Markov Decision Processes

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Outline

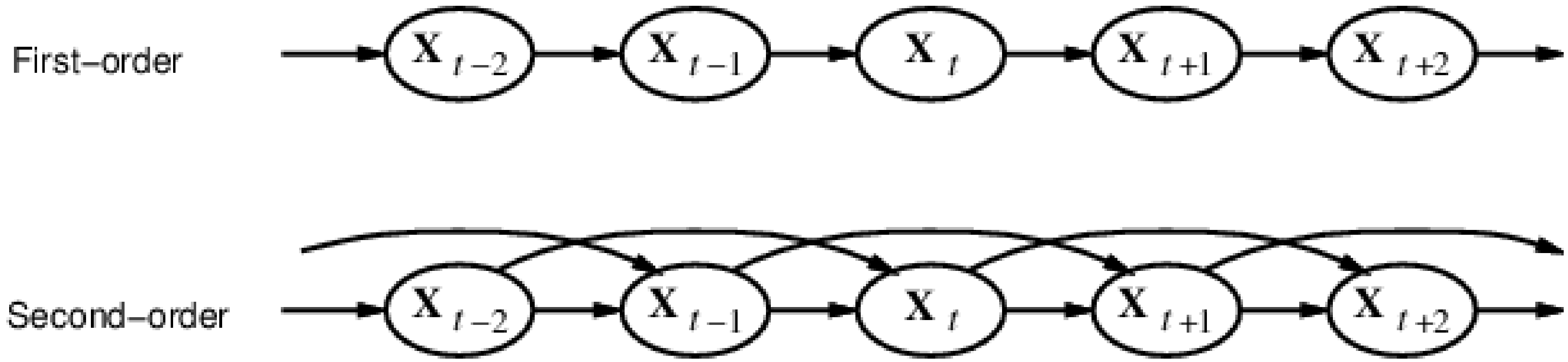
- Hidden Markov models
- Inference: filtering, smoothing, best sequence
- Dynamic Bayesian networks
- Speech recognition

Time and Uncertainty

- The world changes; we need to track and predict it
- Diabetes management vs vehicle diagnosis
- Basic idea: sequence of state and evidence variables
- $\mathbf{X}_t$ = set of unobservable state variables at time t
  e.g., $BloodSugar_t$, $StomachContents_t$, etc.
- $\mathbf{E}_t$ = set of observable evidence variables at time t
  e.g., $MeasuredBloodSugar_t$, $PulseRate_t$, $FoodEaten_t$
- This assumes **discrete time**; step size depends on problem
- Notation: $\mathbf{X}_{a:b} = \mathbf{X}_a, \mathbf{X}_{a+1}, \ldots, \mathbf{X}_{b-1}, \mathbf{X}_b$

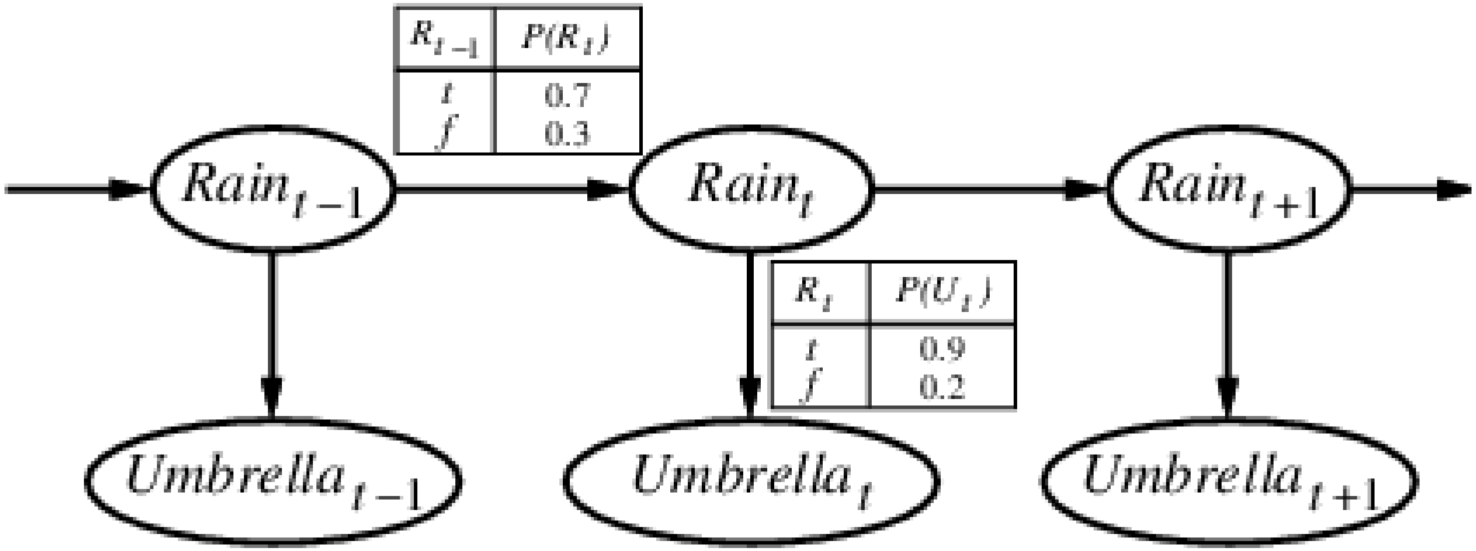
Markov Processes (Markov Chains)

