Homework Assignment 2 – 600.455-655 Fall 2018

Instructions and Score Sheet (hand in with answers)

Name	
Email	
Other contact information (optional)	
Signature (required)	I have followed the rules in completing this assignment ______________________________

Question	Points (445)	Points (645)		Totals	Question	Points (445)	Points (645)		Totals	Grand Total
1A	5	5			3A	10	5			
1B	5	5			3B	5	5			
1C	5	5			3C	10	10			
1D	5	5			4A	10	10			
2A	5	5			4B	10	10			
2B	5	5			4C	5	5			
2C	5	5			4D	10	10			
2D	5	5			4E	10	10			
Total	40	40				70	65			

Note: The total points add up to 110 for undergrads and 105 for grad students. However, the maximum total counted toward your course grades will be 100.

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. Remember to include a sealable 8 ½ by 11 inch self-addressed envelope if you want your assignment

Question 1

A. Recall that we have two alternative ways to represent the uncertainty in some pose $\mathbf{F} = [\mathbf{R}, \vec{\mathbf{p}}]$:

$$\mathbf{F}^* = \Delta\mathbf{F}_L \mathbf{F} = \mathbf{F}\Delta\mathbf{F}_R$$

where

$$\Delta\mathbf{F}_L \approx [\mathbf{I} + skew(\vec{\alpha}_L), \vec{\varepsilon}_L]$$
$$\Delta\mathbf{F}_R \approx [\mathbf{I} + skew(\vec{\alpha}_R), \vec{\varepsilon}_R]$$

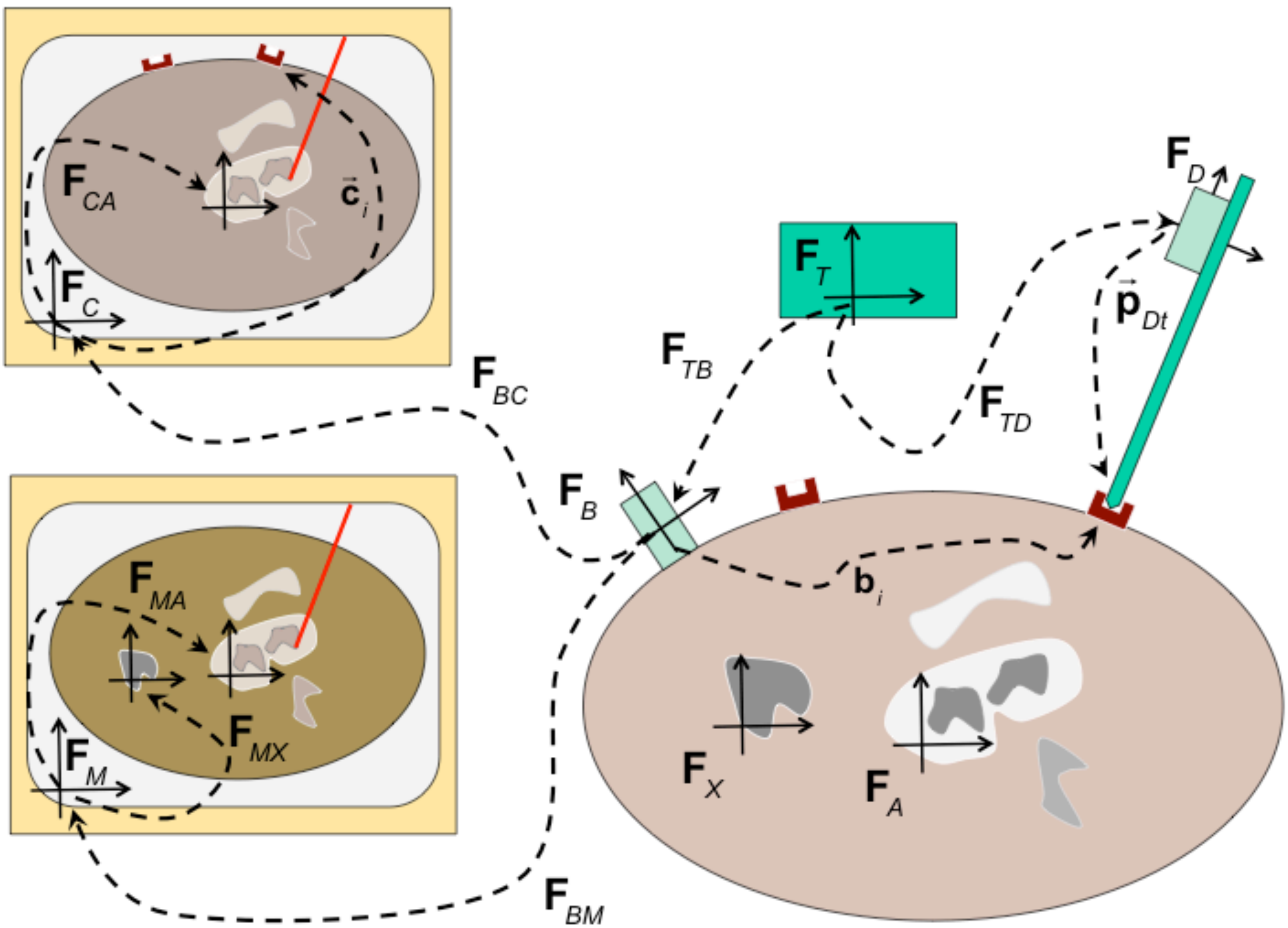
Give expressions for $[\vec{\alpha}_L, \vec{\varepsilon}_L]$ in terms of $[\mathbf{R}, \vec{\mathbf{p}}, \vec{\alpha}_R, \vec{\varepsilon}_R]$. Similarly, give expressions for $[\vec{\alpha}_R, \vec{\varepsilon}_R]$ in terms of $[\mathbf{R}, \vec{\mathbf{p}}, \vec{\alpha}_L, \vec{\varepsilon}_L]$. Show your work in sufficient detail so we can see how you got the results you report. Also, do sufficient algebraic manipulations so that your answers do not include terms like $skew(\vec{\alpha}_{xx})$. **Hint:** Remember that $skew(\vec{\mathbf{a}})\vec{\mathbf{b}}$ is the same as $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$.

