First Order Logic

Philipp Koehn

29 February 2024

Wittgenstein: Tractatus Logico-Philosophicus

1. The world is everything that is the case.

2. What is the case (a fact) is the existence of states of affairs.

3. A logical picture of facts is a thought.

4. A thought is a proposition with a sense.

5. A proposition is a truth-function of elementary propositions. (An elementary proposition is a truth-function of itself.)

6. The general form of a proposition is the general form of a truth function, which is: $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of a proposition.

7. Whereof one cannot speak, thereof one must be silent.

Outline

- Why first order logic?
- Syntax and semantics of first order logic
- Fun with sentences
- Wumpus world in first order logic

why?

Pros and Cons of Propositional Logic

- **PRO:** Propositional logic is **declarative**: pieces of syntax correspond to facts
- **PRO:** Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- **PRO:** Propositional logic is **compositional**: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- **PRO:** Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends on context)
- **CON:** Propositional logic has very limited expressive power (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

First-Order Logic

- Propositional logic: world contains **facts**
- First-order logic: the world contains **objects**, **relations**, and **functions**
- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries ...
- Relations: red, round, bogus, prime, multistoried ...,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...
- Functions: father of, best friend, third inning of, one more than, end of ...

More Logics

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Higher-order logic:

relations and functions operate not only on objects,
but also on relations and functions

syntax and semantics

Syntax of FOL: Basic Elements

- Constants: $KingJohn, 2, UCB, \ldots$
- Predicates: $Brother, >, \ldots$
- Functions: $Sqrt, LeftLegOf, \ldots$
- Variables: $x, y, a, b, \ldots$
- Connectives: $\wedge \ \vee \ \neg \ \implies \ \Leftrightarrow$
- Equality: $=$
- Quantifiers: $\forall \ \exists$

Atomic Sentences

- Atomic sentence = $predicate(term_1, \ldots, term_n)$
  or $term_1 = term_2$
- Term = $function(term_1, \ldots, term_n)$
  or $constant$ or $variable$
- E.g., $Brother(KingJohn, RichardTheLionheart)$
  $> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex Sentences

- Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \implies S_2, \quad S_1 \Leftrightarrow S_2$$

- For instance

$$Sibling(KingJohn, Richard) \implies Sibling(Richard, KingJohn)$$

Truth in First-Order Logic

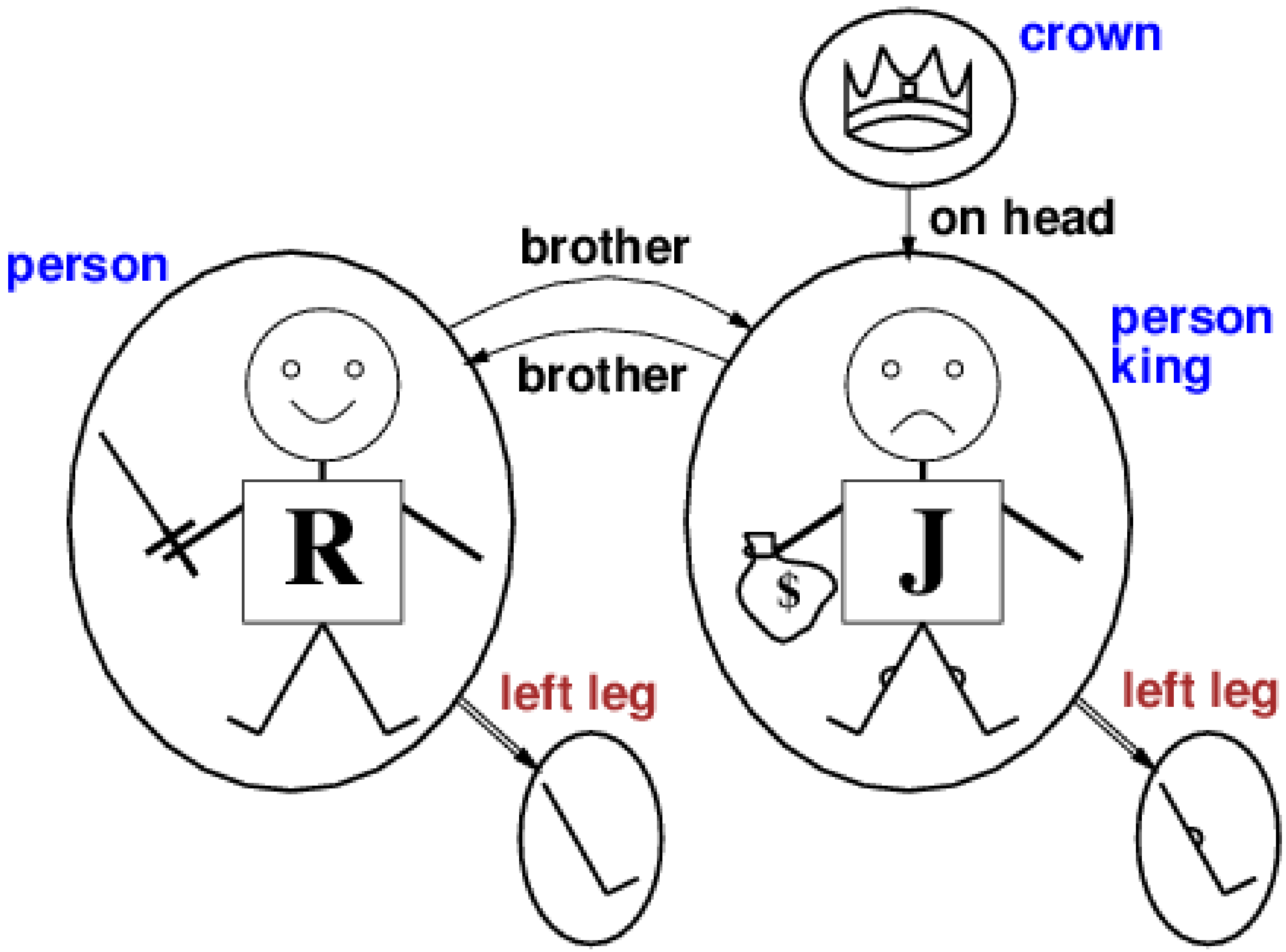
- Sentences are true with respect to a model and an interpretation

