d}\sigma}\frac{\mathrm{d}t}{\mathrm{d}\zeta} & \left(\frac{\mathrm{d}t}{\mathrm{d}\zeta}\right)^2 & 0 & \cdots & 0 \\ \cdots\ \cdots & & & & & \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{2n} \end{bmatrix} = 0. \tag{3.C1}$$

The matrix of the coefficients is a $n \times 2n$ matrix. Note that its rows are linearly independent (since the coefficients of the rth-row vector are zero after the $2r$th component).

The next n equations are given by the matrix of the derivatives at a second point, x_1, that have the same form as Eq. (3.C1), multiplied by the coefficients of the expansion at (x_1):

$$\begin{bmatrix} \frac{\mathrm{d}\bar{x}}{\mathrm{d}\zeta}\ \frac{\mathrm{d}\bar{t}}{\mathrm{d}\zeta} & 0 & 0 & 0 & \cdots & 0 \\ \frac{\mathrm{d}^2\bar{x}}{\mathrm{d}\zeta^2}\ \frac{\mathrm{d}^2\bar{t}}{\mathrm{d}\zeta^2} + \left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}\zeta}\right)^2 & 2\frac{\mathrm{d}\bar{x}}{\mathrm{d}\zeta}\frac{\mathrm{d}\bar{t}}{\mathrm{d}\zeta} & \left(\frac{\mathrm{d}\bar{t}}{\mathrm{d}\zeta}\right)^2 & 0 & \cdots & 0 \\ \cdots\ \cdots & & & & & \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_{2n} \end{bmatrix} = 0. \tag{3.C2}$$

The coefficients K_n can be expressed in terms of the coefficients C_n by the following transformation (see Subsection 3.B):

$$\begin{bmatrix} 1 & x_1 & \frac{{x_1}^2}{2} & \frac{{x_1}^3}{3!} & \cdots & \frac{{x_1}^{2n}}{2n!} \\ 0 & 1 & x_1 & \cdots & & \\ & & & & & \\ 0 & 0 & 1 & x_1 & \cdots & \\ & & & & 1 & \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{2n} \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_{2n} \end{bmatrix}. \tag{3.C3}$$

Equation (3.C3) substituted into Eq. (3.C2) gives, together with Eq. (3.C1), the full set of $2n$ equations in the $2n$ unknowns C_i. The $2n \times 2n$ matrix of the coefficients can be thought of as originating from the first point (the top half) and from the second point (the bottom half) on the zero-crossing curves.

In general, the determinant of this matrix is nonzero. Intuitively, if the filtered signal has nonzero coefficients of order higher than $2n$, the system of $2n$ equations would not have a solution. A proof for this claim is given in Appendix B. The argument is based on the fact that the determinant of the coefficients is a polynomial in x_1. If this vanishes, then x_1 can be expressed in terms of the first n derivatives at the two points. We show, however, that in general it is possible to change x continuously without altering the first n derivatives. This implies that the determinant is almost always different from zero. The argument breaks down if the filtered signal is a polynomial of degree $2n$ or less.

In this case, the determinant must be zero, since the homogeneous set of equations has at least one solution. At this point, we have to show that the solution is unique. We first observe that the determinant of the coefficients of the $2n \times 2n$ system of equations is a polynomial in x_1. This polynomial is nontrivial[23] since the first n and the second n equations separately are independent. It follows that the determinant vanishes at a finite number (at most $2n$) of values of x_1.

Suppose that the determinant is zero. Observe that x_1 is known from the position of the zero-crossing curves (x_1 is the distance between the two points at which derivatives are taken). Typically the roots of the polynomial in x_1 will be distinct, and there will be a unique zero eigenvector of the matrix. Thus we have proved that n derivatives at two points determine uniquely (modulus a common scaling factor) the $2n$ coefficients of a polynomial of degree $2n$. The case of multiple-zero eigenvectors is nongeneric, i.e., an arbitrarily small perturbation in the image would annihilate eventual multiple-zero eigenvectors. Furthermore, multiple-zero eigenvectors of the matrix of degree $2n$ must also be multiple-zero eigenvectors of all higher-order matrices, which is even more unlikely (except on a set of measure zero).

Our proof is limited to filtered function of the polynomial type (albeit of high degree). We now sketch an argument suggesting that the result holds also for most filtered functions $E(x, y)$ that are not polynomials.

