Latest *on* Linear Sketches *for* Large Graphs: Lots *of* Problems, Little Space, *and* Loads *of* Handwaving

Andrew McGregor
University of Massachusetts

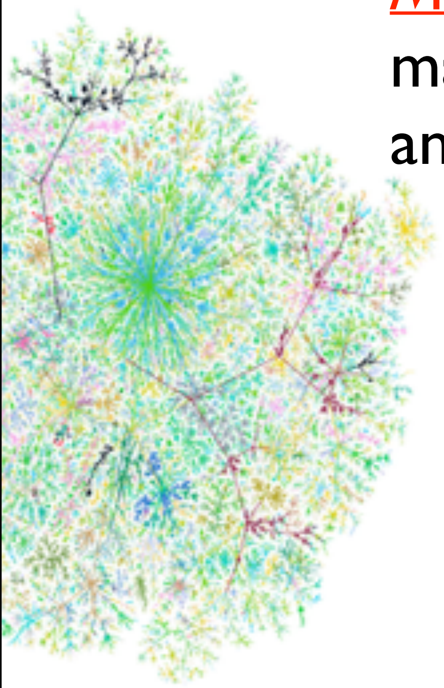
Latest *on* Linear Sketches *for* Large Graphs: Lots *of* Problems, Little Space, *and* Loads *of* Handwaving

Andrew McGregor
University of Massachusetts

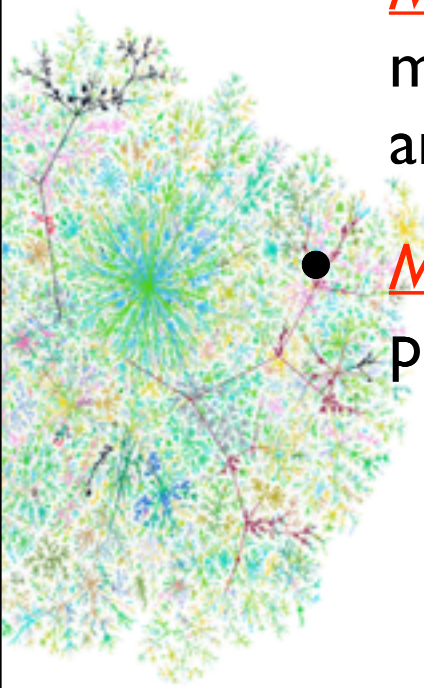
Vertex Connectivity and Sparsification — Guha, McGregor, Tench [PODS 15]

Densest Subgraphs — McGregor, Tench, Vorotnikova, Vu [MFCS 15]

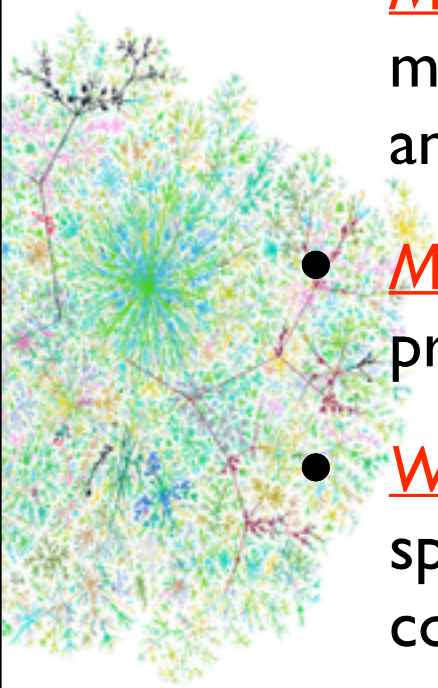
Matching, Vertex Cover, Hitting Set
Chitnis, Cormode, Esfandiari, Hajiaghayi, McGregor, Monemizadeh, Vorotnikova [SODA 16]



- *Motivation* Dynamic Graph Streams. Want to analyze a massive graph defined by a long sequence of edge insertions and deletions. Don't want to have to store the entire graph.

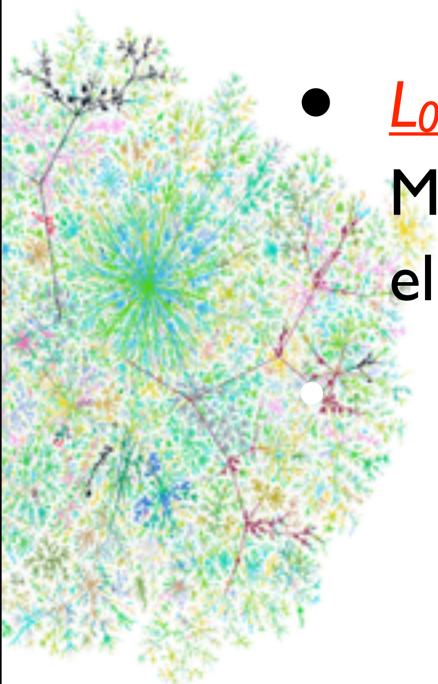


- *Motivation* Dynamic Graph Streams. Want to analyze a massive graph defined by a long sequence of edge insertions and deletions. Don't want to have to store the entire graph.
- *Main Technique* Linear Sketches. Maintain random linear projections of vectors and matrices representing the graph.



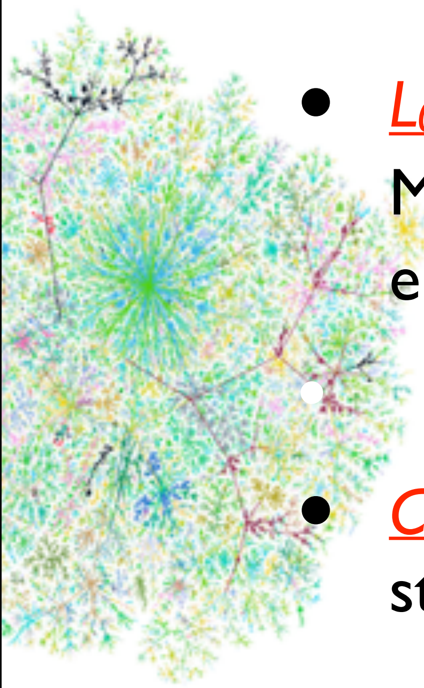
- *Motivation* Dynamic Graph Streams. Want to analyze a massive graph defined by a long sequence of edge insertions and deletions. Don't want to have to store the entire graph.
- *Main Technique* Linear Sketches. Maintain random linear projections of vectors and matrices representing the graph.
- *What's Known* Lots and lots! Edge and vertex connectivity, spectral sparsification, matching, vertex cover, hitting set, correlation clustering, triangles, spanners, densest subgraph…

Graph Streaming Survey
McGregor [SIGMOD Record 14]



- ***L_0 Sampling Primitive*** There's a distribution over matrices $M \in \mathbb{R}^{\text{polylog}(N) \times N}$ such that for any $x \in \mathbb{R}^N$, a random non-zero element of x can be reconstructed from Mx whp.

Jowhari, Saglam, Tardos [PODS 11]



- *L_0 Sampling Primitive* There's a distribution over matrices $M \in \mathbb{R}^{\text{polylog}(N) \times N}$ such that for any $x \in \mathbb{R}^N$, a random non-zero element of x can be reconstructed from Mx whp.

Jowhari, Saglam, Tardos [PODS 11]

- *Corollary* Can uniformly sample an edge in the dynamic graph stream model using O(polylog n) bits of space.

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* $2+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* 2+ε approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

  Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]
- *Our Result* One pass 1+ε approximation using $\tilde{O}(\varepsilon^{-2} n)$ space:

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* $2+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

  Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]

- *Our Result* One pass $1+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space:

  Uniformly sample of $\tilde{O}(\varepsilon^{-2} n)$ edges. Let $\check{D}_S$ be estimate of D_S based on sampled edges. Return $\max_S \check{D}_S$.

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* $2+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

  Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]

- *Our Result* One pass $1+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space:

  Uniformly sample of $\tilde{O}(\varepsilon^{-2} n)$ edges. Let $\check{D}_S$ be estimate of D_S based on sampled edges. Return $\max_S \check{D}_S$.

- *Analysis* For any subset S of size k, with probability $1-n^{-2k}$,

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* $2+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

  Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]

- *Our Result* One pass $1+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space:

  Uniformly sample of $\tilde{O}(\varepsilon^{-2} n)$ edges. Let $\check{D}_S$ be estimate of D_S based on sampled edges. Return $\max_S \check{D}_S$.