- Construct a Bayes net from these variables: parents?
- Markov assumption: $\mathbf{X}_t$ depends on **bounded** subset of $\mathbf{X}_{0:t-1}$
- First-order Markov process: $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$
  Second-order Markov process: $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = \mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-2}, \mathbf{X}_{t-1})$



- Sensor Markov assumption: $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = \mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$
- Stationary process: transition model $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$ and sensor model $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$ fixed for all t

Example



- First-order Markov assumption not exactly true in real world!

- Possible fixes:
  1. **Increase order** of Markov process
  2. **Augment state**, e.g., add $Temp_t$, $Pressure_t$

inference

Inference Tasks

- Filtering: $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$
  belief state—input to the decision process of a rational agent

- Smoothing: $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$
  better estimate of past states, essential for learning

- Most likely explanation: $\arg\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}|\mathbf{e}_{1:t})$
  speech recognition, decoding with a noisy channel

Filtering

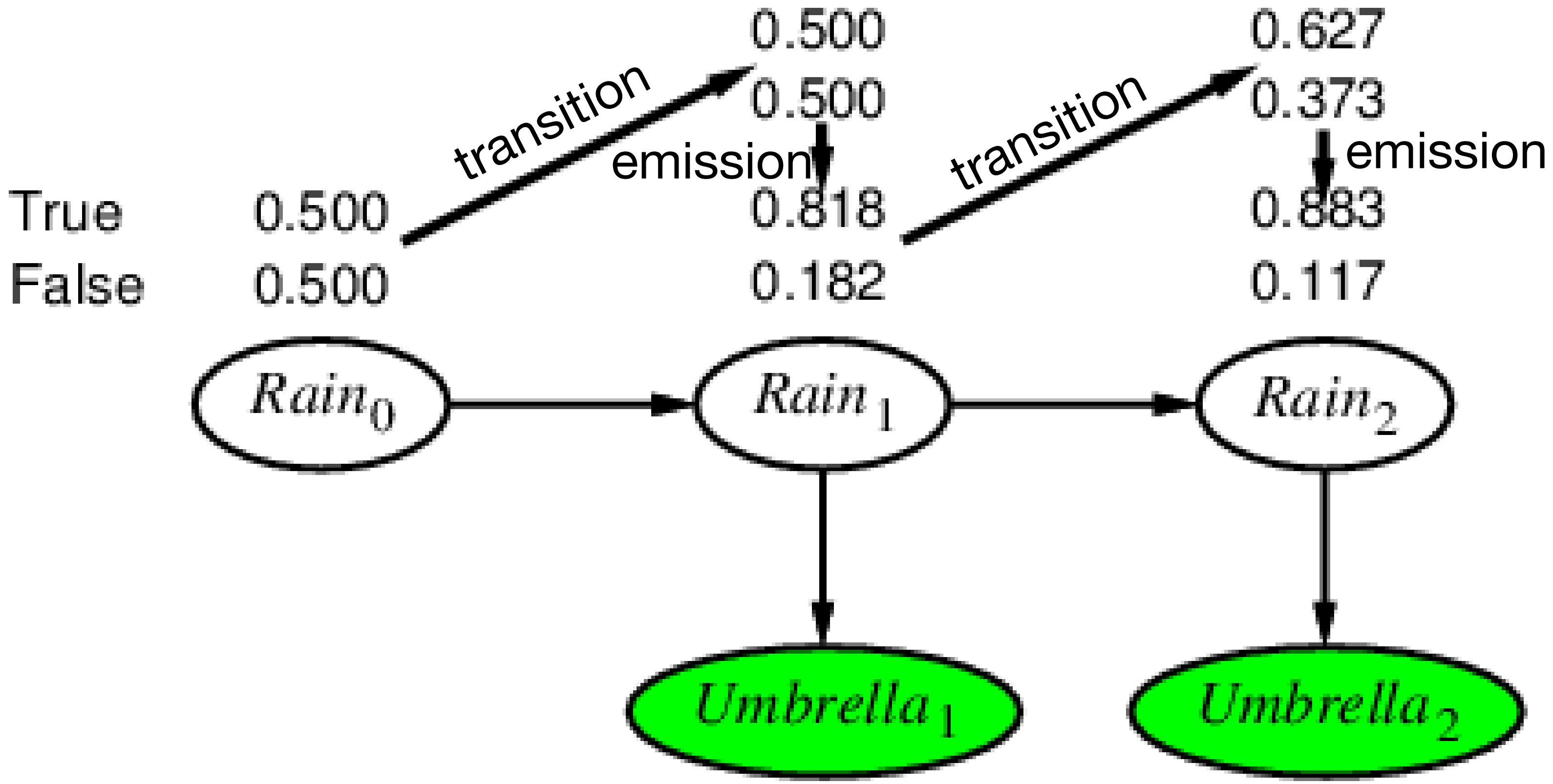
- Aim: devise a **recursive** state estimation algorithm