Consider the homogeneous system of equations obtained from two points up to degree n. Denote by A' the matrix of the coefficients. Let A be the matrix of the coefficients of the inhomogeneous system of equations obtained by dividing all unknowns C_1 to C_{2N} by the first coefficient. The system, $AC' = Z$, where Z is the first column of A' divided by C_1, does not in general have solutions, as we have shown (see Appendix B). Furthermore, A has no null vector (if it has, then A' must also have a null vector, which is impossible since det $A' \neq 0$). Then there is a unique least-squares solution of the equation $\|AC' - Z\| = 0$ given by $C' = A^{+}Z$, where A^{+} is the pseudoinverse of A (see Ref. 24). Thus for every finite C there is a unique least-squares solution to the system of equations $AC' = A$ but no exact solution. As n goes to infinity, however, at least one exact solution must appear.

To summarize, in Subsection 3.A we showed that the coefficients of the Taylor series expansion of the signal are constrained by the derivatives of the zero-crossing contours at one point. In Appendix A we show that the moments are equal to the coefficients of the expansion of the unfiltered signal $F(x)$ in our Hermite-like expansion. In Subsection 3.A we showed how we can combine constraints from two different points on the zero-crossing contours at the same scale. Finally, in Subsection 3.B we demonstrated that the equations obtained in this way from two points determine a unique solution. The stability of the solution was briefly discussed in Section 2.

The theorems are illustrated by a simple, worked-out example in Appendix C.[15]

4. CONCLUSIONS

We conclude with a brief discussion of a few issues that are raised by this paper and that will require further work.

Stability of the Reconstruction

Although we have not yet rigorously addressed the question of numerical stability of the whole reconstruction scheme, there seem to be various ways for designing a robust reconstruction scheme. The first step to consider is the reconstruction of the filtered signal $E(x, t)$. One could exploit the derivatives at n points—at the given σ—and then solve the resulting highly constrained linear equations with least-squares methods. Alternatively, it may be possible to fit a smooth curve through several points on one contour and then obtain the derivatives there in terms of this interpolated curve. The same process must be performed on a second separate zero-crossing contour. This scheme provides a rigorous way of proving that, instead of derivatives at two points, the location of the whole zero-crossing contour across scales can be used directly to reconstruct the signal (since the implicit function theorem shows that the zero-crossing curve is C^{∞}).

The second step involves the reconstruction of the unfiltered signal $F(x)$. We can construct $E(x, 0) = F(x)$ explicitly in terms of Hermite functions.[15]

Degenerate Fingerprints

Our uniqueness result applies to almost all signals: A restricted class of signals with vertical zero crossings in the scale-space diagram, such as a sine or a square wave, corresponds to nonunique fingerprints. These signals, which will be discussed in a forthcoming paper[20] and which correspond to functions antisymmetric about all their zeros, do not belong to the class P introduced in Theorems 1 and 2. Interestingly, level crossings (with a level different from zero) can distinguish between elements of this class.

Extensions

Our main results apply to zero and level crossings of a signal filtered by a Gaussian filter of variable size. They also apply to transformations of a signal under a linear space-invariant operator—in particular, they apply to the linear derivatives of a signal and to linear combinations of them.

Are the Fingerprints Redundant?

The proof of our theorem implies that two points on the fingerprint contours are sufficient. As we mentioned earlier, several points are probably required to make the reconstruction robust and to avoid a nongeneric pair of points. We conjecture, however, that the fingerprints are redundant and that appropriate constraints derived from the process underlying signal generation (the imaging process in the case of images) should be used to characterize how to collapse the fingerprints into more-compact representations. Witkin[11] has already made this point and discussed various heuristic ways to achieve this goal.

Implications of the Results

As we discussed in the Introduction, our results imply that the fingerprint representation is a complete representation of a signal or an image. Zero and level crossings across scales of a filtered signal capture full information about it. These results also suggest a central role for the Gaussian filter in multiscale filtering that ensures that zero and level crossing indeed contain full information. Note, however, that the fingerprint theorems do not constrain or characterize in any way the differential filter that has to be used. The filter may be just the identity operator, provided of course that enough zero-crossing contours exist. Independent arguments, based on the constraints of the signal-formation process, must be exploited to characterize a suitable filter for each class of signals. For images, second-derivative operators such as the Laplacian are suggested by work that takes into account the physical properties of objects and of the imaging process.[25,26] We plan to explore this approach in the near future.

