NSF Engineering Research Center
for Computer Integrated Surgical
Systems and Technology

LABORATORY FOR Computational Sensing + Robotics
THE JOHNS HOPKINS UNIVERSITY

Coherent Point Drift Registration

600.445 Lecture

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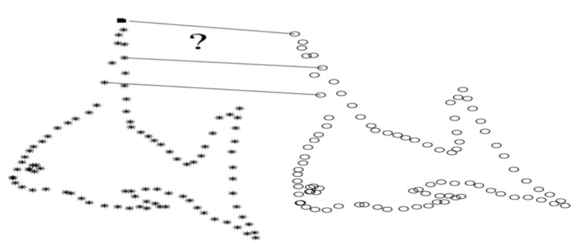
Coherent Point Drift

- A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence,* vol. 32- 12, pp. 2262-2275, 2010.
- Alignment of point clouds
  - Fast method follows “EM” paradigm
  - Tolerates outliers and noise
  - Transformations: Rigid, affine, general deformable

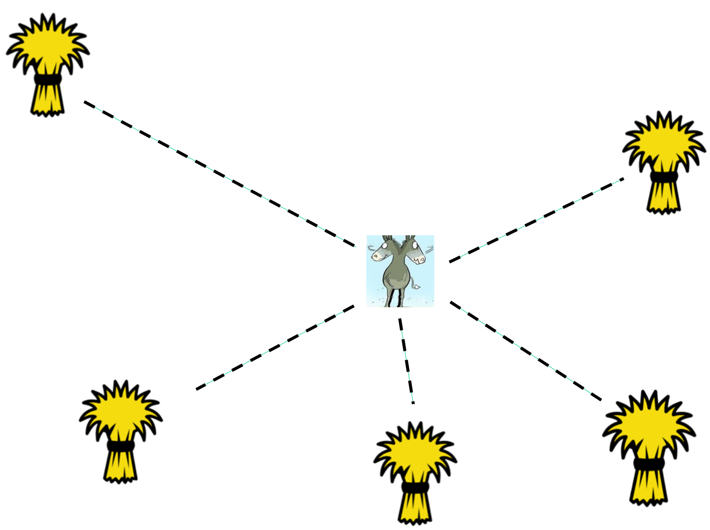


Buridan's Ass: Stuck Between Two Haystacks

Image: www.collegetransitions.com

Article: https://en.wikipedia.org/wiki/Buridan%27s_ass

Many Haystacks



Images: www.collegetransitions.com
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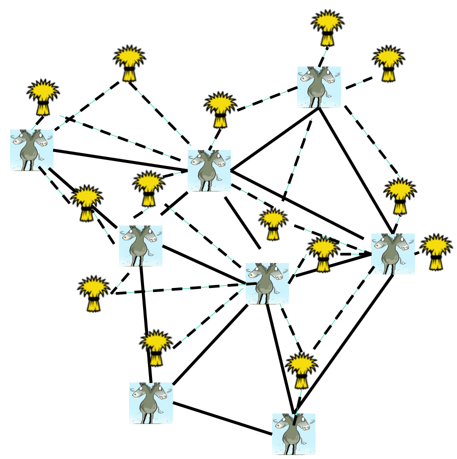
Many Haystacks, Many Donkeys Tied Together



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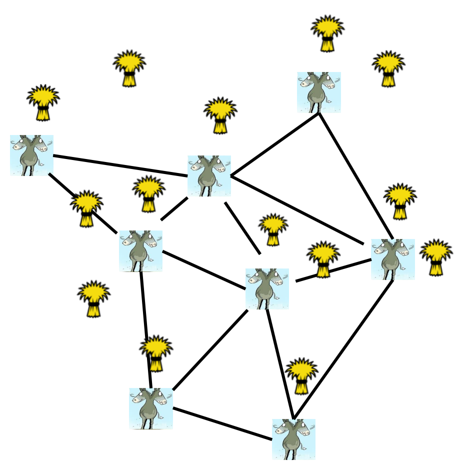
Many Haystacks, Many Donkeys Tied Together