$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t},\mathbf{e}_{t+1})\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1},\mathbf{e}_{1:t})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \quad \textit{(Bayes rule)}\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \quad \textit{(Sensor Markov assumption)}\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t,\mathbf{e}_{1:t})P(\mathbf{x}_t|\mathbf{e}_{1:t}) \quad \textit{(multiplying out)}\\ &= \alpha\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t}) \quad \textit{(first order Markov model)}\end{aligned}$$

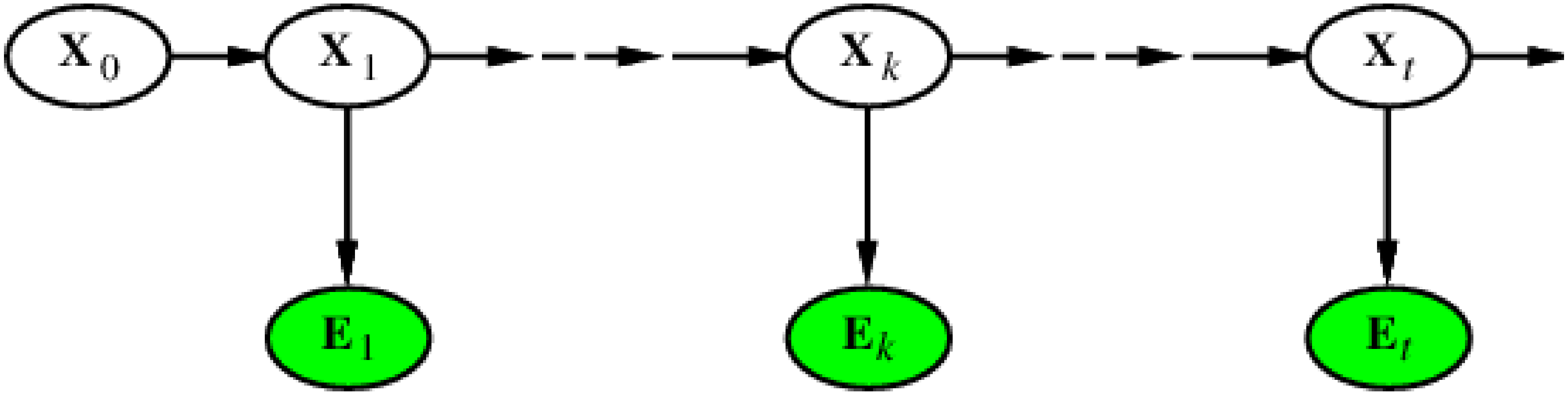
- Summary: $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha \underbrace{\mathbf{P}(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})}_{\text{emission}} \sum_{\mathbf{x}_t} \underbrace{\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{x}_t)}_{\text{transition}} \underbrace{P(\mathbf{x}_t|\mathbf{e}_{1:t})}_{\text{recursive call}}$