Zero Crossings and Slopes

One can ask whether gradient information across scales at the zero crossings, in addition to their location, characterizes uniquely the signal and can be used to reconstruct it. Hummel[27] has recently shown that this is the case, as one would expect in the light of our results.[14] We have been able to simplify and extend the elegant proof by Hummel and obtain the following result[28]: Knowledge of zero-crossing surfaces and magnitude of the x–t gradient over a finite, nonzero interval of the zero-crossing surface is sufficient to determine the image in the usual sense.

Significance of the Proof

It is important to emphasize that our proof is a uniqueness proof only. We stress that we do not suggest reconstructing the signal from its fingerprints.

Fingerprints and the Primal Sketch

The original goal of the primal sketch program was to characterize the different types of intensity changes in the image. A major obstacle, as it turned out, was the lack of a theory of multiple scales. The idea of using filters with different sizes, then at an embryonic stage, has developed into a body of practical and theoretical results. Fingerprints now provide a way of attacking again the main problem posed by the primal sketch. The idea is to identify a small number of primitive image-intensity features—such as step edges and roof edges—and label them partially in terms of the properties of the underlying physical surfaces—distinguishing, for instance, shadows from occlusion boundaries. The initial success of a similar attempt in the realm of 2-D shape representation by Asada and Brady[19] is quite encouraging.

APPENDIX A: RELATING THE TAYLOR EXPANSION TO HERMITE POLYNOMIALS

We first review Hermite functions and polynomials. We will then derive useful relations between Taylor series coefficients and Hermite coefficients.

The set of Hermite functions is defined by

$$\psi_n(x) = \frac{\exp(-x^2/2)H_n(x)}{(2^n!\sqrt{\pi})^{1/2}}, \tag{A1}$$

where H_n are the Hermite polynomials:

$$H_n(x) = (-1)^n \exp(x^2)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2). \tag{A2}$$

The Hermite functions are an orthonormal basis of functions that is complete for L^2 functions. The completeness is expressed by

$$\sum_n \psi_n(x)\psi_n(\zeta) = \delta(x-\zeta) \tag{A3}$$

and the orthonormality by

$$\int \psi_n(x)\psi_m(x)\mathrm{d}x = \delta_{nm}. \tag{A4}$$

In general, the Hermite expansion of a L^2 function does not converge uniformly but only in the L^2 norm. The series will converge to the function except at a set of points of measure zero. At any point, the series can be truncated at a term of order N such that the remainder of the series is arbitrarily small. If we consider only a finite number of points where the series converges, the series can be truncated and the function approximated arbitrarily well by a finite number of Hermite components.

The Hermite polynomials defined in Eqs. (2) and the set of functions $w_n(x)$ defined as [see Eq. (3.B6)]

$$w_n(x) = \frac{1}{2^n n!\sqrt{\pi}}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2) \tag{A5}$$

are biorthogonal sets of functions, i.e.,

$$\int H_n(x)w_m(x)\mathrm{d}x = \delta_{nm}. \tag{A6}$$

They also obey a completeness property

$$\sum_{nm} H_n(x)w_n(\zeta) = \delta(x-\zeta), \tag{A7}$$

and therefore a L^2 function $f(x)$ can be expanded in either set of functions as

$$f(x) = \sum_n a_n H_n(x),$$
$$f(x) = \sum_n b_n w_n(x), \tag{A8}$$

with

$$a_n = \langle f, w_n\rangle,$$
$$b_n = \langle f, H_n\rangle. \tag{A9}$$

We now show that the coefficients C_n defined by Eq. (3.A2) are the coefficients of the expansion of $F(x)$ in functions related to the Hermite polynomials. We expand $F(x)$ in terms of the functions $\varphi_n(x, \sigma)$ related to the Hermite polynomials $H_n(x)$ by

$$\varphi_n(x,\sigma) = (-1)^n\frac{\sigma^{n-1}}{(\sqrt{2})^{n+1}\sqrt{\pi}}H_n\left(\frac{x}{\sqrt{2}\sigma}\right), \tag{A10}$$

