### Homework Assignment 2 – 600.445/645 Fall 2016 (Circle One)

**Instructions and Score Sheet (hand in with answers)**

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* Undergrads (600.445 students) may attempt any or all these problems (which are mandatory for 645 students) for extra credit. We will award up to 10 extra points, but your total score will be limited to 100. I.e., if your total on the remaining problems is $S$ and you score a total of $E$ points on the extra credit problems, your net homework score will be $\min(100, S + \min(E, 10))$. They are good problems, and I would urge people to try them.
1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.

2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.

3. It is otherwise open book, notes, and web. But you should cite any references you consult.

4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.

5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.

6. Sign and hand in the score sheet as the first sheet of your assignment.

7. Remember to include a sealable 8 ½ by 11 inch self-addressed envelope if you want your assignment
Scenario

Consider the pelvic osteotomy situation illustrated in Fig. 1. Here we assume that a three locating pins (similar to those used in Robodoc) have been inserted into the patient’s pelvis, and that a CT scan of the pelvis with the pins inserted has been produced. The patient has been placed onto the operating table. Also, a magnetic navigation system (here, the Northern Digital Aurora) is present in the room. Two surgical tools are available:

- A probe/pointer device
- An osteotome (essentially a fancy chisel) that will be used to cut the pelvis.

6 DOF Aurora tracking sensors have been attached to the handle of each tool and an additional 6 DOF sensor has been affixed rigidly to the pelvis. The Aurora is capable of determining the position and orientation of each sensor relative to the Aurora base unit. We will define the following coordinate systems:

![Fig. 1: Computer-Assisted Osteotomy](image)
\( \mathbf{F}_b = \) Coordinate system of tracking system base unit
\( \mathbf{F}_p = \) Coordinate system of tracking device on pointer handle
\( \mathbf{F}_h = \) Coordinate system of tracking device on osteotome handle
\( \mathbf{F}_g = \) Coordinate system of tracking device attached to pelvis
\( \mathbf{F}_c = \) Coordinate system of CT image

We also have the following relationships
\[
\begin{align*}
\mathbf{F}_{bx} &= \text{Measured 6 DOF pose of tracking device x relative to base unit} \\
\mathbf{F}_{hk} &= \text{6 DOF pose of osteotome blade relative to osteotome handle tracking device} \\
\mathbf{F}_{dk} &= \text{6 DOF pose of pointer tip relative to pointer handle tracking device} \\
\mathbf{\tilde{a}}_k &= \text{Position of the top of pin k in CT coordinates} \\
\mathbf{b}_k &= \text{Position of the top of pin k relative to tracking device G}
\end{align*}
\]

We will follow our usual conventions where frame position and orientation components are represented by \( \mathbf{F} = [\mathbf{R}, \mathbf{p}] \), and errors are represented by \( \Delta \mathbf{F} = [\Delta \mathbf{R}, \Delta \mathbf{p}] \). We will also use the approximation convention \( \Delta \mathbf{R} \approx \mathbf{I} + s\mathbf{k}(\hat{\alpha}) \).

**Question 1**

A. Let \( \mathbf{\tilde{p}}_{sp} = \mathbf{\tilde{p}}_{GE} \) be the position of the tip of the pointer tool relative to the reference marker coordinate system \( \mathbf{F}_g \). Give a formula for computing \( \mathbf{\tilde{p}}_{sp} \), based on the available tracking system measurements \( \mathbf{F}_{bx} \).

B. Suppose that we have touched the tops of the three fiducial pins and used the results to compute a registration transformation \( \mathbf{F}_{gc} \) such that \( \mathbf{F}_{gc} \mathbf{\tilde{a}}_k = \mathbf{b}_k \). Give an expression for computing the position and orientation \( \mathbf{F}_{ck} \) of the osteotome blade in CT coordinates, based on the available tracking system measurements \( \mathbf{F}_{bx} \).