- $\mathbf{f}_{1:t+1} = \textsc{Forward}(\mathbf{f}_{1:t},\mathbf{e}_{t+1})$ where $\mathbf{f}_{1:t} = \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t})$
  Time and space **constant** (independent of t)

Filtering Example



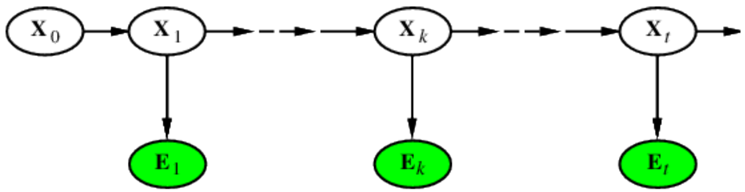
Smoothing



- If full sequence is known
  ⇒ what is the state probability $\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t})$ including future evidence?

- Smoothing: sum over all paths

Smoothing



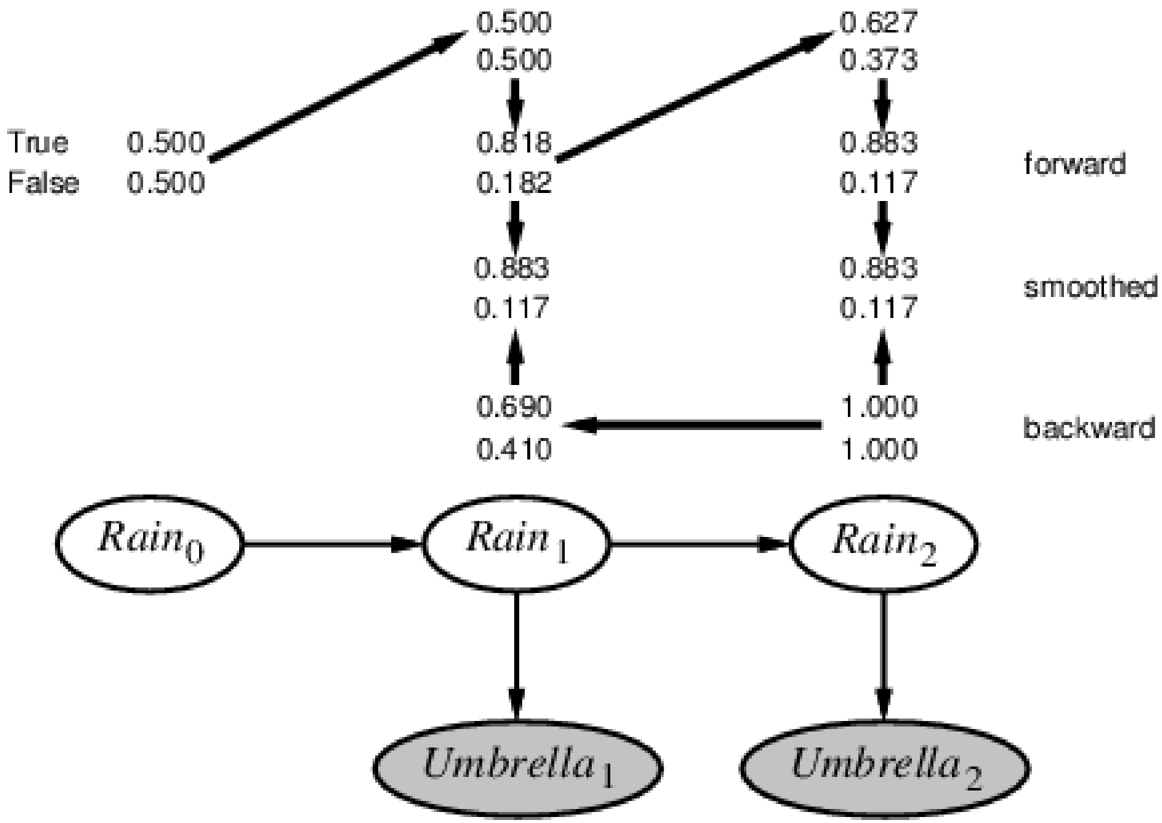
- Divide evidence $\mathbf{e}_{1:t}$ into $\mathbf{e}_{1:k}$, $\mathbf{e}_{k+1:t}$:

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k},\mathbf{e}_{k+1:t}) \\
&= \alpha\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k,\mathbf{e}_{1:k}) \\
&= \alpha\mathbf{P}(\mathbf{X}_k|\mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) \\
&= \alpha\mathbf{f}_{1:k}\mathbf{b}_{k+1:t}
\end{aligned}$$

- Backward message $\mathbf{b}_{k+1:t}$ computed by a backwards recursion

$$\begin{aligned}
\mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t}|\mathbf{X}_k,\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1}|\mathbf{X}_k)
\end{aligned}$$

Smoothing Example



Forward–backward algorithm: cache forward messages along the way
Time linear in t (polytree inference), space $O(t|\mathbf{f}|)$

Most Likely Explanation

- Most likely sequence ≠ sequence of most likely states
- Most likely path to each $\mathbf{x}_{t+1}$
  = most likely path to **some** $\mathbf{x}_t$ plus one more step

$$\max_{\mathbf{x}_1 \ldots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_t, \mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$$
$$= \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} \left(\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \max_{\mathbf{x}_1 \ldots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{x}_t | \mathbf{e}_{1:t}) \right)$$

- Identical to filtering, except $\mathbf{f}_{1:t}$ replaced by

$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \ldots \mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{X}_t | \mathbf{e}_{1:t})$$

  i.e., $\mathbf{m}_{1:t}(i)$ gives the probability of the most likely path to state i.

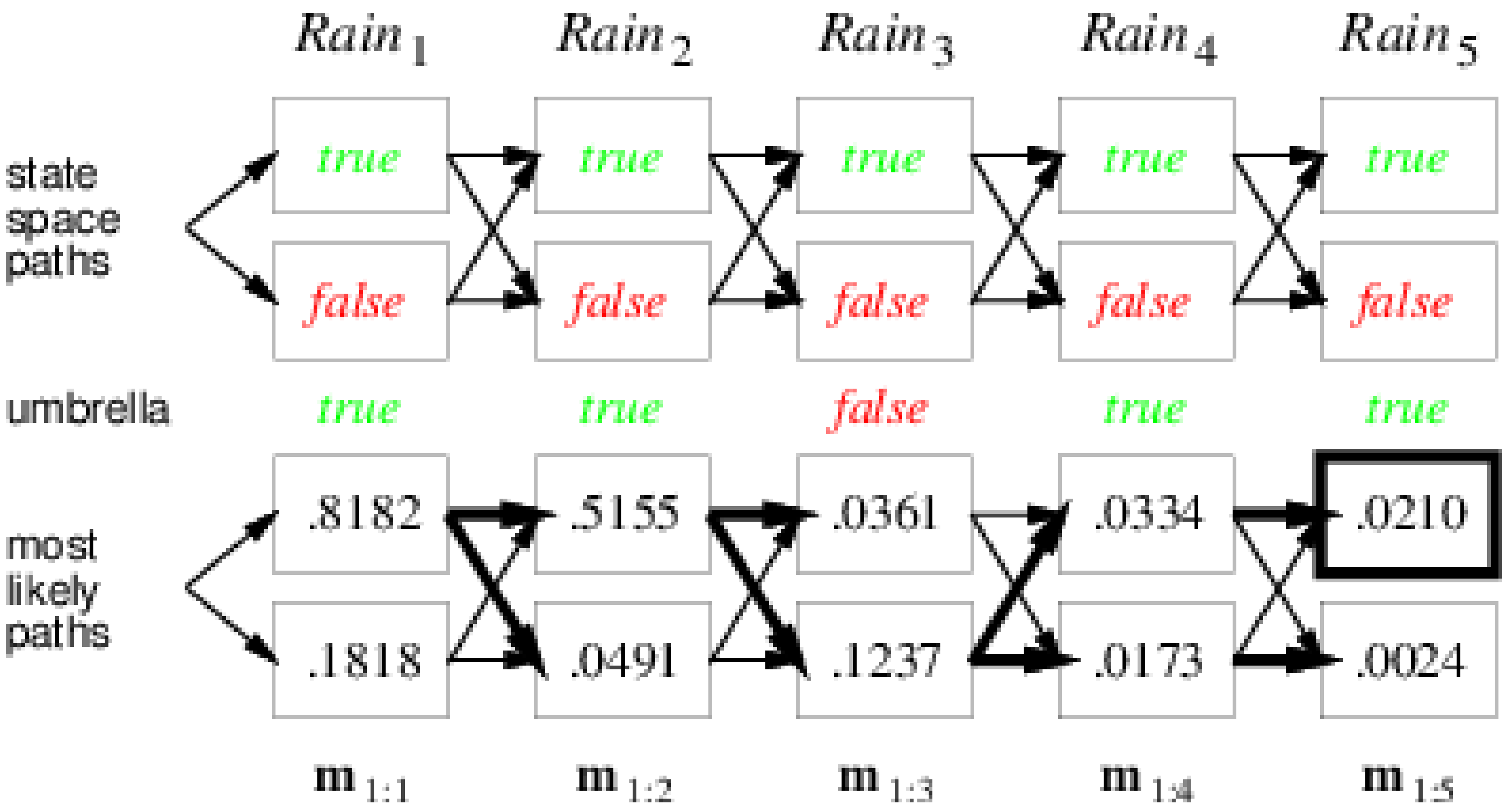
- Update has sum replaced by max, giving the Viterbi algorithm:

$$\mathbf{m}_{1:t+1} = \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} (\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \mathbf{m}_{1:t})$$

  Also requires back-pointers for backward pass to retrieve best sequence

$$\mathbf{b}_{\mathbf{X}_{t+1}, t+1} = \text{argmax}_{\mathbf{x}_t} (\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \mathbf{m}_{1:t})$$

Viterbi Example



Hidden Markov Models

- $\mathbf{X}_t$ is a single, discrete variable (usually $\mathbf{E}_t$ is too)
  Domain of X_t is $\{1, \ldots, S\}$

- Transition matrix $\mathbf{T}_{ij} = P(X_t = j | X_{t-1} = i)$, e.g., $\begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$

- Sensor matrix $\mathbf{O}_t$ for each time step, diagonal elements $P(e_t | X_t = i)$
  e.g., with $U_1 = true$, $\mathbf{O}_1 = \begin{pmatrix} 0.9 & 0.1 \\ 0.8 & 0.2 \end{pmatrix}$

- Forward and backward messages as column vectors:

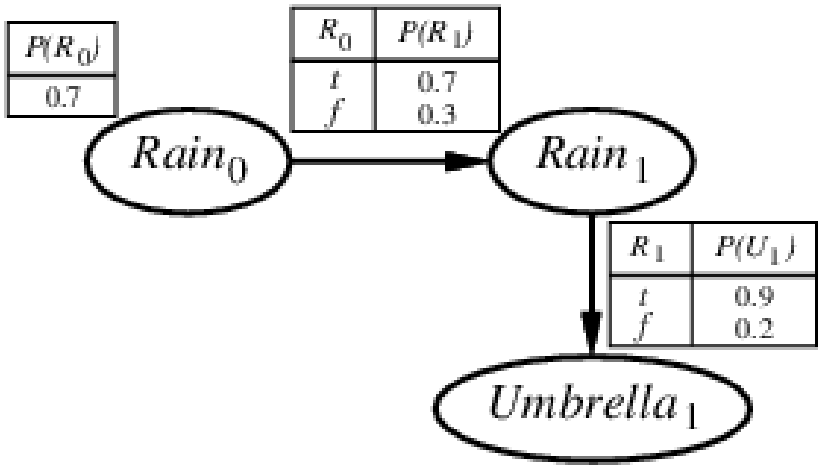
$$\begin{aligned} \mathbf{f}_{1:t+1} &= \alpha \mathbf{O}_{t+1} \mathbf{T}^\top \mathbf{f}_{1:t} \\ \mathbf{b}_{k+1:t} &= \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t} \end{aligned}$$