- **Model** contains ≥ 1 objects (domain elements) and relations among them

- **Interpretation** specifies referents for
  - constant symbols → objects
  - predicate symbols → relations
  - function symbols → functional relations

- An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
  iff the objects referred to by $term_1, \ldots, term_n$
  are in the relation referred to by $predicate$

Models for FOL: Example



Truth Example

- Object symbols
  - $Richard$ → Richard the Lionheart
  - $John$ → the evil King John
- Predicat symbol
  - $Brother$ → the brotherhood relation
- Atomic sentence
  - $Brother(Richard, John)$

  true iff Richard the Lionheart and the evil King John
  are in the brotherhood relation in the model

Models for FOL: Lots!

- Entailment in propositional logic can be computed by enumerating models

- We **can** enumerate the FOL models for a given KB vocabulary:

- For each number of domain elements n from 1 to ∞
  For each k-ary predicate P_k in the vocabulary
    For each possible k-ary relation on n objects
      For each constant symbol C in the vocabulary
        For each choice of referent for C from n objects ...

- Computing entailment by enumerating FOL models is not easy!

Universal Quantification

- Syntax: $\forall \langle variables \rangle \; \langle sentence \rangle$

- Everyone at JHU is smart:
  $\forall\, x \;\; At(x, JHU) \implies Smart(x)$

- $\forall\, x \;\; P$ is true in a model m iff P is true with x being **each** possible object in the model

- **Roughly** speaking, equivalent to the conjunction of instantiations of P

$$\begin{array}{ll} & (At(KingJohn, JHU) \implies Smart(KingJohn)) \\ \wedge & (At(Richard, JHU) \implies Smart(Richard)) \\ \wedge & (At(Jane, JHU) \implies Smart(Jane)) \\ \wedge & \ldots \end{array}$$

A Common Mistake to Avoid

- Typically " $\implies$ " is the main connective with $\forall$

- Common mistake: using "$\wedge$" as the main connective with $\forall$:

$$\forall\, x \;\; At(x, JHU) \wedge Smart(x)$$

means "Everyone is at JHU and everyone is smart"

- Correct

$$\forall\, x \;\; At(x, JHU) \implies Smart(x)$$

means "For everyone, if she is at JHU, then she is smart"

Existential Quantification

- Syntax: $\exists \langle variables \rangle\ \langle sentence \rangle$
- Someone at JHU is smart:

  $\exists x\ \ At(x, JHU) \wedge Smart(x)$
- $\exists x\ \ P$ is true in a model m iff P is true with x being **some** possible object in the model
- **Roughly** speaking, equivalent to the disjunction of instantiations of P

$$
\begin{array}{ll}
 & (At(KingJohn, JHU) \wedge Smart(KingJohn)) \\
\vee & (At(Richard, JHU) \wedge Smart(Richard)) \\
\vee & (At(JHU, JHU) \wedge Smart(JHU)) \\
\vee & \ldots
\end{array}
$$