$$F(x) = \sum_{n=0} a_n(\sigma)\varphi_n(x,\sigma). \tag{A11}$$

The coefficients $a_n(\sigma)$ of the expansion are given by

$$a_n(\sigma) = \langle w_n(x,\sigma), F(x)\rangle, \tag{A12}$$

where $\langle,\rangle$ denote inner product in L^2 and $\{w_n(x, \sigma)\}$ is the set of functions biorthogonal to $\{\varphi_n(x, \sigma)\}$. The $\{\varphi_n(x, \sigma)\}$ are given explicitly by

$$\varphi_n(x,\sigma) = \frac{\sigma^{2n-1}}{n!\sqrt{2}\sqrt{\pi}}\exp(x^2/2\sigma^2)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2/2\sigma^2) \tag{A13}$$

and the $w_n(x, \sigma)$ by

$$w_n(x,\sigma) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2/2\sigma^2). \tag{A14}$$

Convolving the $\{\varphi_n(x, \sigma)\}$ with a Gaussian $G(x, \sigma)$ gives

$$G(x,\sigma)*\varphi_n(x,\sigma) = \frac{(-1)^n(x)^n}{n!}; \tag{A15}$$

hence the a_n are given by

$$a_n = (-1)^n C_n. \tag{A16}$$

Therefore knowledge of the image is equivalent to knowing the a_n.

APPENDIX B: INDEPENDENCE

We will show that the $2n$th-order determinant is generally nonzero. Recall that the determinant is a polynomial in x_1 (of degree at most $2n$) with the coefficients being functions of the first n derivatives of the curves at the two points. If this determinant always vanished, it would mean that the distance between any two curves with prescribed values of their first n derivatives could only take a finite set of values (at most $2n$), whatever the values of the higher order derivation of the curves. We will show that, by changing the values of the higher-order derivatives, it is possible to alter the value of x_1 continuously while keeping the first n derivation of the curves constant.

We take two points $(0, t_1)$ and (x_1, t_1) lying on zero-crossing curves. At these points, we assume that we know the derivatives $\partial E/\partial x$, $\partial E/\partial t$, $\partial^2 E/\partial x \partial t, \ldots,$ up to order n. (This means that we can reconstruct $\mathrm{d}x/\mathrm{d}\sigma, \ldots, \mathrm{d}^n x/\mathrm{d}\sigma^n$ from the implicit function theorem.) We can use the diffusion equation to write these as $\partial E/\partial x, \partial^2 E/\partial x^2, \ldots, \partial^{2n} E/\partial x^{2n}$.

So we have

$$
\begin{aligned}
&E(0, t_1) = 0,\\
&\frac{\partial E}{\partial x}(0, t_1) = K_1,\\
&\ldots\\
&\frac{\partial^{2n} E}{\partial^{2n} x}(0, t_1) = K_{2n},
\end{aligned}
\tag{B1}
$$

and

$$
\begin{aligned}
&E(x_1, t_1) = 0,\\
&\frac{\partial E}{\partial x}(x_1, t_1) = C_1,\\
&\ldots\\
&\frac{\partial^{2n} E}{\partial x^{2n}}(x_1, t_1) = C_{2n},
\end{aligned}
\tag{B2}
$$

where $K_1, \ldots, K_{2n}$ and $C_1, \ldots, C_{2n}$ are specified. Now we will try to alter the value of x_1 while keeping $K_1, \ldots, K_{2n}$ and $C_1, \ldots, C_{2n}$ constant.

We have

$$E(x, t) = \int \exp(i\omega x)\exp(-\omega^2 t)\omega^2 I(\omega)\mathrm{d}\omega. \tag{B3}$$

Introduce a deformation parameter λ and a function $Y(\omega, \lambda)$, where

$$Y(\omega, 0) = \omega^2 I(\omega) \tag{B4}$$

and $x_1 = x_2(\lambda)$.

Let

$$E(x, t, \lambda) = \int \exp(i\omega x)\exp(-\omega^2 t)Y(\omega, \lambda)\mathrm{d}\omega. \tag{B5}$$

Allow $x_1(\lambda)$ to vary while maintaining Eqs. (B1) and (B2). For the first point this gives

$$
\begin{aligned}
&\int \exp(-\omega^2 t)\frac{\partial}{\partial \lambda} Y(\omega, \lambda)\mathrm{d}\omega = 0,\\
&\ldots\\
&\int \exp(-\omega^2 t)\omega^{2n}\frac{\partial Y}{\partial \lambda}(\omega, \lambda)\mathrm{d}\omega = 0.
\end{aligned}
\tag{B6}
$$