- *Analysis* For any subset S of size k, with probability $1-n^{-2k}$,

  $$\check{D}_S \approx_\varepsilon D_S \text{ if } D_S \approx D^* \quad \text{and} \quad \check{D}_S \ll D^* \text{ if } D_S \ll D^*$$

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* 2+ε approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

  Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]

- *Our Result* One pass 1+ε approximation using $\tilde{O}(\varepsilon^{-2} n)$ space:

  Uniformly sample of $\tilde{O}(\varepsilon^{-2} n)$ edges. Let $\check{D}_S$ be estimate of D_S based on sampled edges. Return $\max_S \check{D}_S$.

- *Analysis* For any subset S of size k, with probability $1-n^{-2k}$,

  $$\check{D}_S \approx_\varepsilon D_S \text{ if } D_S \approx D^* \quad \text{and} \quad \check{D}_S \ll D^* \text{ if } D_S \ll D^*$$

  Use union bound over $O(n^k)$ subsets of size k for each k.

Densest Subgraph

- Density of node set S is $D_S = |E_S|/|S|$. Estimate $D^* = \max_S D_S$.
- *Previous Result* $2+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space.

  Bhattycharya et al. [STOC 15], *Bahmani et al.* [PVLDB 12]

- *Our Result* One pass $1+\varepsilon$ approximation using $\tilde{O}(\varepsilon^{-2} n)$ space:

  Uniformly sample of $\tilde{O}(\varepsilon^{-2} n)$ edges. Let $\check{D}_S$ be estimate of D_S based on sampled edges. Return $\max_S \check{D}_S$.

- *Analysis* For any subset S of size k, with probability $1-n^{-2k}$,

  $$\check{D}_S \approx_\varepsilon D_S \text{ if } D_S \approx D^* \quad \text{and} \quad \check{D}_S \ll D^* \text{ if } D_S \ll D^*$$

  Use union bound over $O(n^k)$ subsets of size k for each k.

  see also *Mitzenmacher et al.* [KDD 15], *Esfandiari et al.* [ArXiv 15]

What other types of sampling are there that a) are useful for solving graph problems and b) can be supported on dynamic graph streams?

I. Graph Matching
via **SNAPE** Sampling

II. Graph Connectivity
via **DEALS** Sampling

Graph Matchings

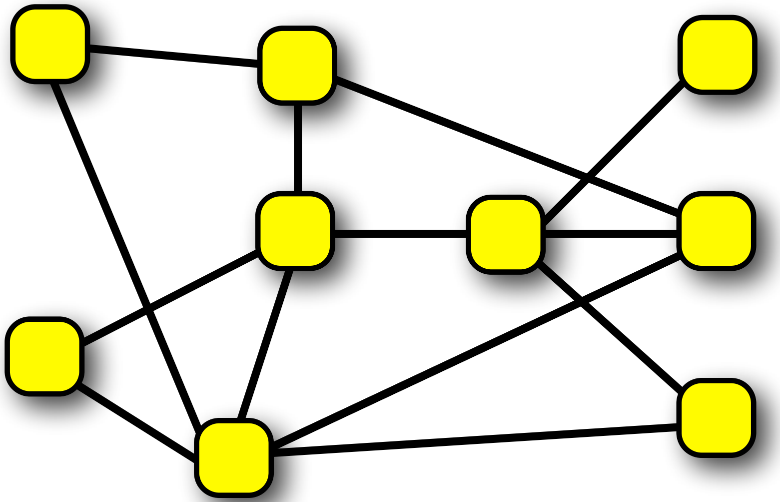
- *1st Result* If max matching has size $\leq$k, can solve exactly exact max matching in dynamic stream model using $\tilde{O}(k^2)$ space.
  - *Optimal & Simple.* Extends to hypergraph matching, vertex cover, hitting set… but gets a lot more complicated.
  - *Basic Idea:* "SNAPE" sampling primitive.

Graph Matchings

- *1st Result* If max matching has size $\leq k$, can solve exactly exact max matching in dynamic stream model using $\tilde{O}(k^2)$ space.
  - *Optimal & Simple.* Extends to hypergraph matching, vertex cover, hitting set… but gets a lot more complicated.
  - *Basic Idea:* "SNAPE" sampling primitive.
- *2nd Result* If max matching has size $\geq k$, can find matching of size $\Omega(k/t)$ in the dynamic stream model using $\tilde{O}(k^2/t^3)$ space.
  - *Application:* Guessing k gives O(t)-approx for max matching using $\tilde{O}(n^2/t^3)$ space. This is also optimal; ask Grigory!