Another Common Mistake to Avoid

- Typically "$\wedge$" is the main connective with $\exists$
- Common mistake: using " $\implies$ " as the main connective with $\exists$:

$$\exists\, x \;\; At(x, JHU) \implies Smart(x)$$

is true if there is anyone who is not at JHU

- Correct

$$\exists\, x \;\; At(x, JHU) \wedge Smart(x)$$

is true if there is someone who is at JHU and smart

Properties of Quantifiers

- $\forall\, x \;\; \forall\, y$ is the same as $\forall\, y \;\; \forall\, x$
- $\exists\, x \;\; \exists\, y$ is the same as $\exists\, y \;\; \exists\, x$
- $\exists\, x \;\; \forall\, y$ is **not** the same as $\forall\, y \;\; \exists\, x$
- $\exists\, x \;\; \forall\, y \;\; Loves(x, y)$
  "There is a person who loves everyone in the world"
- $\forall\, y \;\; \exists\, x \;\; Loves(x, y)$
  "Everyone in the world is loved by at least one person"
- Quantifier duality: each can be expressed using the other
- $\forall\, x \;\; Likes(x, IceCream) \qquad \neg\exists\, x \;\; \neg Likes(x, IceCream)$
- $\exists\, x \;\; Likes(x, Broccoli) \qquad \neg\forall\, x \;\; \neg Likes(x, Broccoli)$

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
- For instance
  - $1 = 2$ and $\forall\, x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
  - $2 = 2$ is true

  (note: syntax does not imply anything about the semantics of 1, 2, $Sqrt(x)$, etc.)
- Definition of (full) $Sibling$ in terms of $Parent$

$$\forall\, x, y \;\; Sibling(x,y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$$

fun with sentences

Fun with Sentences

- Brothers are siblings

  $\forall\, x, y \;\; Brother(x, y) \implies Sibling(x, y)$

- "Sibling" is symmetric

  $\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$

- One's mother is one's female parent

  $\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \land Parent(x, y))$

- A first cousin is a child of a parent's sibling

  $\forall\, x, y \;\; FirstCousin(x, y) \;\Leftrightarrow\; \exists\, p, ps \;\; Parent(p, x) \land Sibling(ps, p) \land Parent(ps, y)$

Lincoln Quote

You can fool all the people some of the time,
and some of the people all the time,
but you cannot fool all the people all the time.

$$\forall\, p\ \exists\, t\ \ Fool(p,t)$$
$$\wedge$$
$$\exists\, p\ \forall\, t\ \ Fool(p,t)$$
$$\wedge$$
$$\neg\ \forall\, p\ \forall\, t\ \ Fool(p,t)$$

Donkey Sentences

- *Every farmer owns a donkey.*
  - $\forall\, f \ (Farmer(f) \implies \exists\, d \ (Donkey(d) \land Own(f,d)))$
  - $\exists\, d \ (Donkey(d) \land \forall\, f \ (Farmer(f) \land Own(f,d)))$
- *Every human lives on a planet.*
  - $\exists\, p \ (Planet(p) \land \forall\, h \ (Human(h) \land LivesOn(h,p)))$
- *Every farmer who owns a donkey beats it.*
  - $\forall\, f \ \ Farmer(f) \land \exists\, d \ (Donkey(d) \land Own(f,d) \implies Beats(f,d))$
    but what if a farmer has a donkey d_1 and a pig d_2 and he beats neither
    $Donkey(d_2) \land Own(f,d_2) \implies Beats(f,d_2)$ is true ($false \land true \implies false$)
  - $\forall\, f \ \forall\, d \ (Farmer(f) \land Donkey(d) \land Own(f,d) \implies Beats(f,d))$
    but this means "Every farmer beats every donkey he owns."

Natural Language

- First order logic is close to the semantics of natural language
- But there are limitations
  - *"There is at least one thing John has in common with Peter."*
    Requires a quantifier over predicates.
  - *"The cake is very good."*
    $\exists\, c \;\; Cake(c) \land Good(c)$ but not $Very(c)$
    Functions and relations cannot be qualified.
- Natural language sentences are often intentionally vague and ambiguous

wampus world

Knowledge Base for the Wumpus World

- "Perception": at current square, three perceptions expressed as variables
  - s either $Smell$ or $\neg Smell$
  - b either $Breeze$ or $\neg Breeze$
  - g either $Glitter$ or $\neg Glitter$

  $Percept([s, b, g], t)$

- Shorthands
  - $\forall\, b, g, t \;\; Percept([Smell, b, g], t) \implies Smelt(t)$
  - $\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \implies AtGold(t)$

- Reflex: $\forall\, t \;\; AtGold(t) \implies Action(Grab, t)$

- Reflex with internal state: do we have the gold already?
  $\forall\, t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \implies Action(Grab, t)$