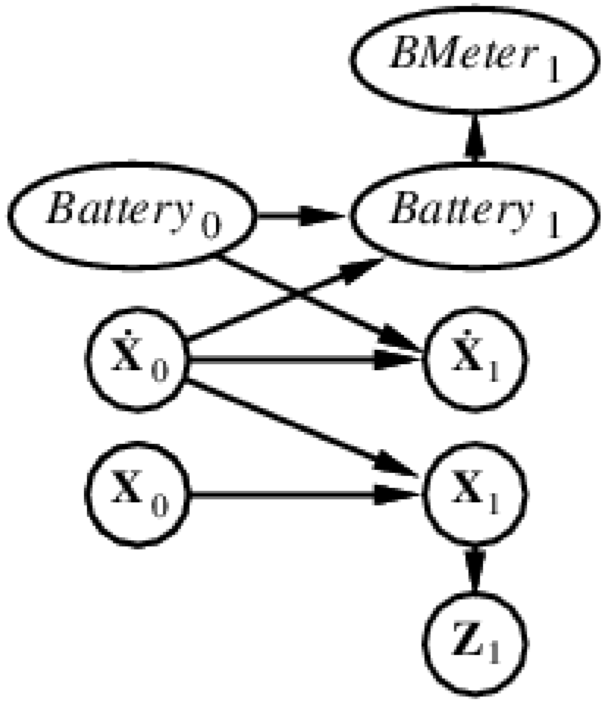
- Forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$

dynamic baysian networks

Dynamic Bayesian Networks

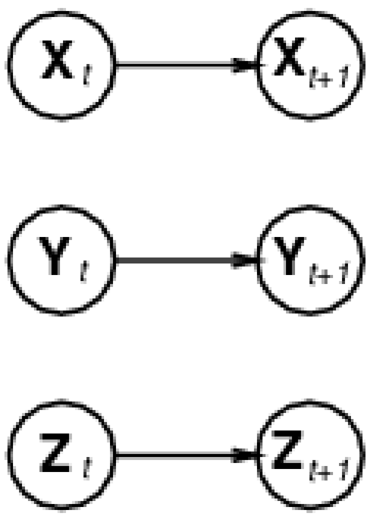
- $\mathbf{X}_t$, $\mathbf{E}_t$ contain arbitrarily many variables in a sequentialized Bayes net

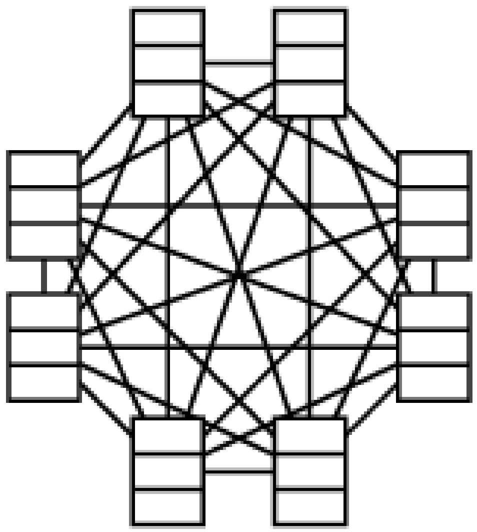




DBNs vs. HMMs

- Every HMM is a single-variable DBN; every discrete DBN is an HMM





- Sparse dependencies ⇒ exponentially fewer parameters;
  e.g., 20 state variables, three parents each
  DBN has $20 \times 2^3 = 160$ parameters, HMM has $2^{20} \times 2^{20} \approx 10^{12}$

speech recognition

Speech as Probabilistic Inference

It's not easy to wreck a nice beach

- Speech signals are noisy, variable, ambiguous
- What is the **most likely** word sequence, given the speech signal?
  I.e., choose $Words$ to maximize $P(Words|signal)$
- Use Bayes' rule:
  $$P(Words|signal) = \alpha P(signal|Words)P(Words)$$
  i.e., decomposes into acoustic model + language model
- $Words$ are the hidden state sequence, $signal$ is the observation sequence