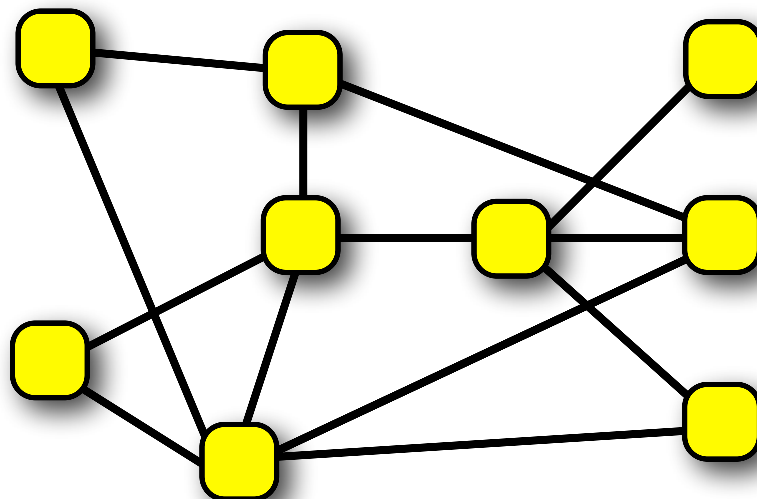
SNAPE Sampling

Sample-Nodes-And-Pick-Edge



SNAPE Sampling

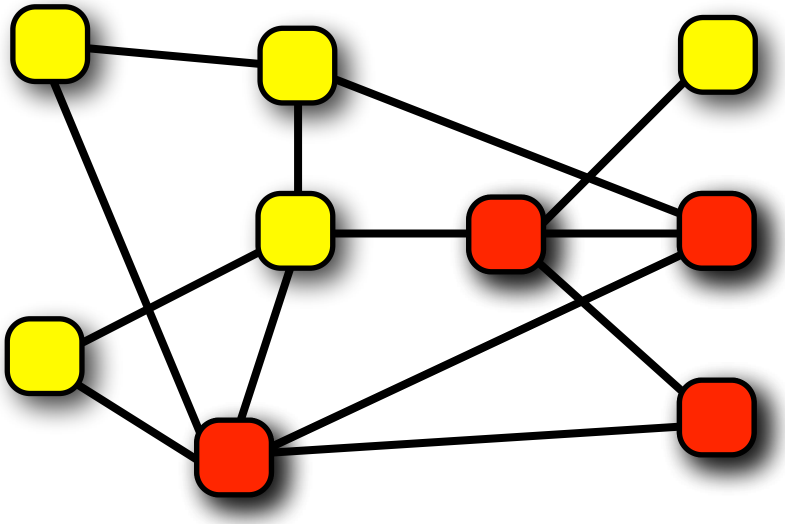
Sample-Nodes-And-Pick-Edge



- Sample each node with probability $\Theta(1/k)$ and delete the rest

SNAPE Sampling

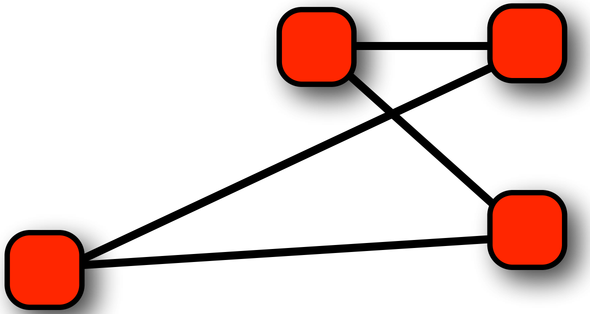
Sample-Nodes-And-Pick-Edge



- Sample each node with probability $\Theta(1/k)$ and delete the rest

SNAPE Sampling

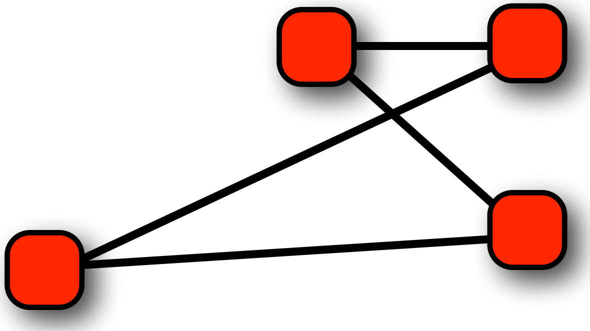
Sample-Nodes-And-Pick-Edge



- Sample each node with probability $\Theta(1/k)$ and delete the rest

SNAPE Sampling

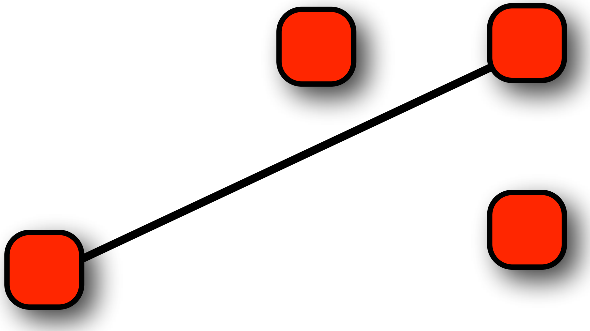
Sample-Nodes-And-Pick-Edge



- Sample each node with probability $\Theta(1/k)$ and delete the rest
- Return a random edge amongst those that remain. If no edges remain, return "null"

SNAPE Sampling

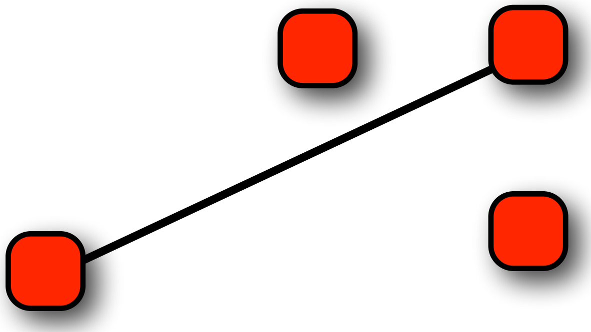
Sample-Nodes-And-Pick-Edge



- Sample each node with probability $\Theta(1/k)$ and delete the rest
- Return a random edge amongst those that remain. If no edges remain, return "null"

SNAPE Sampling

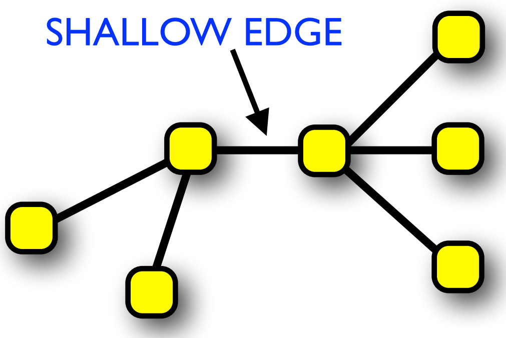
Sample-Nodes-And-Pick-Edge

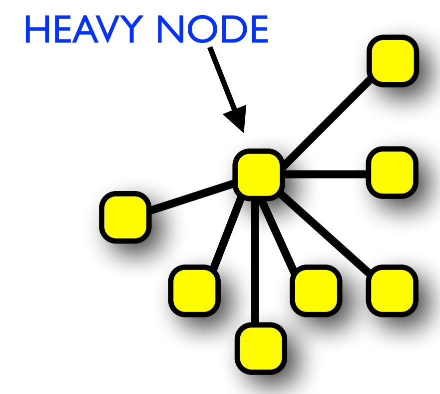


- Sample each node with probability Θ(1/k) and delete the rest
- Return a random edge amongst those that remain. If no edges remain, return "null"
- *Theorem* If G has max matching size ≤k then $O(k^2 \log k)$ SNAPE samples will include a max matching from G.

Small Matching Analysis: Basic Idea

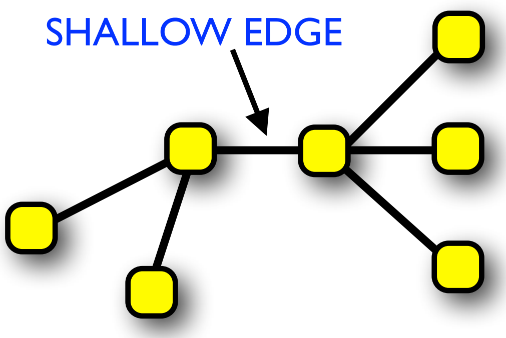
- Let G have max matching of size $\leq$k. Say node is heavy if degree is $\geq$10k and edge is shallow if both endpoints aren't heavy.

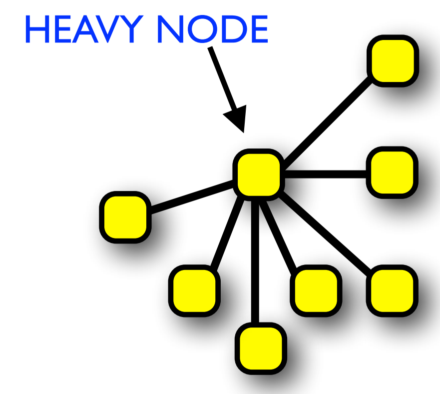




Small Matching Analysis: Basic Idea

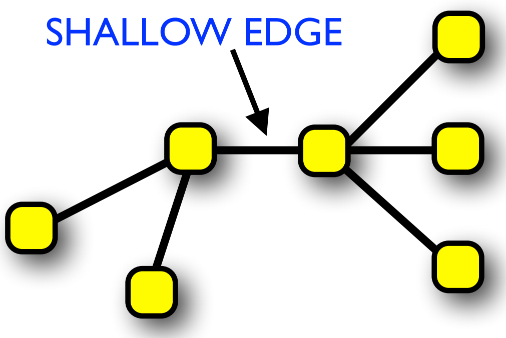
- Let G have max matching of size $\leq$k. Say node is heavy if degree is $\geq$10k and edge is shallow if both endpoints aren't heavy.

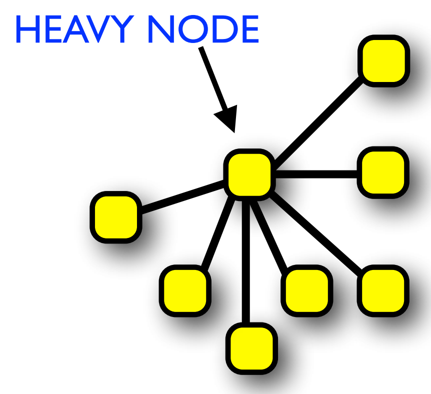




Small Matching Analysis: Basic Idea

- Let G have max matching of size $\leq$k. Say node is heavy if degree is $\geq$10k and edge is shallow if both endpoints aren't heavy.

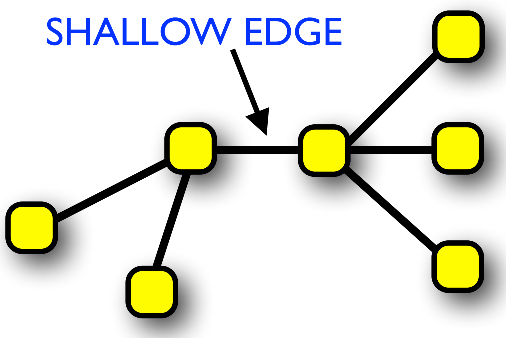


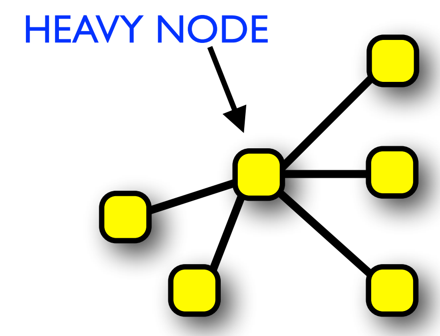


- *Lemma* G' includes a max matching of G if:
  - G' includes all shallow edges in G.
  - Every heavy node in G has degree at least 5k in G'.

Small Matching Analysis: Basic Idea

- Let G have max matching of size $\leq$k. Say node is heavy if degree is $\geq$10k and edge is shallow if both endpoints aren't heavy.