- $Holding(Gold, t)$ cannot be observed
  $\Rightarrow$ keeping track of change is essential

Deducing Hidden Properties

- Properties of locations:
  $\forall\, x,t \;\; At(Agent,x,t) \wedge Smelt(t) \implies Smelly(x)$
  $\forall\, x,t \;\; At(Agent,x,t) \wedge Breeze(t) \implies Breezy(x)$

- Squares are breezy near a pit

- Diagnostic rule—infer cause from effect
  $$\forall\, y \;\; Breezy(y) \implies \exists\, x \;\; Pit(x) \wedge Adjacent(x,y)$$

- Causal rule—infer effect from cause
  $$\forall\, x,y \;\; Pit(x) \wedge Adjacent(x,y) \implies Breezy(y)$$

- Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

- Definition for the $Breezy$ predicate:
  $$\forall\, y \;\; Breezy(y) \Leftrightarrow [\exists\, x \;\; Pit(x) \wedge Adjacent(x,y)]$$

States and Fluents

- By acting, the agent moves through a sequence of situations s

- Fluents: aspects of the world that may change
  - current position
  - having an arrow
  - holding the gold

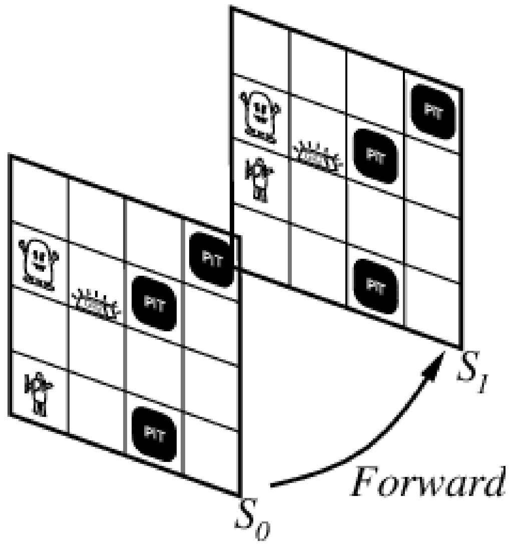
- Taking actions requires updates to the fluents

Keeping Track of Change

- Facts hold in situations, rather than eternally
  E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

- Situation calculus is one way to represent change in FOL:
  Adds a situation argument to each non-eternal predicate
  E.g., Now in $Holding(Gold, Now)$ denotes a situation

- Situations are connected by the $Result$ function
  $s' = Result(a, s)$ is the situation that results from doing a in s



Describing Actions

- "Effect" axiom—describe changes due to action
  $\forall\, s \;\; AtGold(s) \implies Holding(Gold, Result(Grab, s))$

- "Frame" axiom—describe **non-changes** due to action
  $\forall\, s \;\; HaveArrow(s) \implies HaveArrow(Result(Grab, s))$

- Frame problem: find an elegant way to handle non-change
  (a) representation: too many frame axioms
  (b) inference: too many repeated "copy-overs" to keep track of state

- Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

- Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, . . .

Describing Actions

- Successor-state axioms solve the representational frame problem

- Each axiom is "about" a **predicate** (not an action per se):

$$\begin{array}{lcl} \text{P true afterwards} & \Leftrightarrow & [\text{an action made P true} \\ & \vee & \text{P true already and no action made P false}] \end{array}$$

- For holding the gold:

$$\forall\, a, s \;\; Holding(Gold, Result(a, s)) \;\Leftrightarrow$$
$$[(a = Grab \wedge AtGold(s))$$
$$\vee\, (Holding(Gold, s) \wedge a \neq Release)]$$

Making Plans

- Initial condition in KB:

  $At(Agent, [1, 1], S_0)$
  $At(Gold, [1, 2], S_0)$

- Query: $Ask(KB, \exists\, s \;\; Holding(Gold, s))$
  i.e., in what situation will I be holding the gold?

- Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$
  i.e., go forward and then grab the gold

- This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making Plans: A Better Way

- Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

- $PlanResult(p, s)$ is the result of executing p in s

- Then the query $Ask(KB, \exists p\ \ Holding(Gold, PlanResult(p, S_0)))$ has the solution $\{p/[Forward, Grab]\}$

- Definition of $PlanResult$ in terms of $Result$:

  $\forall\, s\ \ PlanResult([\,], s) = s$

  $\forall\, a, p, s\ \ PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

- Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

- First-order logic
  - objects and relations are semantic primitives
  - syntax: constants, functions, predicates, equality, quantifiers

- Increased expressive power: sufficient to define wumpus world

- Situation calculus
  - conventions for describing actions and change in FOL
  - can formulate planning as inference on a situation calculus KB