For the second point we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}\int \exp(i\omega x_1)\exp(-\omega^2 t)(i\omega)Y(\omega, \lambda)\mathrm{d}\omega\\
&\quad + \int \exp(i\omega x_1)\exp(-\omega^2 t)\frac{\partial}{\partial \lambda}Y(\omega, \lambda)\mathrm{d}\omega = 0,\\
&\ldots\\
&\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}\int \exp(i\omega x_1)\exp(-\omega^2 t)(i\omega)\omega^{2n}Y(\omega, \lambda)\mathrm{d}\omega\\
&\quad + \int \exp(i\omega x_1)\exp(-\omega^2 t)\omega^{2n}\frac{\partial}{\partial \lambda}Y(\omega, \lambda)\mathrm{d}\omega = 0.
\end{aligned}
\tag{B7}
$$

We want to solve Eqs. (B6) and (B7) for $(\partial Y/\partial\lambda)(\omega, \lambda)$ in terms of $\mathrm{d}x_1/\mathrm{d}\lambda$. Then the result follows.

Equation (B6) implies that the first $2n$ moments of $(\partial Y/\partial\lambda)(\omega, \lambda)$ are zero. Equation (B7) means that the first $2n$ moments of $\exp(i\omega x_1)(\partial Y/\partial\lambda)(\omega, \lambda)$ take prescribed values. [We assume that $Y(\omega, \lambda)$ is known but that $(\partial/\partial\lambda)Y(\omega, \lambda)$ is not.]

Expanding $\exp(i\omega x_1)$ as a Taylor series [and using Eq. (B6)], we write Eq. (B7) as

$$
\begin{aligned}
&\int \left[\sum_{m=2n+1}^{\infty} (i\omega)^m \frac{x_1{}^m}{m!}\right]\exp(-\omega^2 t)\frac{\partial}{\partial\lambda}Y(\omega, \lambda)\mathrm{d}\omega\\
&\quad = -\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}\int \exp(i\omega x_1)\exp(-\omega^2 t)(i\omega)Y(\omega, \lambda)\mathrm{d}\omega,\\
&\ldots\\
&\int \left[\sum_{m=1}^{\infty} (i\omega)^m \frac{x_1{}^m}{m!}\right]\omega^{2n}\exp(-\omega^2 t)\frac{\partial}{\partial\lambda}Y(\omega, \lambda)\mathrm{d}\omega\\
&\quad = -\frac{\mathrm{d}x_1}{\mathrm{d}\lambda}\int \exp(i\omega x_1)\exp(-\omega^2 t)(i\omega)\omega^{2n}Y(\omega, \lambda)\mathrm{d}\omega.
\end{aligned}
\tag{B8}
$$

The moments of $(\partial/\partial\lambda)Y(\omega, \lambda)$ are

$$W_m = \int \exp(-\omega^2 t)\frac{\partial}{\partial\lambda}Y(\omega, \lambda)\omega^m \mathrm{d}\omega \tag{B9}$$

and define

$$A_p = \frac{-\mathrm{d}x_1}{\mathrm{d}\lambda}\int \exp(i\omega x_1)\exp(-\omega^2 t)(i\omega)\omega^p Y(\omega, \lambda)\mathrm{d}\omega. \tag{B10}$$

Using Eqs. (B9) and (B10), we rewrite Eqs. (B8) as

$$
\begin{aligned}
&\sum_{m=2n+1}^{\infty} \frac{(ix_1)^m}{m!} W_m = A_1\\
&\ldots\\
&\sum_{m=2n+1}^{\infty} \frac{(ix_1)^{m-2n}}{(m-2n)!} W_m = A_{2n+1}.
\end{aligned}
\tag{B11}
$$

It will always be possible to solve these equations for W_m, and there will be infinitely many solutions. To see this, we set

$$W_m = 0, \qquad m > 4n + 1 \tag{B12}$$

and write Eqs. (B11) as

$$\begin{bmatrix} \dfrac{(ix_1)}{(2n+1)!} & \cdots & \dfrac{(ix_1)^{4n+1}}{(4n+1)!} \\ \vdots & & \\ \dfrac{(ix_1)}{1!} & \cdots & \dfrac{(ix_1)^{2n+1}}{2n!} \end{bmatrix} \begin{bmatrix} W_{2n+1} \\ \vdots \\ W_{4n+1} \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_{2n+1} \end{bmatrix}. \qquad \text{(B13)}$$