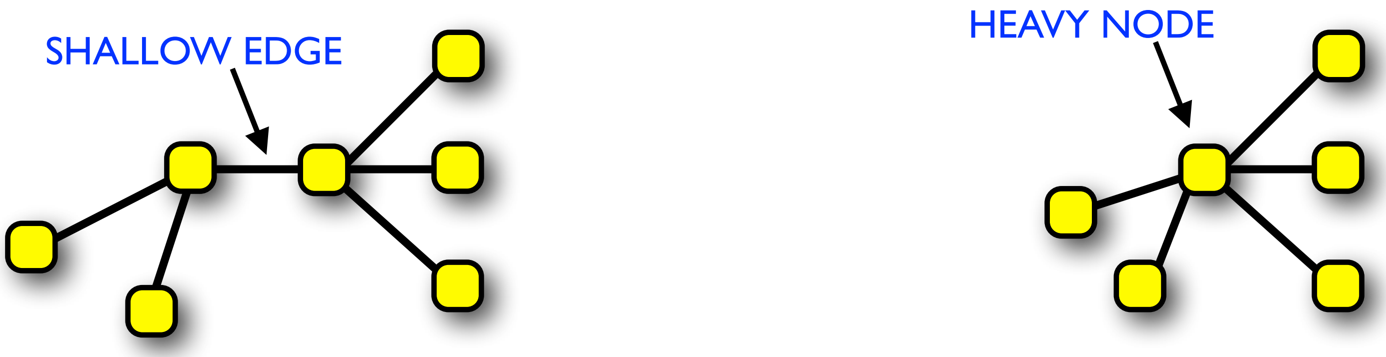




- *Lemma* G' includes a max matching of G if:
  - G' includes all shallow edges in G.
  - Every heavy node in G has degree at least 5k in G'.

Small Matching Analysis: Basic Idea

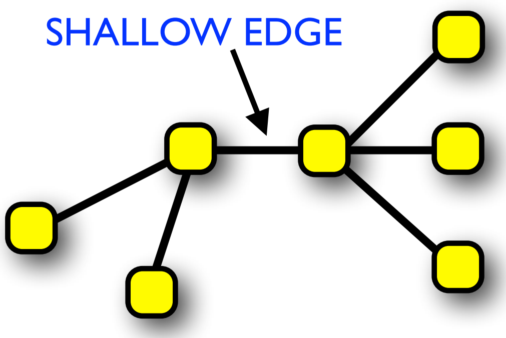
- Let G have max matching of size ≤k. Say node is heavy if degree is ≥10k and edge is shallow if both endpoints aren't heavy.

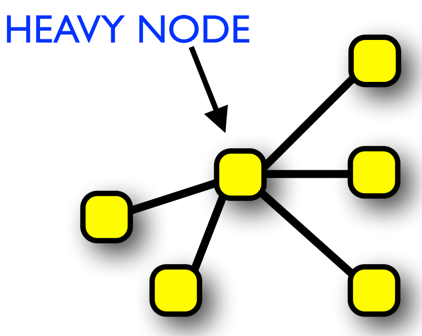


- *Lemma* G' includes a max matching of G if:
  - G' includes all shallow edges in G.
  - Every heavy node in G has degree at least 5k in G'.
- *Proof Idea* Each missing edge is incident to some heavy node but you still have plenty of other edges on that node.

Small Matching Analysis: Basic Idea

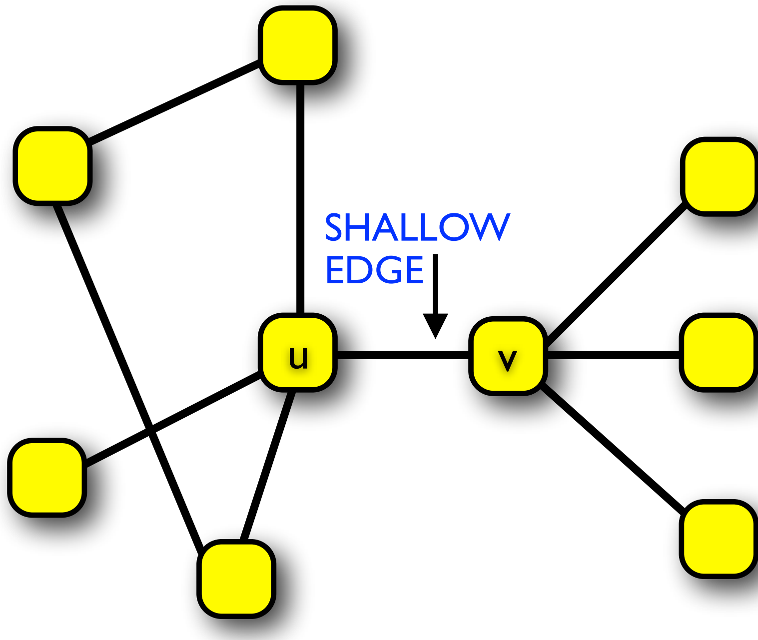
- Let G have max matching of size $\leq k$. Say node is heavy if degree is $\geq 10k$ and edge is shallow if both endpoints aren't heavy.



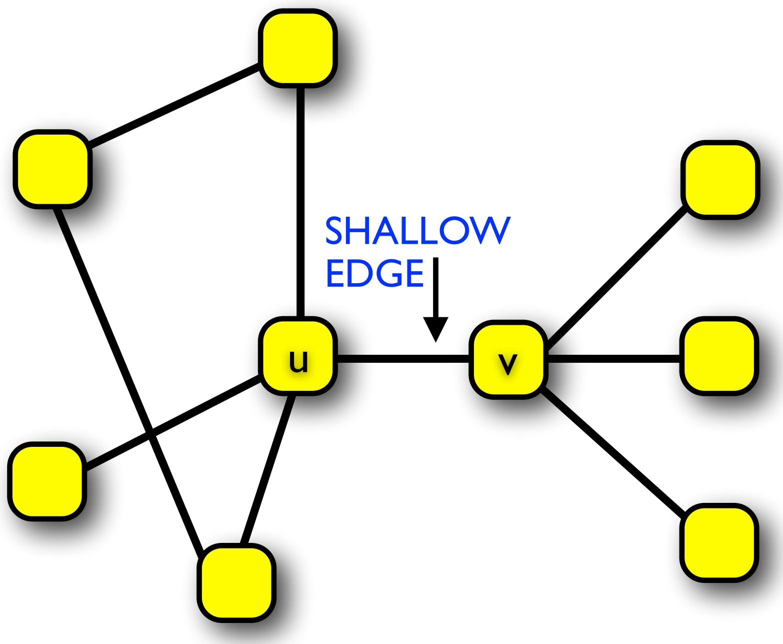


- *Lemma* G' includes a max matching of G if:
  - G' includes all shallow edges in G.
  - Every heavy node in G has degree at least 5k in G'.
- *Proof Idea* Each missing edge is incident to some heavy node but you still have plenty of other edges on that node.
- *Useful Fact* G has a vertex cover W of size at most 2k.

Small Matching Analysis: Shallow Edges

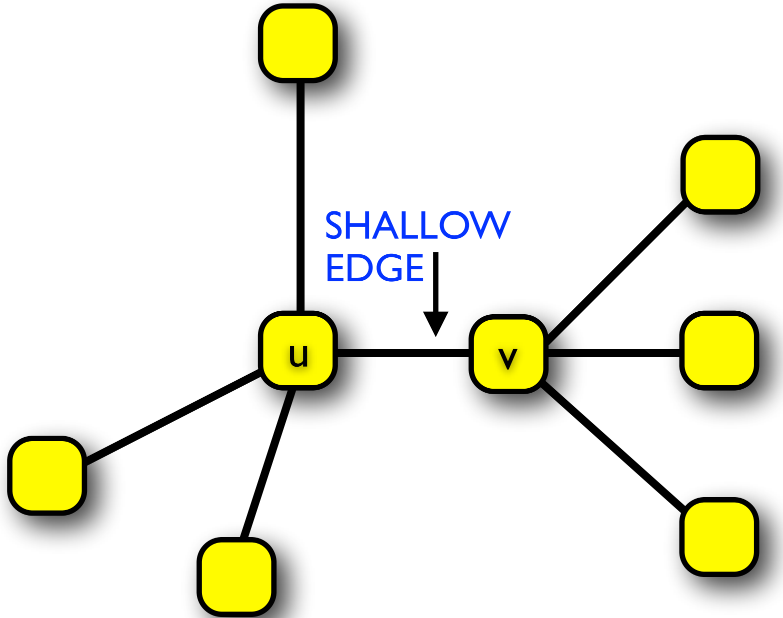


Small Matching Analysis: Shallow Edges



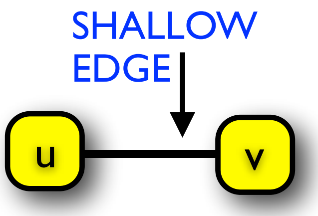
- If u and v are sampled but no other nodes in vertex cover W or $\Gamma(u)$ or $\Gamma(v)$ are sampled then uv is only edge remaining!