C. Suppose now that the tracking system is not perfectly accurate, so that at any given time the actual value \( \mathbf{F}_{bx} \) of a measurement \( \mathbf{F}_{bx} \) is given by \( \mathbf{F}_{bx} = \mathbf{F}_{bx} + \Delta \mathbf{F}_{bx} \). Thus, the measurements will include some "common mode" errors \( \Delta \mathbf{F}_p \) as well as "place specific" errors associated with each marker. In both cases, these errors may come from many different sources. The specific causes are irrelevant for your analysis. In any case, they will introduce some error in your computation of \( \mathbf{\tilde{p}}_{sp} \), so that \( \mathbf{\tilde{p}}_{sp} = \mathbf{\tilde{p}}_{sp} + \Delta \mathbf{\tilde{p}}_{sp} \). Give an expression for \( \Delta \mathbf{\tilde{p}}_{sp} \) in terms of the \( \mathbf{R} \)'s, \( \mathbf{p} \)'s, \( \Delta \mathbf{R} \)'s, and \( \Delta \mathbf{p} \)'s.

D. Now, make use of linear approximations to re-express your answer for 1C in terms of the \( \hat{\alpha} \)'s and \( \hat{\varepsilon} \)'s and simplify the result.

E. Now assume again that the tracking system is not perfectly accurate, so that at any given time the actual value \( \mathbf{F}_{bx} \) of a measurement \( \mathbf{F}_{bx} \) is given by \( \mathbf{F}_{bx} = \mathbf{F}_{bx} + \Delta \mathbf{F}_{bx} \). Assume that a registration has been done somehow, but that there is some error in the
registration transformation, such that $F_{GC} = F_{GC}^* + \Delta F_{GC}$. Produce an expression for
\[
\Delta F_{CK} = [\Delta R_{CK}, \Delta \vec{p}_{ck}] \text{ in terms of the } R \text{'s, } \vec{p} \text{'s, } \Delta R \text{'s, and } \Delta \vec{p} \text{'s.}
\]

F. Apply linear approximations to provide expressions for $\alpha_{ck}$ and $\varepsilon_{ck}$.

G. Suppose that our pointer system has been used to touch the tops of the fiducial pins and the system has produced values for the $\vec{b}_k$. But there is some error, so that $\vec{b}_k^* = \vec{b}_k + \Delta \vec{b}_k$. Suppose that we have some constraint function $f(\cdot)$ such that
\[
0 \leq f(\Delta \vec{b}_k) \leq \eta_k \quad \text{but that the CT segmentation is good enough so that we can assume that the } \vec{a}_k \text{ are known exactly. Develop a system of equations and inequalities that will express what can be known about } \alpha_{ck} \text{ and } \varepsilon_{ck}.
\]

H. How would your answer to 1G change if there is also some segmentation error, so that $\vec{a}_k^* = \vec{a}_k + \Delta \vec{a}_k$ and $0 \leq f(\Delta \vec{a}_k) \leq \xi_k$?

Question 2

![Diagram](image)

**Fig. 2: Osteotomy with preformed plates.** (Left) surgical plan and holes drilled; (Right) acetabular fragment relocated and affixed to predrilled and preformed plates.

Consider now the periacetabular osteotomy strategy shown in Fig. 2. The goal is to free the acetabulum and reposition it to a pose $F_{CG}$ relative to the rest of the pelvis. In other words, and point on the acetabulum that has a position $\vec{p}_{Aj}$ in the original CT coordinates will have a new position $F_{CG} \vec{p}_{Aj}$ after relocation. In this case, the osteotomy plan includes the places that the pelvis must be cut to free the acetabulum from the rest of the pelvis. The planning software determines where holes should be drilled (before any cuts are made) into the acetabular fragment at positions $\vec{g}_k$ in CT coordinates, It also determines where other holes are to be drilled into the other part of the pelvis at positions $\vec{h}_k$ in CT coordinates. The directions of the holes in the CT coordinate system are $\vec{n}_{g,k}$ and $\vec{n}_{h,k}$, respectively. The plan assumes that a custom plate is available with holes drilled at predrilled positions $\vec{q}_k$ and $\vec{r}_k$ corresponding to the acetabular and main pelvis holes, respectively. For convenience, the plate coordinate system
has been choses so that \( \hat{\mathbf{r}}_k = \mathbf{r}_k \). The pointer of Question 1 has been upgraded to a drill or awl that can drill holes in bone.