It is possible to solve Eq. (B13) if the determinant is nonzero. The determinant is of form $\lambda(x_1)^{(2n+1)/2}$ (this follows directly from the form of the matrix) and so is either zero for all x_1 or else never zero. The determinant is also the Wronskian of the function $(ix_1)^{2n+1}/2n!, \ldots, (ix_1)^{4n+1}/(4n+1)!$, and, as these functions are linearly independent, it cannot vanish everywhere. Hence the determinant never vanishes, and we can solve for the W_m's in terms of the A_p's. Relaxing condition (B12) gives us infinitely many solutions.

Thus we have shown that it is possible to alter x_1 continuously without changing the values of the first n derivatives at both points. This means that the determinant of the $2n$-order matrix in the moments will in general be nonzero; it can be zero only for a finite set of x_1, and there is an infinite set of possible values for x_1 compatible with the first n derivatives at the points.

APPENDIX C: EXAMPLES

We now illustrate the theorems by considering some special cases. If the signal is a low-order polynomial in X, it is possible to obtain the zero-crossing curves explicitly. We then use the derivatives of these curves to reconstruct the image, as in the theorem. These examples also suggest that the derivatives of the curves at a single point will usually give sufficient information to reconstruct the signal.

Suppose the signal $F(X)$ is a second-order polynomial in X. In this case our result might seem trivial, since the two roots of the polynomial are known. We are assuming, however, that only the fingerprint of the polynomial is given but not its order. The polynomial is

$$F(X) = 1 + AX + BX^2, \qquad \text{(C1)}$$

where A and B are arbitrary coefficients. All the moments of the signal are zero except for the first two. We convolve this signal with a Gaussian at scale σ and obtain

$$E(X, \sigma) = 1 + AX + BX^2 + B\sigma^2. \qquad \text{(C2)}$$

We consider the curves given by

$$E(X, \sigma) = 0. \qquad \text{(C3)}$$

We write these in the form

$$\sigma^2 + [X^2 + (A/B)X + 1/B] = 0 \qquad \text{(C4)}$$

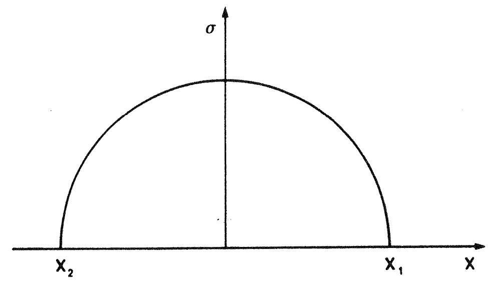

Fig. 2. The zero crossings of the function $(x - x_1)^2 + (x - x_2)^2$, with $x_1 = x_2$, are a semicircle in scale space.

and see that they correspond to circles in the (X, σ) plane. Define X_1 and X_2 by

$$X_1 X_2 = 1/B,$$
$$-(X_1 + X_2) = A/B. \qquad \text{(C5)}$$

Then we can rewrite the equations as

$$\sigma^2 + \left[X - \left(\frac{X_1 + X_2}{2}\right)\right]^2 = \left(\frac{X_2 - X_1}{2}\right)^2. \qquad \text{(C6)}$$

Thus the zero-crossing curve corresponds to a semicircle that intersects the X axis at X_1 and X_2 (see Fig. 2).

We now parameterize the curve by an angle θ so that

$$X(\theta) = \left(\frac{X_1 + X_2}{2}\right) + \left(\frac{X_2 - X_1}{2}\right)\cos\theta,$$
$$\sigma(\theta) = \left(\frac{X_2 - X_1}{2}\right)\sin\theta. \qquad \text{(C7)}$$

We calculate the derivatives

$$\frac{dX}{d\theta} = \left(\frac{X_2 - X_1}{2}\right)\sin\theta,$$
$$\frac{d\sigma}{d\theta} = \left(\frac{X_2 - X_1}{2}\right)\cos\theta. \qquad \text{(C8)}$$

Recalling that $t = \sigma^2/2$, we combine Eqs. (C7) and (C8) to obtain

$$\frac{dt}{d\theta} = \left(\frac{X_2 - X_1}{2}\right)^2 \sin\theta\cos\theta. \qquad \text{(C9)}$$