B. Given $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1]$, $\mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$, and $\mathbf{F}_3 = \mathbf{F}_1\mathbf{F}_2\mathbf{F}_1^{-1}$, give formulas for $\mathbf{R}_3$ and $\vec{\mathbf{p}}_3$ in terms of $\left\{\mathbf{R}_1, \mathbf{R}_2, \vec{\mathbf{p}}_1, \vec{\mathbf{p}}_2\right\}$.

C. Suppose that you know that $\mathbf{F}_2$ in Question 1B is "small", so that $\mathbf{R}_2 \approx \mathbf{I} + skew(\vec{\alpha}_2)$ and $\vec{\mathbf{p}}_2 = \vec{\varepsilon}_2$ is a small displacement. Produce estimates for $\mathbf{R}_3$ and $\vec{\mathbf{p}}_3$ in terms of $\left\{\mathbf{R}_1, \vec{\alpha}_2, \vec{\mathbf{p}}_1, \vec{\varepsilon}_2\right\}$. Show your work in sufficient detail so we can see how you got the results you report. Also, do sufficient algebraic manipulations so that your answer for $\vec{\mathbf{p}}_3$ does not include terms like $skew(\vec{\alpha}_{xx})$.

D. Given a unit quaternion $\vec{\mathbf{q}} = \left[\cos\left(\theta/2\right), \sin\left(\theta/2\right)\vec{\mathbf{n}}\right]$, where $\left\|\vec{\mathbf{n}}\right\| = 1$ and a vector $\vec{\mathbf{p}}$ show that $\left[0, Rot(\vec{\mathbf{n}},\theta)\vec{\mathbf{p}}\right] = \vec{\mathbf{q}} \bullet [0, \vec{\mathbf{p}}] \bullet \vec{\mathbf{q}}^*$. Again give sufficient detail so that we can follow your reasoning.