Small Matching Analysis: Shallow Edges



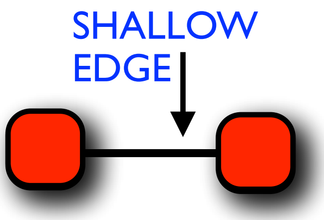
- If u and v are sampled but no other nodes in vertex cover W or Γ(u) or Γ(v) are sampled then uv is only edge remaining!

Small Matching Analysis: Shallow Edges



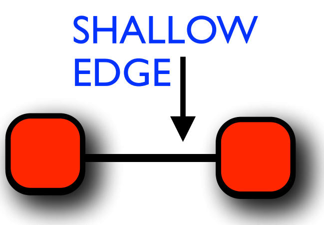
- If u and v are sampled but no other nodes in vertex cover W or Γ(u) or Γ(v) are sampled then uv is only edge remaining!

Small Matching Analysis: Shallow Edges



- If u and v are sampled but no other nodes in vertex cover W or Γ(u) or Γ(v) are sampled then uv is only edge remaining!

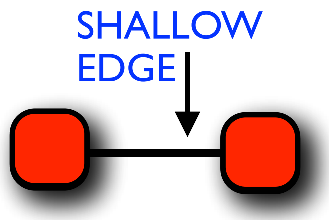
Small Matching Analysis: Shallow Edges



- If u and v are sampled but no other nodes in vertex cover W or Γ(u) or Γ(v) are sampled then uv is only edge remaining!
- Hence, if uv is shallow:

$$\Pr[\text{uv is only remaining edge}] \geq p^2(1-p)^{|\Gamma(u)|+|\Gamma(v)|+|W|} = \Omega(k^{-2})$$

Small Matching Analysis: Shallow Edges

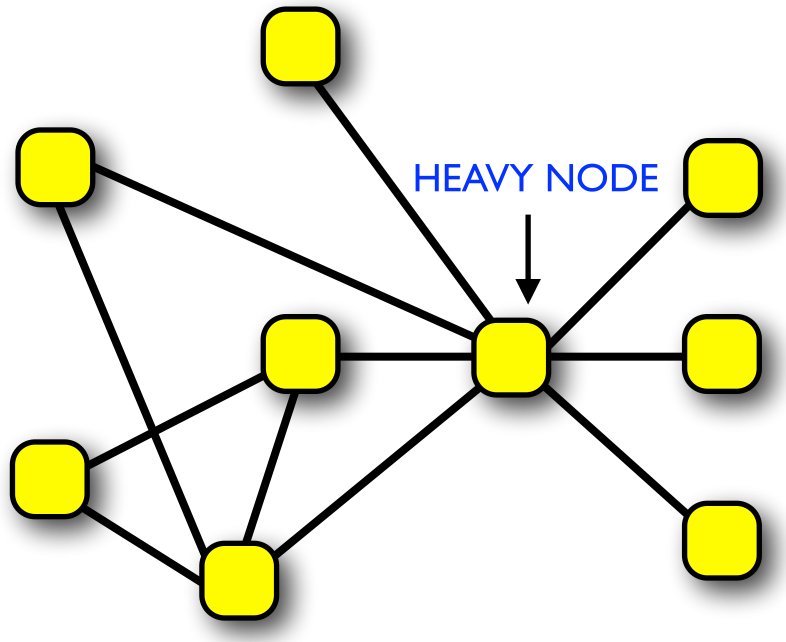


- If u and v are sampled but no other nodes in vertex cover W or Γ(u) or Γ(v) are sampled then uv is only edge remaining!
- Hence, if uv is shallow:

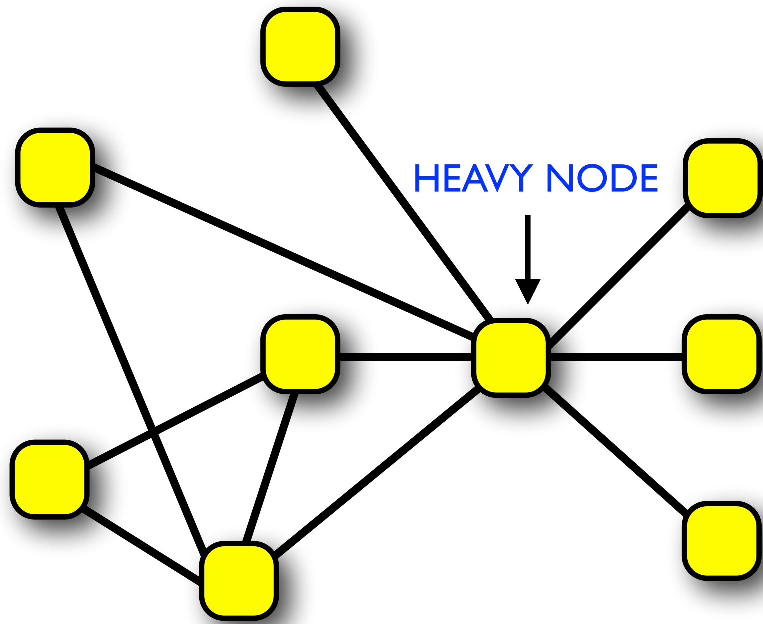
$$\Pr[\text{uv is only remaining edge}] \geq p^2(1-p)^{|\Gamma(u)|+|\Gamma(v)|+|W|} = \Omega(k^{-2})$$

- After $O(k^2 \log k)$ repetitions, have sampled edge uv whp.

Small Matching Analysis: Edges on Heavy Nodes

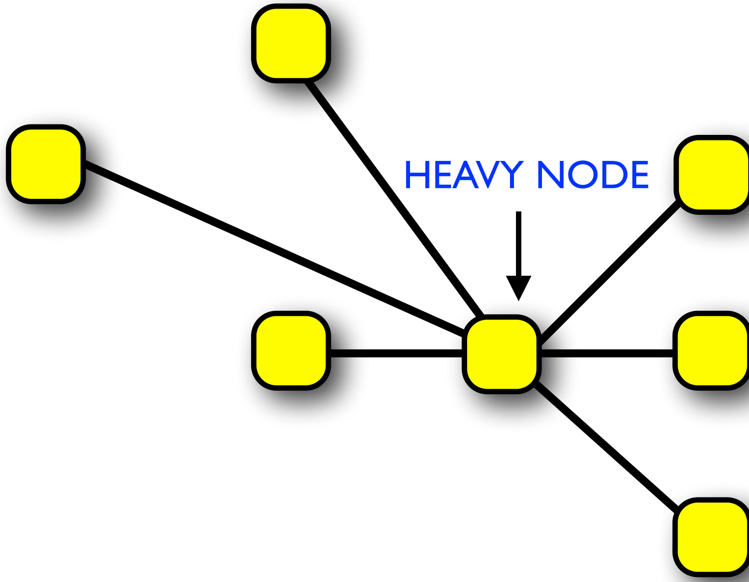


Small Matching Analysis: Edges on Heavy Nodes



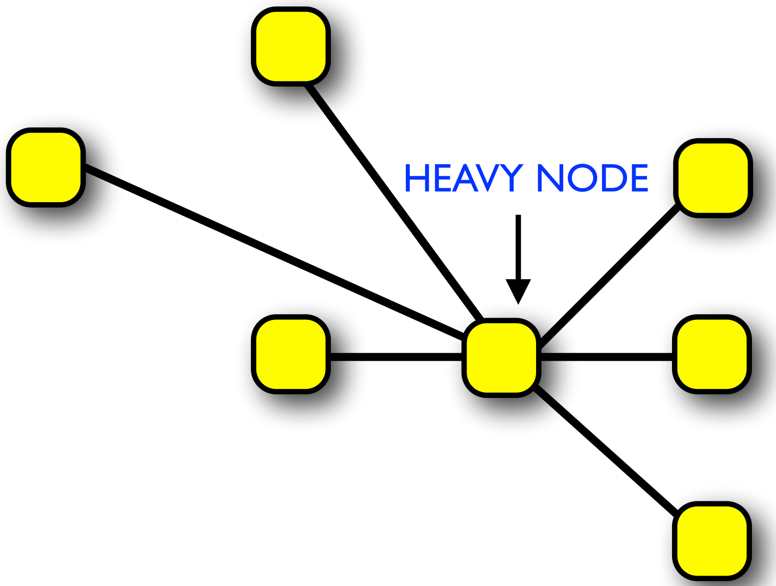
- For heavy u, deleting W\{u} leaves star on u with ≥ 8k leaves.