In surgery, a registration step is performed. After registration, surgical navigation is performed to assist the surgeon in drilling the holes in the desired places in the pelvis. Then, navigation is used to assist the surgeon in cutting the acetabulum free from the pelvis. Then the plate is attached to the acetabulum fragment with screws. Finally, the fragment is manipulated so that the plate holes are aligned with the holes in the main part of the pelvis, and the plate is attached with screws.

A. Given the design of the plate, where should the holes \( \mathbf{g}_k \) be drilled in order to achieve the desired realignment?

B. Assume that the tracking system is now very accurate, but that there has been some other source of registration error, so that \( \mathbf{F}_{GC} = \mathbf{F}_{GC} \Delta \mathbf{F}_{GC} \). How accurately can the surgical plan be carried out? I.e., what can you say about the accuracy with which the surgeon achieve the desired alignment?

C. Suppose that the holes are slightly over-sized, so that if each screw has diameter \( r \), then each hole has diameter \( r + \rho \). Can you say anything about how this will affect the accuracy with which the acetabulum can be aligned? **Hints:** Let \( \mathbf{F}_{AP} \) represent the coordinate transformation between the acetabular fragment and the plate and \( \mathbf{F}_{PC} \) represent the transformation between the plate and the rest of the pelvis. Consider the effect of the hole clearance on the worst case accuracy of these transformations, then put them together. Also, remember that we are after only an approximate answer, so an estimate within a factor of 1.5 would still be OK if this helps you make some simplifying assumptions.

**Question 3**

Suppose that the three locating pins have been replaced by two cannulated pins inserted into the pelvis in different directions, as shown in Fig. 3. Suppose, further, that

- The diameter of the cannulas matches the outside diameter of the probe, so that when the probe is inserted into the cannula, the axis of the probe shaft is coincident with the axis of the hole.

- The axis of the cannula is coincident to the z-axis of the coordinate system for the probe when the probe is inserted into the cannula. Further, the pose \( \mathbf{F}_{C_k} \) of each cannulated pin has been of each pin determined by the CT image processing software used in surgical planning. The origin of this coordinate system associated with each cannulated pin is at the bottom center of the cannula, and the z-axis runs along the cannula center toward the top of the pin.

- A dynamic reference body \( \mathbf{F}_G \) similar to that described in Question 1 has been affixed to the pelvis at the start of surgery.

![Fig. 3: Cannulated Pins](image_url)
• The insertion depth of the probe into the hole is not always consistently known. I.e., it may not be practical to ensure that the probe is always inserted to the “bottom” of the cannula.

A. Outline a procedure and calculation method for performing the registration step. Provide sufficient detail and formulas so that it is clear what data you are collecting and how you are computing the required information.

B. Assume that it is now possible to place the probe to a consistent depth into the pins. However, the probe has been damaged, so that $F_{DE}$ is no longer known exactly. Suppose that the probe has been bent, so that $F_{DE}$ is no longer known exactly. Outline a procedure for simultaneously determining the new value of $F_{DE}$ and locating the axes of the pins relative to $F_g$. (I.e., determine the direction of the pin cannula center axis, together with the location of a point on the center axis). Provide sufficient detail and formulas so that it is clear what data you are collecting and how you are computing the required information.

Question 4

As discussed in class, a unit quaternion $q = \cos \theta/2, \sin \theta/2$ can be used to represent a rotation by an angle $\theta$ about an axis $\mathbf{n}$, where $\mathbf{n}$ is a unit vector. Provide an algebraic proof of this fact. I.e., demonstrate that

$$[0,\text{Rot}((\mathbf{n}, \theta) \cdot \mathbf{p})] = \begin{bmatrix} \cos \theta/2, \sin \theta/2 \end{bmatrix} \circ [0, \mathbf{p}] \circ \begin{bmatrix} \cos \theta/2, -\sin \theta/2 \end{bmatrix}$$

for an arbitrary vector $\mathbf{p}$. **Hint:** Carry out various algebraic manipulations to show that this is equivalent to Rodrigues’ formula.