Scenario for remaining questions



Consider the stereotactic navigation scenario illustrated in the figure above. Here, we have a stereotactic tracking system whose coordinate system is represented by $\mathbf{F}_T$, capable of tracking the pose $\mathbf{F}_{TB}$ of a tracker body $\mathbf{F}_B$ attached to the patient and the pose $\mathbf{F}_{TD}$ of another tracker body $\mathbf{F}_D$ attached to a pointer tool. The position of the tip of the pointer tool has been calibrated to be at a position $\vec{\mathbf{p}}_{Dt}$ relative to $\mathbf{F}_D$.

CT and MRI images of the patient are available. The anatomic structure $\mathbf{F}_X$ of greatest interest (which you may think of as a tumor or other malformation) is visible in the MRI image at pose $\mathbf{F}_{MX}$ in MRI coordinates but not in the CT image. However, another anatomic structure $\mathbf{F}_A$ is visible at pose $\mathbf{F}_{MA}$ in MRI coordinates and $\mathbf{F}_{CA}$ in CT coordinates.

After the MRI image was acquired, but before the CT image was acquired, small fiducial objects were pasted to the patient's skin. These markers are visible at locations $\vec{\mathbf{c}}_i$ in CT coordinates. During a registration step, the tracked pointer is placed on each of the small fiducials, and the corresponding position $\vec{\mathbf{b}}_i$ relative to $\mathbf{F}_B$ is computed (see questions, below). The corresponding values of $\vec{\mathbf{b}}_i$ and $\vec{\mathbf{c}}_i$ are used to compute the registration transformation $\mathbf{F}_{BC}$ between CT and patient tracker body coordinates, so that $\mathbf{F}_{BC}\vec{\mathbf{c}}_i \approx \vec{\mathbf{b}}_i$. At some point, we will want to know the registration $\mathbf{F}_{BM}$ between MRI and tracker body coordinates, but this has not yet been computed.

Question 2

A. Given values for $\mathbf{F}_{TB}$ and $\mathbf{F}_{TD}$ when the pointer tip is touching fiducial i, give a formula for computing $\vec{\mathbf{p}}_{Bt} = \vec{\mathbf{b}}_i$. Give the answer first in terms of the "$\mathbf{F}_{pq}$" variables and then in terms of the corresponding **"$\mathbf{R}_{pq}$"** and "$\vec{\mathbf{p}}_{pq}$" variables.

B. Assuming that $\mathbf{F}_{BC}$ has been computed by some registration process, give a formula for computing $\mathbf{F}_{BM}$, given the other information available. Give the answer first in terms of the "$\mathbf{F}_{pq}$" variables and then in terms of the corresponding **"$\mathbf{R}_{pq}$"** and "$\vec{\mathbf{p}}_{pq}$" variables.

C. Given values for $\mathbf{F}_{TB}$ and $\mathbf{F}_{TD}$, give a formula for computing the position $\vec{\mathbf{p}}_{Mt}$ in MRI coordinates corresponding to the current position $\vec{\mathbf{p}}_{tip}$ of the pointer tip. **NOTE:** For this problem, you can assume that your answer for Question 2A gives a way to compute the position $\vec{\mathbf{p}}_{Bt}$ of the pointer tip relative to $\mathbf{F}_B$ and that question 2B has given you a way to compute $\mathbf{F}_{BM}$. Express your answer first in terms of $\mathbf{F}_{BM}$ and $\vec{\mathbf{p}}_{Bt}$ and then in terms of $\mathbf{R}_{BM}$, $\vec{\mathbf{p}}_{BM}$, and $\vec{\mathbf{p}}_{Bt}$.

D. Suppose that the surgeon is now operating on the patient and has identified an anatomic feature located at position $\vec{\mathbf{p}}_{Xf}$ relative to a local coordinate system $\mathbf{F}_X$ associated with the tumor. Given values for $\mathbf{F}_{TB}$ and $\mathbf{F}_{TD}$, give a formula for computing the distance between the current position $\vec{\mathbf{p}}_{tip}$ of the pointer tip and the anatomic feature. Give the answer in terms of $\mathbf{F}_{BM}$, $\mathbf{F}_{BX}$, $\vec{\mathbf{p}}_{Xf}$, and $\vec{\mathbf{p}}_{Bt}$.