Small Matching Analysis: Edges on Heavy Nodes



- For heavy u, deleting W\{u} leaves star on u with ≥ 8k leaves.

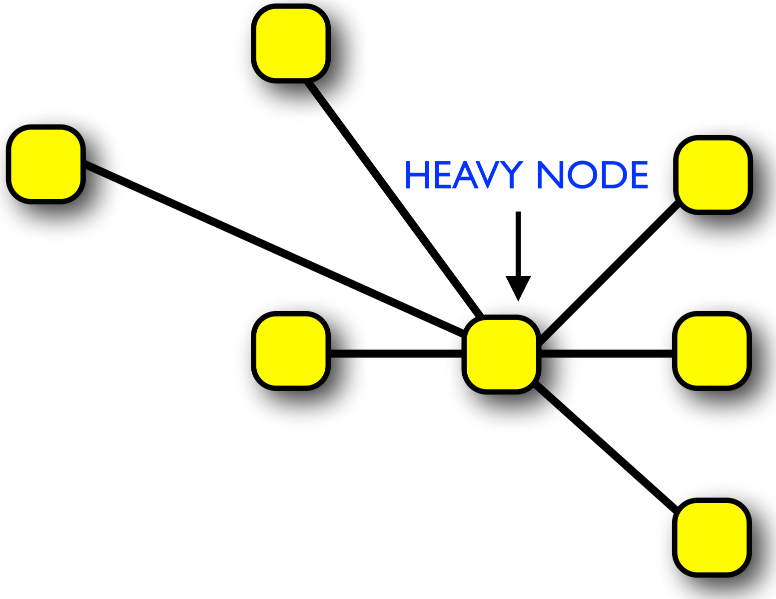
Small Matching Analysis: Edges on Heavy Nodes



- For heavy u, deleting W\{u} leaves star on u with ≥ 8k leaves.
- Hence,

$$\Pr[\text{edge incident to } u \text{ is sampled}] \approx \Omega(kp^2) \cdot (1-p)^{|W|} = \Omega(k^{-1})$$

Small Matching Analysis: Edges on Heavy Nodes



- For heavy u, deleting W\{u} leaves star on u with ≥ 8k leaves.
- Hence,

$$\Pr[\text{edge incident to } u \text{ is sampled}] \approx \Omega(kp^2) \cdot (1-p)^{|W|} = \Omega(k^{-1})$$

- After $O(k^2 \log k)$ repetitions, have sampled 5k edges on u.

Approximate Matching: Basic Idea

- *Theorem* If G has matching $\geq k$ then $O(k^2/t^3)$ SNAPE samples with $p=\Theta(t/k)$ has matching of size $\Omega(k/t)$ with high probability.

Approximate Matching: Basic Idea

- *Theorem* If G has matching $\geq k$ then $O(k^2/t^3)$ SNAPE samples with $p=\Theta(t/k)$ has matching of size $\Omega(k/t)$ with high probability.
- *Proof*
  - Let $e_1, e_2, e_3, \ldots$ be sequence of SNAPE samples and consider constructing greedy matching M.

Approximate Matching: Basic Idea

- *Theorem* If G has matching $\geq$k then $O(k^2/t^3)$ SNAPE samples with $p=\Theta(t/k)$ has matching of size $\Omega(k/t)$ with high probability.
- *Proof*
  - Let $e_1, e_2, e_3, \ldots$ be sequence of SNAPE samples and consider constructing greedy matching M.
  - Assuming $|M|=o(k/t)$ then

$$\Pr[e_i \text{ added to } M] \approx \Pr[e_i \text{ isn't a NULL}] \cdot \Pr[\text{all endpoints in } M \text{ are deleted}]$$
$$= \Omega(kp^2) \cdot (1-p)^{o(k/t)} = \Omega(t^2/k)$$

Approximate Matching: Basic Idea

- *Theorem* If G has matching $\geq$k then $O(k^2/t^3)$ SNAPE samples with p=$\Theta(t/k)$ has matching of size $\Omega(k/t)$ with high probability.
- *Proof*
  - Let $e_1, e_2, e_3, \ldots$ be sequence of SNAPE samples and consider constructing greedy matching M.
  - Assuming $|M|=o(k/t)$ then

$$\Pr[e_i \text{ added to } M] \approx \Pr[e_i \text{ isn't a NULL}] \cdot \Pr[\text{all endpoints in } M \text{ are deleted}]$$
$$= \Omega(kp^2) \cdot (1-p)^{o(k/t)} = \Omega(t^2/k)$$

  - After $O(k^2/t^3)$ SNAPE samples we have $|M|= \Omega(k/t)$

I. Graph Matching
via **SNAPE** Sampling

II. Graph Connectivity
via **DEALS** Sampling

Graph Connectivity

Graph Connectivity

- ***1st Result*** Test if k-edge-connected using Õ(kn) space.
  - Basic Idea: "DEALS" sampling primitive.

Graph Connectivity

- *<u>1st Result</u>* Test if k-edge-connected using $\tilde{O}(kn)$ space.
    - Basic Idea: "DEALS" sampling primitive.
- *<u>2nd Result</u>* Distinguish node connectivity $\leq k$ from $\geq (1+\varepsilon)k$ using $\tilde{O}(\varepsilon^{-1}kn)$ space.
    - Basic Idea: Combine node sampling and DEALS sampling.
    - Open: Testing exact node connectivity?

Graph Connectivity

- *1st Result* Test if k-edge-connected using $\tilde{O}(kn)$ space.
  - Basic Idea: "DEALS" sampling primitive.
- *2nd Result* Distinguish node connectivity $\leq k$ from $\geq (1+\varepsilon)k$ using $\tilde{O}(\varepsilon^{-1}kn)$ space.
  - Basic Idea: Combine node sampling and DEALS sampling.
  - Open: Testing exact node connectivity?
- *3rd Result* $(1+\varepsilon)$-approx every cut using $\tilde{O}(\varepsilon^{-2}n)$ space.
  - Basic Idea: Combine edge sampling and DEALS sampling.
  - Hypergraph Sparsifiers: Extends *Kogan, Krauthgamer* [ITCS 15]

DEALS Sampling

Direct-Edges-Add-L_0-Sketches

DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.

DEALS Sampling

Direct-Edges-Add-L_0-Sketches

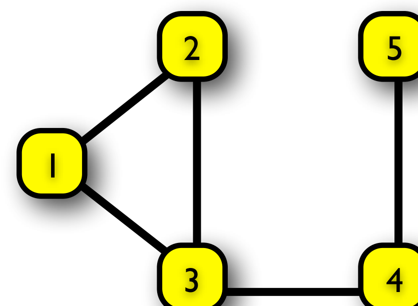
- ***Problem*** Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.

- ***Algorithm*** Construct $M\mathbf{a}_1$, $M\mathbf{a}_2$, … , $M\mathbf{a}_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.
- *Algorithm* Construct Ma_1, Ma_2, … , Ma_n where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.



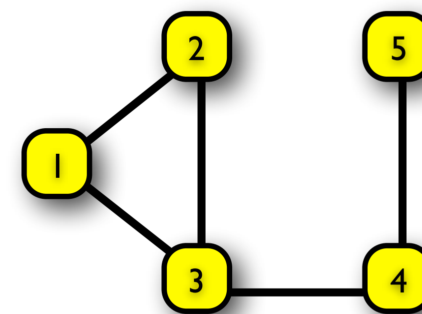
DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use $\tilde{O}(n)$ space.
- *Algorithm* Construct $Ma_1, Ma_2, \ldots, Ma_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}	{4,5}
$\mathbf{a_1} =$	1	1	0	0	0	0	0	0	0	0

$$\mathbf{a_1} = (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$



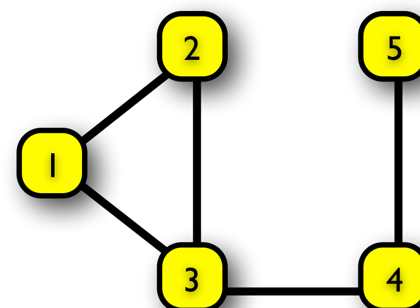
DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use $\tilde{O}(n)$ space.