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CPD – Basic EM Paradigm

- Initialization
  - Given initial guess of registration, compute the variance of distances between all possible point pairs
  - Assumes independent isotropic Gaussian distribution for matches, uniform distribution for outliers

$$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_m + \mathbf{t})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_k + \mathbf{t})\|^2} + (2\pi\sigma^2)^{D/2}\frac{w}{1-w}\frac{M}{N}}$$

- “E Step”
  - Based on current variance, compute probability of matches of all possible point pairs, decide what are outliers
- “M Step”
  - Compute new transformation that increases probability
  - Update probabilities based on registration

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence,* vol. 32- 12, pp. 2262-2275, 2010.

CPD Inputs

- D—dimension of the point sets,
- N, M—number of points in the point sets,
- $\mathbf{X}_{N\times D} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)^T$—the first point set (the data points),
- $\mathbf{Y}_{M\times D} = (\mathbf{y}_1, \ldots, \mathbf{y}_M)^T$—the second point set (the GMM centroids),
- $\mathcal{T}(\mathbf{Y}, \theta)$—Transformation $\mathcal{T}$ applied to $\mathbf{Y}$, where θ is a set of the transformation parameters,
- $\mathbf{I}$—identity matrix,
- $\mathbf{1}$—column vector of all ones,
- $\mathrm{d}(\mathbf{a})$—diagonal matrix formed from the vector $\mathbf{a}$.

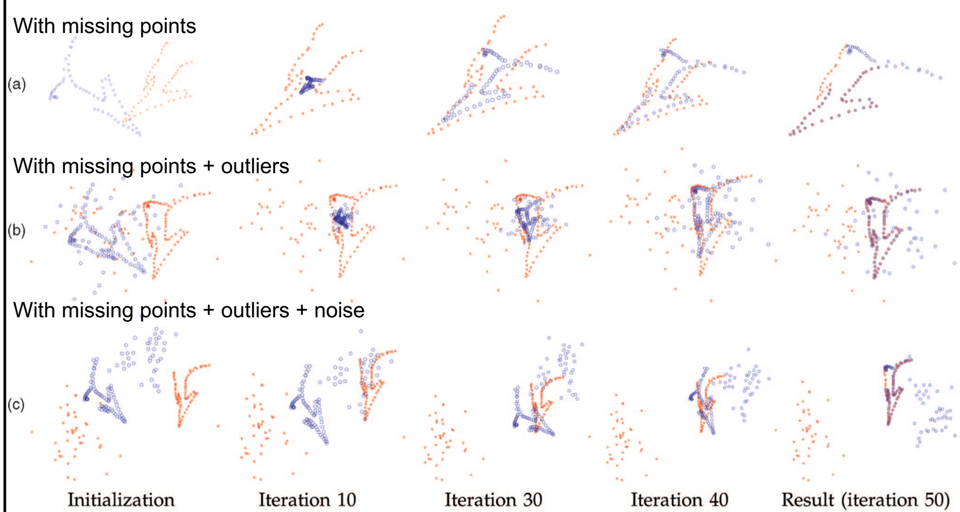
Rigid and Similarity Transform CPD

Rigid point set registration algorithm:

- Initialization: $\mathbf{R} = \mathbf{I}, \mathbf{t} = 0, s = 1, 0 \le w \le 1$
  $\sigma^2 = \frac{1}{DNM} \sum_{n=1}^{N} \sum_{m=1}^{M} \|\mathbf{x}_n - \mathbf{y}_m\|^2$
- EM optimization, repeat until convergence:
  - E-step: Compute $\mathbf{P}$,
    $$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_m + \mathbf{t})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (s\mathbf{R}\mathbf{y}_k + \mathbf{t})\|^2} + (2\pi\sigma^2)^{D/2} \frac{w}{1-w} \frac{M}{N}}$$
  - M-step: Solve for $\mathbf{R}, s, \mathbf{t}, \sigma^2$:
    - $\cdot$ $N_{\mathbf{P}} = \mathbf{1}^T \mathbf{P} \mathbf{1}, \mu_{\mathbf{x}} = \frac{1}{N_{\mathbf{P}}} \mathbf{X}^T \mathbf{P}^T \mathbf{1}, \mu_{\mathbf{y}} = \frac{1}{N_{\mathbf{P}}} \mathbf{Y}^T \mathbf{P} \mathbf{1}$,
    - $\cdot$ $\hat{\mathbf{X}} = \mathbf{X} - \mathbf{1}\mu_{\mathbf{x}}^T, \ \hat{\mathbf{Y}} = \mathbf{Y} - \mathbf{1}\mu_{\mathbf{y}}^T$,
    - $\cdot$ $\mathbf{A} = \hat{\mathbf{X}}^T \mathbf{P}^T \hat{\mathbf{Y}}$, compute SVD of $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^T$,
    - $\cdot$ $\mathbf{R} = \mathbf{U}\mathbf{C}\mathbf{V}^T$, where $\mathbf{C} = \mathrm{d}(1, .., 1, \det(\mathbf{U}\mathbf{V}^T))$,
    - $\cdot$ $s = \frac{\mathrm{tr}(\mathbf{A}^T \mathbf{R})}{\mathrm{tr}(\hat{\mathbf{Y}}^T \mathrm{d}(\mathbf{P}\mathbf{1})\hat{\mathbf{Y}})}$,
    - $\cdot$ $\mathbf{t} = \mu_{\mathbf{x}} - s\mathbf{R}\mu_{\mathbf{y}}$,
    - $\cdot$ $\sigma^2 = \frac{1}{N_{\mathbf{P}} D}(\mathrm{tr}(\hat{\mathbf{X}}^T \mathrm{d}(\mathbf{P}^T \mathbf{1})\hat{\mathbf{X}}) - s\,\mathrm{tr}(\mathbf{A}^T \mathbf{R}))$.
- The aligned point set is $\mathcal{T}(\mathbf{Y}) = s\mathbf{Y}\mathbf{R}^T + \mathbf{1}\mathbf{t}^T$,
- The probability of correspondence is given by $\mathbf{P}$.

A. Myronenko and X. Song, "Point-Set Registration: Coherent Point Drift", *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 32- 12, pp. 2262-2275, 2010.

Rigid and Similarity Transform CPD



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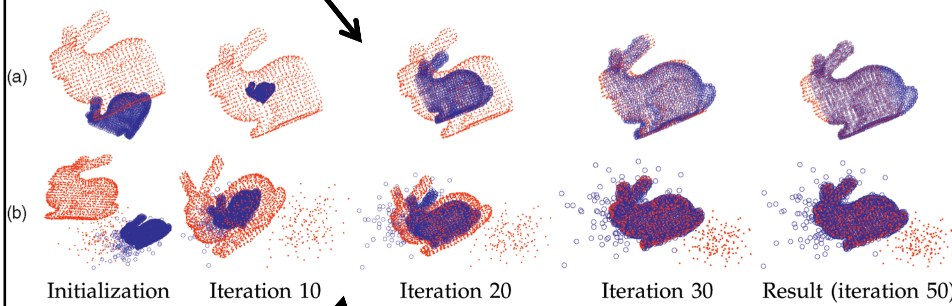
Rigid and Similarity Transform CPD