Question 3

So far, we have assumed that the tracking system, images, etc. are perfect. Now, we will consider what happens when reality departs from this happy situation. We will adopt our usual notation, so that the actual value $\mathbf{F}_{qr}^*$ of a measured or computed pose $\mathbf{F}_{qr} = \left[\mathbf{R}_{qr}, \vec{\mathbf{p}}_{pq}\right]$ will be given by $\mathbf{F}_{qr}^* = \mathbf{F}_{qr}\Delta\mathbf{F}_{qr}$, where $\Delta\mathbf{F}_{qr} = \left[\Delta\mathbf{R}_{qr}, \Delta\vec{\mathbf{p}}_{qr}\right]$.

A. Given measurement errors $\Delta\mathbf{F}_{TB} = \left[\Delta\mathbf{R}_{TB}, \Delta\vec{\mathbf{p}}_{TB}\right]$ and $\Delta\mathbf{F}_{TD} = \left[\Delta\mathbf{R}_{TD}, \Delta\vec{\mathbf{p}}_{TD}\right]$ and pointer tip calibration error $\vec{\mathbf{p}}_{Dt}^{\ *} = \vec{\mathbf{p}}_{Dt} + \Delta\vec{\mathbf{p}}_{Dt}$, give a formula for the error $\Delta\mathbf{p}_{Bt}$ in localizing the tip of the pointer relative to $\mathbf{F}_B$. Hint: You might first consider finding the position $\vec{\mathbf{p}}_{Tt}$ of the pointer tip relative to the tracking system and its associated error $\Delta\vec{\mathbf{p}}_{Tt}$. You may also find it convenient to express part of your answer in terms of $\mathbf{F}_{BD}$.

B. Similarly, suppose that the image processing algorithms used to find $\mathbf{F}_A$ in CT and MRI images are subject to some error to that $\mathbf{F}_{CA}^{\ *} = \mathbf{F}_{CA}\Delta\mathbf{F}_{CA}$ and $\mathbf{F}_{MA}^{\ *} = \mathbf{F}_{MA}\Delta\mathbf{F}_{MA}$, and that there has been some error in computing the registration transformation $\mathbf{F}_{BC}$, so that $\mathbf{F}_{BC}^{\ *} = \mathbf{F}_{BC}\Delta\mathbf{F}_{BC}$. Give a formula for the error $\Delta\mathbf{F}_{BM}$ in the formula developed in Question 2B for $\mathbf{F}_{BM}$. Here, I want you to give separate expressions for $\Delta\mathbf{R}_{BM}$ and $\Delta\vec{\mathbf{p}}_{BM}$. Show your work in sufficient detail so we can see how you got the results you report.

C. Under the assumptions above, give a formula for the error $\Delta\vec{\mathbf{p}}_{Mt}$ in computing the computing the position $\vec{\mathbf{p}}_{Mt}$ in MRI coordinates corresponding to the current position $\vec{\mathbf{p}}_{tip}$ of the pointer tip in terms of $\left\{\Delta\mathbf{p}_{Bt}, \Delta\mathbf{R}_{BM}, \Delta\vec{\mathbf{p}}_{BM}\right\}$.

Question 4

Now, consider linearized approximations to the errors:
$\Delta\mathbf{R}_{pq} \approx \mathbf{I} + sk(\vec{\alpha}_{pq})$ and $\Delta\vec{\mathbf{p}}_{pq} = \vec{\varepsilon}_{pq}$.