We differentiate again to obtain

$$\frac{d^2X}{d\theta^2} = -\left(\frac{X_2 - X_1}{2}\right)\cos\theta$$
$$\frac{d^2t}{d\theta^2} = \left(\frac{X_2 - X_1}{2}\right)^2 (\cos^2\theta - \sin^2\theta). \qquad \text{(C10)}$$

We set $X_2 - X_1/2 = b$. Then we write the first two equations at $\theta = \theta_1$ as

$$\begin{bmatrix} -b\sin\theta_1 & b^2\sin\theta_1\cos\theta_1 & 0 & 0 \\ -b\cos\theta_1 & b^2\cos^2\theta_1 & -2b^3\sin^2\theta_1\cos\theta_1 & b^4\sin^2\theta_1\cos^2\theta_1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \text{(C11)}$$

We pick another point on the curve with the same value of σ. This point has parameter $\theta_2 = \pi - \theta_1$ (with $0 < \theta_2 < \pi/2$). This gives us a second equation:

$$\begin{bmatrix} -b\sin\theta_1 & -b^2\sin\theta_1\cos\theta_1 & 0 & 0 \\ b\cos\theta_1 & 2\cos^2\theta_1 & 2b^3\sin^2\theta_1\cos^2\theta_1 & b^4\sin^2\theta_1\cos^2\theta_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & X_1 & \dfrac{X_1^2}{2!} & X_1^3/3! \\ 0 & 1 & X_1 & X_1^2/2! \\ 0 & 0 & 1 & X_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{(C12)}$$

Now we consider the equation for the first two moments obtained by taking the first derivative at both points. From Eqs. (C11) and (C12), this becomes

$$\begin{bmatrix} -b\sin\theta_1 & b^2\sin\theta_1\cos\theta_1 \\ -b\sin\theta_1 & -b^2\sin\theta_1\cos\theta_1 - X_1 b\sin\theta_1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \text{(C13)}$$

The condition for there to be a solution of Eq. (C13) is that the determinant of the matrix vanish. This occurs at

$$X_1 = -2b\cos\theta_1. \quad \text{(C14)}$$

From Eq. (C7), we see that this is indeed the distance between the two points, and so we can solve for M_1 and M_2. We obtain

$$M_1 = b\cos\theta_1 M_2. \quad \text{(C15)}$$

Substituting for $b\cos\theta_1$ from Eq. (C7) yields

$$M_1 = \left[X_0 - \left(\frac{X_1 + X_2}{2}\right)\right] M_2, \quad \text{(C16)}$$

where X_0 is the position of the first point. The reconstructed function is

$$F(X) = -M_1\varphi_1(X - X_0, \sigma_1) + M_2\varphi_2(X - X_0, \sigma). \quad \text{(C17)}$$

Without loss of generality, set $X_0 = 0$. Then, up to a scale factor,

$$F(X) = \frac{(X_1 + X_2)}{2}\frac{X}{\sqrt{2\pi}\sigma_1} + \frac{1}{2\sqrt{2\pi}}\left(-\sigma_1 + \frac{X^2}{\sigma_1}\right). \quad \text{(C18)}$$

Now σ_1 lies on the circle

$$\sigma^2 + \left[X - \left(\frac{X_1 + X_2}{2}\right)\right]^2 = \left(\frac{X_2 - X_1}{2}\right)^2 \quad \text{(C19)}$$

at the point where $X = 0$. Hence

$$\sigma_1^2 = -X_1X_2. \quad \text{(C20)}$$

Note that X_1 and X_2 have opposite signs if $X = 0$ lies on the circle. Substituting Eq. (C20) into Eq. (C18) gives

$$F(X) = \frac{1}{2\sqrt{2\pi}(-X_1X_2)^{1/2}}[X_1X_2 - (X_1 + X_2)X + X^2]. \quad \text{(C21)}$$

From Eqs. (C1) and (C5), we see that this is indeed the original function up to a scaling factor. Thus we have demonstrated how to reconstruct the signal.