- *Algorithm* Construct $Ma_1, Ma_2, \ldots, Ma_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}	{4,5}
$\mathbf{a_1} = ($	1	1	0	0	0	0	0	0	0	0)
$\mathbf{a_2} = ($	-1	0	0	0	1	0	0	0	0	0)

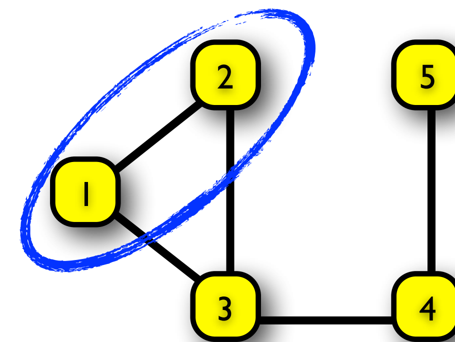


DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.
- *Algorithm* Construct $Ma_1, Ma_2, \ldots, Ma_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}	{4,5}
$\mathbf{a_1} = ($	1	1	0	0	0	0	0	0	0	0$)$
$\mathbf{a_2} = ($	-1	0	0	0	1	0	0	0	0	0$)$



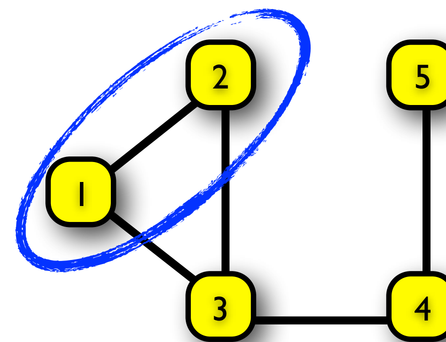
DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.

- *Algorithm* Construct $Ma_1, Ma_2, \ldots, Ma_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

$$
\begin{array}{rcccccccccccc}
 & & \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} & \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} & \\
\mathbf{a_1} = & (& 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &) \\
\mathbf{a_2} = & (& -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 &) \\
\mathbf{a_1} + \mathbf{a_2} = & (& 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 &)
\end{array}
$$

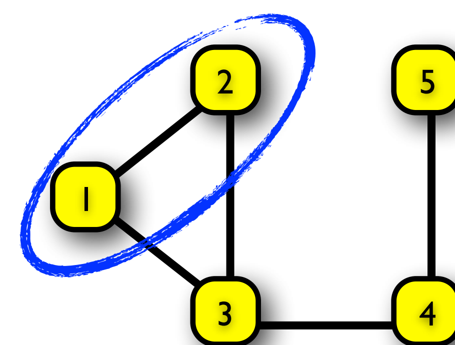


DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.
- *Algorithm* Construct $Ma_1, Ma_2, \ldots, Ma_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,4}	{2,5}	{3,4}	{3,5}	{4,5}
$\mathbf{a}_1 =$	1	1	0	0	0	0	0	0	0	0
$\mathbf{a}_2 =$	−1	0	0	0	1	0	0	0	0	0
$\mathbf{a}_1 + \mathbf{a}_2 =$	0	1	0	0	1	0	0	0	0	0



- *Lemma* Non-zero entries of $\sum_{i \in S} \mathbf{a}_i$ = edges across (S,V\S) and hence $\sum_{i \in S} M\mathbf{a}_i = M(\sum_{i \in S} \mathbf{a}_i)$ yields random edge across (S,V\S).

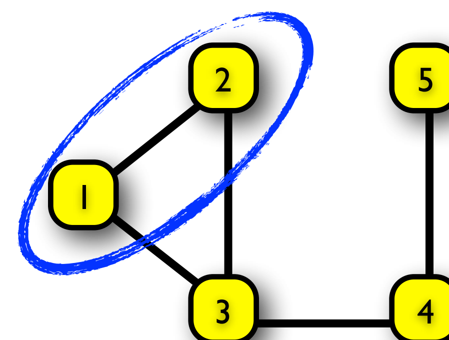
DEALS Sampling

Direct-Edges-Add-L_0-Sketches

- *Problem* Sample edge across cut (S,V\S) where cut is specified at end of the stream. May use Õ(n) space.

- *Algorithm* Construct $Ma_1, Ma_2, \ldots, Ma_n$ where M is L_0-sampling sketch and a_i is vector encoding neighborhood of node i.

$$
\begin{array}{rcccccccccccl}
 & & \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} & \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} & \\
\mathbf{a_1} = & (& 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &) \\
\mathbf{a_2} = & (& -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 &) \\
\mathbf{a_1} + \mathbf{a_2} = & (& 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 &)
\end{array}
$$



- *Lemma* Non-zero entries of $\sum_{i \in S} \mathbf{a_i}$ = edges across (S,V\S) and hence $\sum_{i \in S} M\mathbf{a_i} = M(\sum_{i \in S} \mathbf{a_i})$ yields random edge across (S,V\S).

- *Application* Find spanning trees and edges in light cuts.

Application to Node Connectivity…

Application to Node Connectivity…

- *Simplified Result* Can answer queries of form "are u and v connected after removal of set of k nodes S" using $\tilde{O}(kn)$ space.

Application to Node Connectivity...

- *Simplified Result* Can answer queries of form "are u and v connected after removal of set of k nodes S" using Õ(kn) space.
- *Algorithm*
  - Sample some edges and answer "no" iff there's no S-avoiding path between u and v amongst sampled edges.

Application to Node Connectivity...

- *Simplified Result* Can answer queries of form "are u and v connected after removal of set of k nodes S" using $\tilde{O}(kn)$ space.
- *Algorithm*
    - Sample some edges and answer "no" iff there's no S-avoiding path between u and v amongst sampled edges.
    - How to sample: Pick each node with probability 1/k and find spanning forest on these nodes. Repeat $\tilde{O}(k^2)$ times.

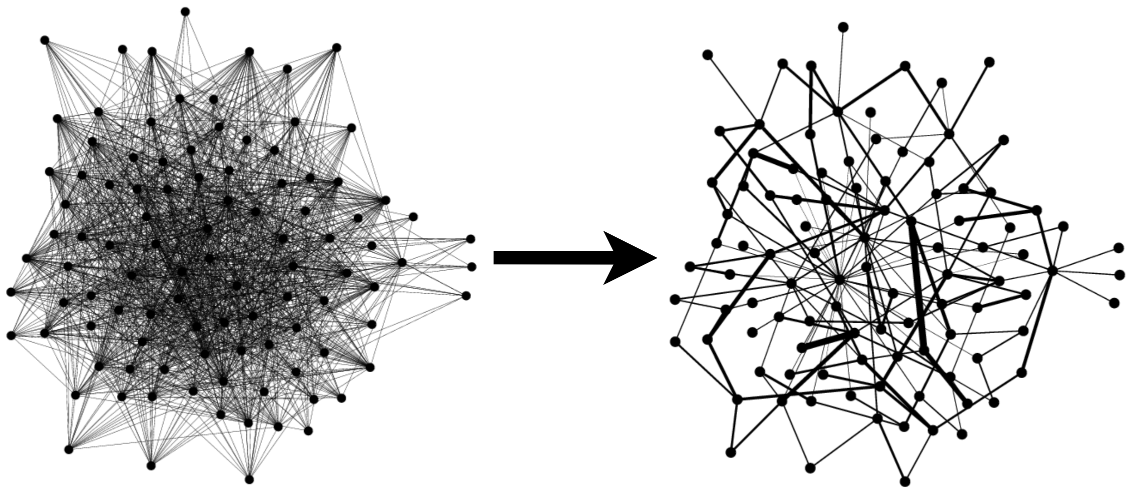
Application to Node Connectivity…

- *Simplified Result* Can answer queries of form "are u and v connected after removal of set of k nodes S" using Õ(kn) space.
- *Algorithm*
  - Sample some edges and answer "no" iff there's no S-avoiding path between u and v amongst sampled edges.
  - How to sample: Pick each node with probability 1/k and find spanning forest on these nodes. Repeat $\tilde{O}(k^2)$ times.
- *Analysis* Let u-x_1-x_2-….-x_t-v be S-avoiding path in input graph.