With missing points



With missing points, outliers and noise

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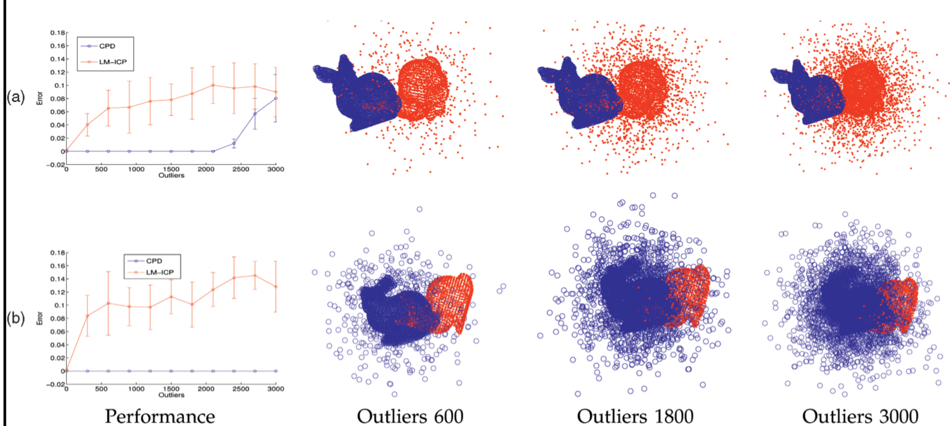
Rigid and Similarity Transform CPD

Comparison to ICP with outliers and noise



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Affine CPD

Affine point set registration algorithm:

- Initialization: $\mathbf{B} = \mathbf{I}, \mathbf{t} = 0, 0 \leq w \leq 1$
  $\sigma^2 = \frac{1}{DNM} \sum_{n=1}^{N} \sum_{m=1}^{M} \|\mathbf{x}_n - \mathbf{y}_m\|^2$
- EM optimization, repeat until convergence:
  - E-step: Compute $\mathbf{P}$,
    $$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{B}\mathbf{y}_m + \mathbf{t})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{B}\mathbf{y}_k + \mathbf{t})\|^2} + (2\pi\sigma^2)^{D/2}\frac{w}{1-w}\frac{M}{N}}$$
  - M-step: Solve for $\mathbf{B}, \mathbf{t}, \sigma^2$:
    - $\cdot\ N_{\mathbf{P}} = \mathbf{1}^T\mathbf{P}\mathbf{1}, \mu_{\mathbf{x}} = \frac{1}{N_{\mathbf{P}}}\mathbf{X}^T\mathbf{P}^T\mathbf{1}, \mu_{\mathbf{y}} = \frac{1}{N_{\mathbf{P}}}\mathbf{Y}^T\mathbf{P}\mathbf{1},$
    - $\cdot\ \hat{\mathbf{X}} = \mathbf{X} - \mathbf{1}\mu_{\mathbf{x}}^T,\ \hat{\mathbf{Y}} = \mathbf{Y} - \mathbf{1}\mu_{\mathbf{y}}^T,$
    - $\cdot\ \mathbf{B} = (\hat{\mathbf{X}}^T\mathbf{P}^T\hat{\mathbf{Y}})(\hat{\mathbf{Y}}^T \mathrm{d}(\mathbf{P}\mathbf{1})\hat{\mathbf{Y}})^{-1},$
    - $\cdot\ \mathbf{t} = \mu_{\mathbf{x}} - \mathbf{B}\mu_{\mathbf{y}},$
    - $\cdot\ \sigma^2 = \frac{1}{N_{\mathbf{P}}D}(\mathrm{tr}(\hat{\mathbf{X}}^T \mathrm{d}(\mathbf{P}^T\mathbf{1})\hat{\mathbf{X}}) - \mathrm{tr}(\hat{\mathbf{X}}^T\mathbf{P}^T\hat{\mathbf{Y}}\mathbf{B}^T)).$
- The aligned point set is $\mathcal{T}(\mathbf{Y}) = \mathbf{Y}\mathbf{B}^T + 1\mathbf{t}^T$,
- The probability of correspondence is given by $\mathbf{P}$.

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Deformable CPD

Non-rigid point set registration algorithm:

- Initialization: $\mathbf{W} = 0, \sigma^2 = \frac{1}{DNM} \sum_{m,n=1}^{M,N} \|\mathbf{x}_n - \mathbf{y}_m\|^2$
- Initialize $w (0 \leq w \leq 1)$, $\beta > 0$, $\lambda > 0$,
- Construct $\mathbf{G}$: $g_{ij} = \exp^{-\frac{1}{2\beta^2}\|\mathbf{y}_i - \mathbf{y}_j\|^2}$,
- EM optimization, repeat until convergence:
  - E-step: Compute $\mathbf{P}$,
    $$p_{mn} = \frac{\exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{y}_m + \mathbf{G}(m,\cdot)\mathbf{W})\|^2}}{\sum_{k=1}^{M} \exp^{-\frac{1}{2\sigma^2}\|\mathbf{x}_n - (\mathbf{y}_k + \mathbf{G}(k,\cdot)\mathbf{W})\|^2} + \frac{w}{1-w}\frac{(2\pi\sigma^2)^{D/2}M}{N}}$$
  - M-step:
    - $\cdot$ Solve $(\mathbf{G} + \lambda\sigma^2 d(\mathbf{P}\mathbf{1})^{-1})\mathbf{W} = d(\mathbf{P}\mathbf{1})^{-1}\mathbf{P}\mathbf{X} - \mathbf{Y}$
    - $\cdot\ N_{\mathbf{P}} = \mathbf{1}^T\mathbf{P}\mathbf{1}$, $\mathbf{T} = \mathbf{Y} + \mathbf{G}\mathbf{W}$,
    - $\cdot\ \sigma^2 = \frac{1}{N_{\mathbf{P}}D}(\mathrm{tr}(\mathbf{X}^T \mathrm{d}(\mathbf{P}^T\mathbf{1})\mathbf{X}) - 2\,\mathrm{tr}((\mathbf{P}\mathbf{X})^T\mathbf{T}) +$
      $\mathrm{tr}(\mathbf{T}^T \mathrm{d}(\mathbf{P}\mathbf{1})\mathbf{T})),$
- The aligned point set is $\mathbf{T} = \mathcal{T}(\mathbf{Y}, \mathbf{W}) = \mathbf{Y} + \mathbf{G}\mathbf{W}$,
- The probability of correspondence is given by $\mathbf{P}$.

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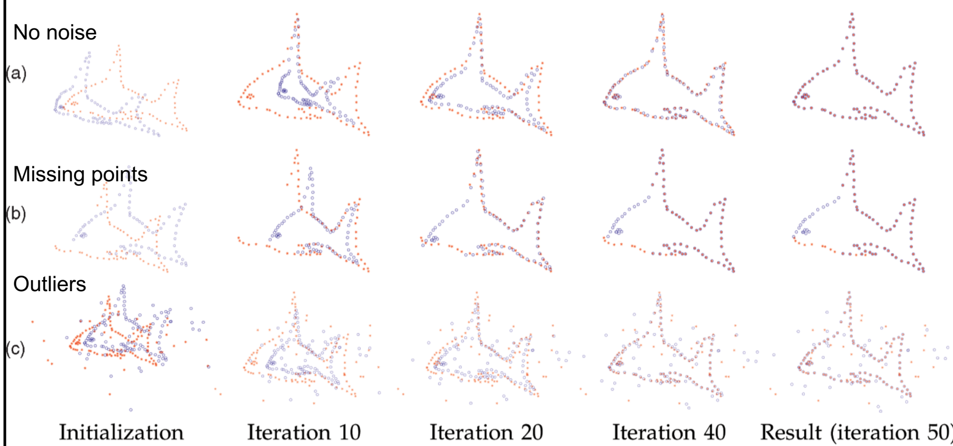
Deformable CPD



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Fast Implementation

- Uses the “Fast Gauss Transform” to reduce the computational complexity to linear time (up to a multiplicative constant).

Compute $\mathbf{P}^T\mathbf{1}, \mathbf{P1}$ and $\mathbf{PX}$:
- Compute $\mathbf{K}^T\mathbf{1}$ (using FGT),
- $\mathbf{a} = 1./(\mathbf{K}^T\mathbf{1} + c\mathbf{1})$,
- $\mathbf{P}^T\mathbf{1} = \mathbf{1} - c\mathbf{a}$,
- $\mathbf{P1} = \mathbf{Ka}$ (using FGT),
- $\mathbf{PX} = \mathbf{K}(\mathbf{a}. * \mathbf{X})$ (using FGT),

- See the paper for more details

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