We should check that $X_1 = -2b\cos\theta_1$ remains a root of the determinant for the higher-order determinants. We will calculate the result for the case $n = 2$. From Eqs. (C11) and (C12), the determinant equation becomes (in unconventional notation)

$$\begin{bmatrix} -b\sin\theta_1 & b^2\sin\theta_1\cos\theta_1 & 0 & 0 \\ -b\cos\theta_1 & b^2\cos^2\theta_1 & -2b^3\sin^2\theta_1\cos\theta_1 & b^4\sin^2\theta_1\cos^2\theta_1 \\ -b\sin\theta_1 & -X_1b\sin\theta_1 - b^2\sin^2\theta_1\cos\theta_1 & (-X_1^2/2!)b^2\sin\theta_1\cos\theta_1 - X_1b^2\sin\theta_1\cos\theta_1 & C \\ b\cos\theta_1 & X_1b\cos\theta_1 + b^2\cos^2\theta_1 & X_1^2/2!b\cos\theta_1 + X_1b^2\cos^2\theta_1 + 2b^3\sin^2\theta_1\cos^2\theta_1 & B \end{bmatrix} = 0, \quad \text{(C22)}$$

where $C = (-X_1^3/3!)b\sin\theta_1 - (X_1^2/2!)b^2\sin\theta_1\cos\theta_1$ and $B = (X_1^3/3!)b\cos\theta_1 + (X_1^2/2!)\cos^2\theta_1 + X_12b^3\sin^2\theta_1\cos^2\theta_1 + b^4\sin^2\theta_1\cos^2\theta_1$. By dividing the matrix by factors common to rows, this becomes

$$\begin{bmatrix} -1 & b\cos\theta_1 & 0 & 0 \\ -1 & b\cos\theta_1 & -2b^2\sin^2\theta_1 & b^3\sin^2\theta_1\cos\theta_1 \\ -1 & -X_1 - b\cos\theta_1 & -X_1^2/2! - X_1b\cos\theta_1 & -X_1^3/3! - X_1^2/2!b\cos\theta_1 \\ 1 & X_1 + b\cos\theta_1 & X_1^2/2! + X_1b\cos\theta_1 + 2b^2\sin^2\theta_1\cos\theta_1 & V \end{bmatrix} = 0, \quad \text{(C23)}$$

where $V = (X_1^3/3!) + (X_1^2/2!)b\cos\theta_1 + X_12b^2\sin^2\theta_1\cos\theta_1 + b^3\sin^2\theta_1\cos\theta_1$. By adding and subtracting rows, this reduces to

$$\begin{bmatrix} -1 & b\cos\theta_1 & 0 & 0 \\ 0 & 0 & -2b^2\sin^2\theta_1 & b^3\sin^2\theta_1\cos\theta_1 \\ 0 & -X_1-2b\cos\theta_1 & -X_1{}^2/2!-X_1b\cos\theta_1 & -X_1{}^3/3!-X_1{}^2/2!b\cos\theta_1 \\ 0 & 0 & 2b^2\sin^2\theta_1\cos\theta_1 & S \end{bmatrix} = 0, \quad \text{(C24)}$$

where $S = X_1 2b^2 \sin^2\theta_1 \cos\theta_1 + b^3 \sin^2\theta_1 \cos\theta_1$. Thus the equation becomes (removing common factors)

$$(X_1 + 2b\cos\theta_1)\begin{vmatrix} -1 & b \\ \cos\theta_1 & 2X_1 + b \end{vmatrix} = 0 \qquad \text{(C25)}$$

and can be expressed as

$$(X_1 + 2b\cos\theta_1)(2X_1 + b + b\cos\theta_1) = 0. \qquad (26)$$

Hence $X_1 = -2b\cos\theta_1$ remains a root for the $n = 2$ case.

It is clear that the zeros of a polynomial function at one scale are enough to determine the function up to a scaling factor. In the general case, however, many of these zeros are complex and hence are not available from the scale-space map at just one scale.

ACKNOWLEDGMENTS

Support for work done at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology is provided in part by the Advanced Research Projects Agency of the U.S. Department of Defense under U.S. Office of Naval Research contract N00014-80-C-0505. Additionally, T. Poggio was, in part, supported under U.S. Air Force Office of Sponsored Research contract F49620/83/C and a grant from the Sloan Foundation. We are also grateful to E. Grimson, M. Kass, C. Koch, K. Nishihara, and D. Terzopoulos for useful discussions and suggestions. Carol Bonomo worked on what text there was and the equations. The research described in this paper originally appeared in a longer version as a technical paper, Artificial Intelligence Memo 730, Massachusetts Institute of Technology, in October 1983.

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