Application to Node Connectivity…

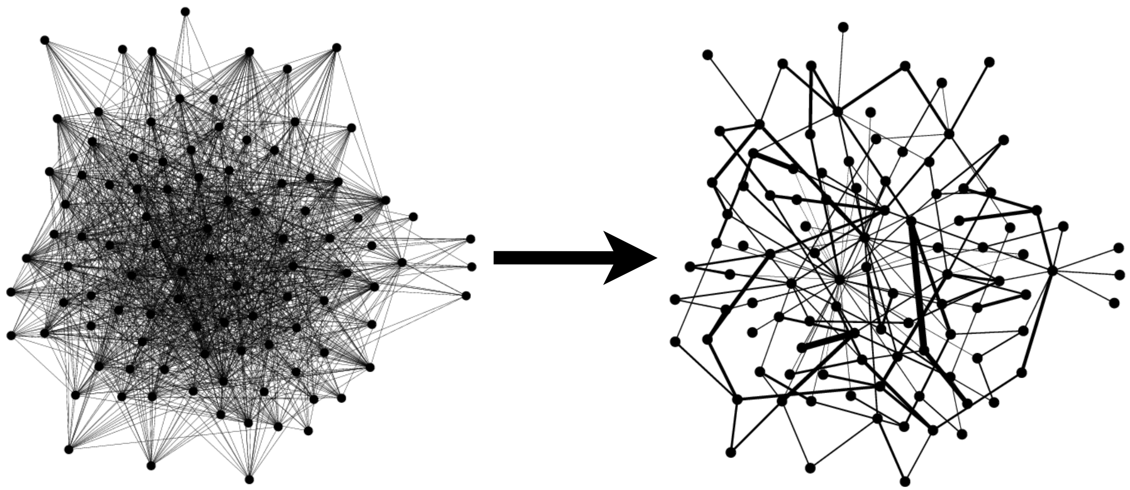
- *Simplified Result* Can answer queries of form "are u and v connected after removal of set of k nodes S" using $\tilde{O}(kn)$ space.
- *Algorithm*
  - Sample some edges and answer "no" iff there's no S-avoiding path between u and v amongst sampled edges.
  - How to sample: Pick each node with probability 1/k and find spanning forest on these nodes. Repeat $\tilde{O}(k^2)$ times.
- *Analysis* Let u-x_1-x_2-….-x_t-v be S-avoiding path in input graph.
  - Spanning forest on sampled nodes contains an S-avoiding path between x_i and x_{i+1} with prob. $p^2(1-p)^k \approx k^{-2}$. After $\tilde{O}(k^2)$ repeats we have S-avoiding path in E' with high probability.

Application to Cut Sparsification…



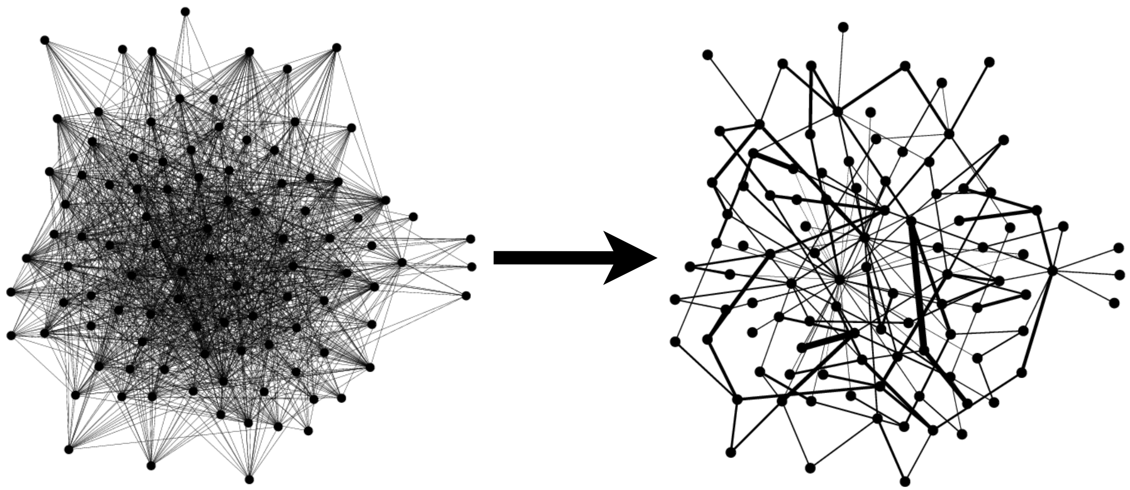
- *Result* Can 1+ε approximate all cuts using $\tilde{O}(\varepsilon^{-2}n)$ space.

Application to Cut Sparsification…



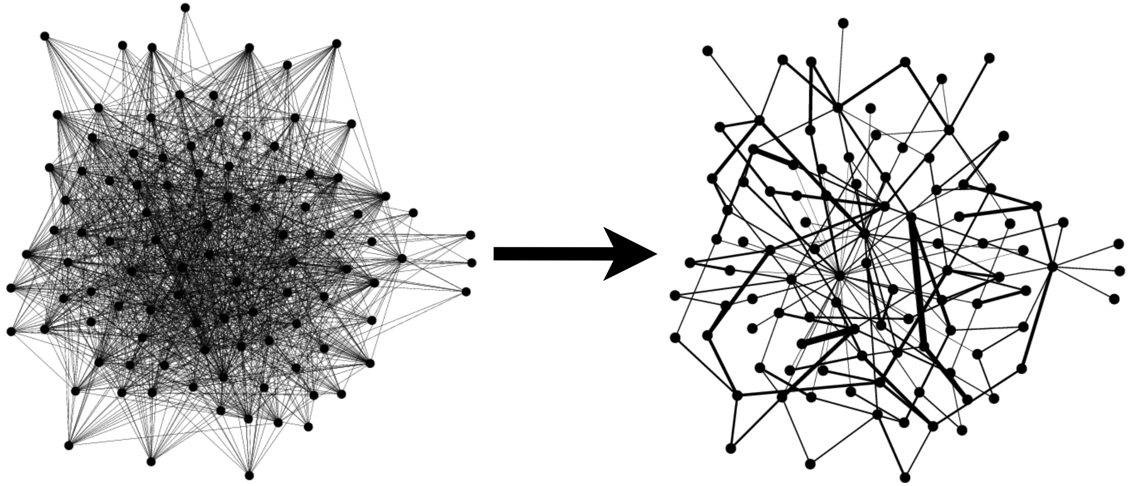
- *Result* Can 1+ε approximate all cuts using $\tilde{O}(\varepsilon^{-2} n)$ space.
- *Basic Idea*
  - Sampling edges with probability ≥ $(c\varepsilon^{-2} \log n)/\lambda_e$ preserves all cut sizes where λ_e is the edge connectivity. *Fung et al.* [STOC 11]

Application to Cut Sparsification…



- *Result* Can $1+\varepsilon$ approximate all cuts using $\tilde{O}(\varepsilon^{-2}n)$ space.
- *Basic Idea*
  - Sampling edges with probability $\geq (c\varepsilon^{-2} \log n)/\lambda_e$ preserves all cut sizes where λ_e is the edge connectivity. *Fung et al.* [STOC 11]
  - Use DEALS sampling to pick all edges with $\lambda_e \leq 2c\varepsilon^{-2} \log n$ and sample each remaining edge with probably 1/2.

Application to Cut Sparsification…



- *Result* Can $1+\varepsilon$ approximate all cuts using $\tilde{O}(\varepsilon^{-2}n)$ space.
- *Basic Idea*
  - Sampling edges with probability $\geq (c\varepsilon^{-2} \log n)/\lambda_e$ preserves all cut sizes where λ_e is the edge connectivity. *Fung et al.* [STOC 11]
  - Use DEALS sampling to pick all edges with $\lambda_e \leq 2c\varepsilon^{-2} \log n$ and sample each remaining edge with probably 1/2.
  - Recurse O(log n) times in parallel until we have sparse graph.

Thanks!

Graph Streaming Survey McGregor [SIGMOD Record 14]

Vertex Connectivity and Sparsification Guha, McGregor, Tench [PODS 15]

Densest Subgraphs McGregor, Tench, Vorotnikova, Vu [MFCS 15]

Matching, Vertex Cover, Hitting Set

Chitnis, Cormode, Esfandiari, Hajiaghayi, McGregor, Monemizadeh, Vorotnikova [SODA 16]

Thanks!

Graph Streaming Survey — McGregor [SIGMOD Record 14]

Vertex Connectivity and Sparsification — Guha, McGregor, Tench [PODS 15]

Densest Subgraphs — McGregor, Tench, Vorotnikova, Vu [MFCS 15]

Matching, Vertex Cover, Hitting Set

Chitnis, Cormode, Esfandiari, Hajiaghayi, McGregor, Monemizadeh, Vorotnikova [SODA 16]