A. Give a formula for estimating $\vec{\varepsilon}_i = \Delta\vec{\mathbf{b}}_i$, assuming all relevant $\vec{\alpha}_{pq}$ and $\vec{\varepsilon}_{pq}$ are sufficiently small so that our linear approximations are valid, as discussed in class. Give your answer in terms of $\left\{\mathbf{R}_{BD}, \vec{\varepsilon}_{Dt}, \vec{\alpha}_{TD}, \vec{\mathbf{p}}_{Dt}, \vec{\alpha}_{TB}, \vec{\mathbf{b}}_i, \vec{\varepsilon}_{TB}\right\}$. Show your work in sufficient detail so we can see how you got the results you report. Also, do sufficient algebraic manipulations so that your answers do not include terms like *skew*($\vec{\alpha}_{xx}$). Your final answers should be sums of terms of the general form $\mathbf{M}\vec{\alpha}$ or $\mathbf{M}\vec{\varepsilon}$ where the **M**'s are 3x3 matrices.

B. Give formulas for estimating $\vec{\alpha}_{BM}$ and $\vec{\varepsilon}_{BM}$ under the assumptions above. Show your work in sufficient detail so we can see how you got the results you report. Also, do sufficient algebraic manipulations so that your answers do not include terms like *skew*$(\vec{\alpha}_{xx})$. Your final answers should be sums of terms of the general form $\mathbf{M}\vec{\alpha}$ or $\mathbf{M}\vec{\varepsilon}$ where the **M**'s are 3x3 matrices.

C. Give a formula for estimating $\vec{\varepsilon}_{Mt}$ under the assumptions above. Express your answer in terms of $\left\{\vec{\alpha}_{BM},\vec{\varepsilon}_{BM},\vec{\varepsilon}_{Bt}\right\}$. Show your work in sufficient detail so we can see how you got the results you report. Also, do sufficient algebraic manipulations so that your answers do not include terms like *skew*$(\vec{\alpha}_{xx})$. Your final answer should be a sum of terms of the general form $\mathbf{M}\vec{\alpha}$ or $\mathbf{M}\vec{\varepsilon}$ where the **M**'s are 3x3 matrices.

D. Suppose we know that the error in localizing fiducials $\vec{\mathbf{c}}_i \in \left\{\vec{\mathbf{c}}_1,\cdots,\vec{\mathbf{c}}_N\right\}$ in the CT image was negligibly small, and that the errors $\vec{\varepsilon}_i$ in localizing some corresponding physical fiducials $\vec{\mathbf{b}}_i \in \left\{\vec{\mathbf{b}}_1,\cdots,\vec{\mathbf{b}}_N\right\}$ relative to the patient reference body coordinates $\mathbf{F}_B$

are also small, with $\left|\vec{\varepsilon}_i\right| = \left[\left|\vec{\varepsilon}_{i,x}\right|,\left|\vec{\varepsilon}_{i,y}\right|,\left|\vec{\varepsilon}_{i,z}\right|\right]^T \leq \vec{\varepsilon}_i^{\,\max}$. The errors in the $\vec{\varepsilon}_i$ will naturally introduce some error in the computation of $\mathbf{F}_{BC}$, so that $\mathbf{F}_{BC}{}^* = \mathbf{F}_{BC}\Delta\mathbf{F}_{BC}$, with $\Delta\mathbf{F}_{BC} \approx \left[\mathbf{I} + skew(\vec{\alpha}_{BC}), \vec{\varepsilon}_{BC}\right]$. Derive a system of linear constraints limiting the elements of $\vec{\alpha}_{BC}$ and $\vec{\varepsilon}_{BC}$. Show your work in sufficient detail so we can see how you got the results you report. Also, do sufficient algebraic manipulations so that your answers do not include terms like $skew(\vec{\alpha}_{xx})$.

E. Suppose that we know the following information:

$$\left\|\vec{\alpha}_{TD}\right\| \leq 0.002 \text{ radians} \quad \left\|\vec{\varepsilon}_{TD}\right\| \leq 0.2 \text{ mm} \quad \vec{\mathbf{p}}_{Dt} = [0,0,200] \text{ mm}$$

<u>Estimate</u> an upper bound on the magnitude of the L$_2$ error $\left\|\vec{\varepsilon}_{Tt}\right\|_2$ in computing the position $\vec{\mathbf{p}}_{Tt}$ of the pointer tip relative to the tracker base unit. Show your work in sufficient detail so we can see how you got the results you report.