Phones

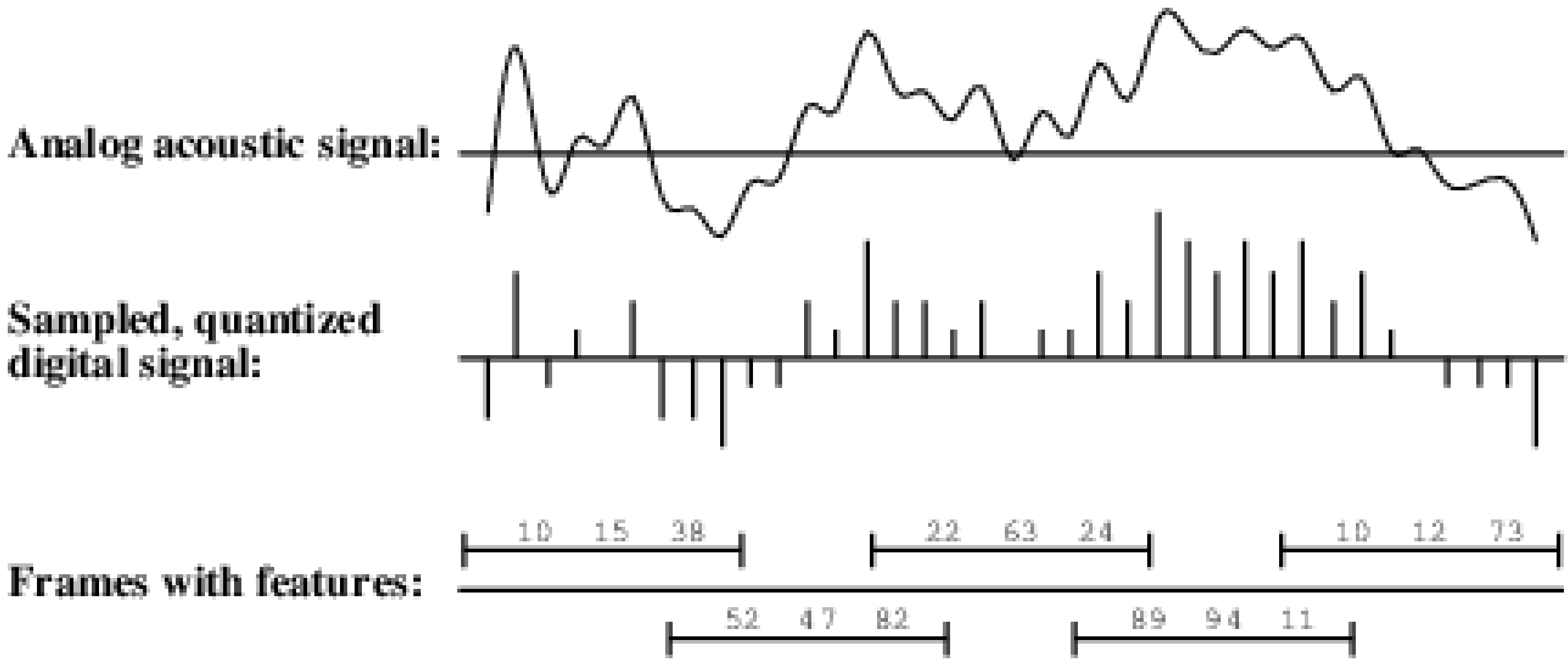
- All human speech is composed from 40-50 phones, determined by the configuration of articulators (lips, teeth, tongue, vocal cords, air flow)
- Form an intermediate level of hidden states between words and signal
  $\Rightarrow$ acoustic model = pronunciation model + phone model
- ARPAbet designed for American English

[iy]	b**ea**t	[b]	**b**et	[p]	**p**et
[ih]	b**i**t	[ch]	**Ch**et	[r]	**r**at
[ey]	b**e**t	[d]	**d**ebt	[s]	**s**et
[ao]	b**ough**t	[hh]	**h**at	[th]	**th**ick
[ow]	b**oa**t	[hv]	**h**igh	[dh]	**th**at
[er]	B**er**t	[l]	**l**et	[w]	**w**et
[ix]	ros**e**s	[ng]	si**ng**	[en]	butt**on**
⋮	⋮	⋮	⋮	⋮	⋮

e.g., "ceiling" is [s iy l ih ng] / [s iy l ix ng] / [s iy l en]

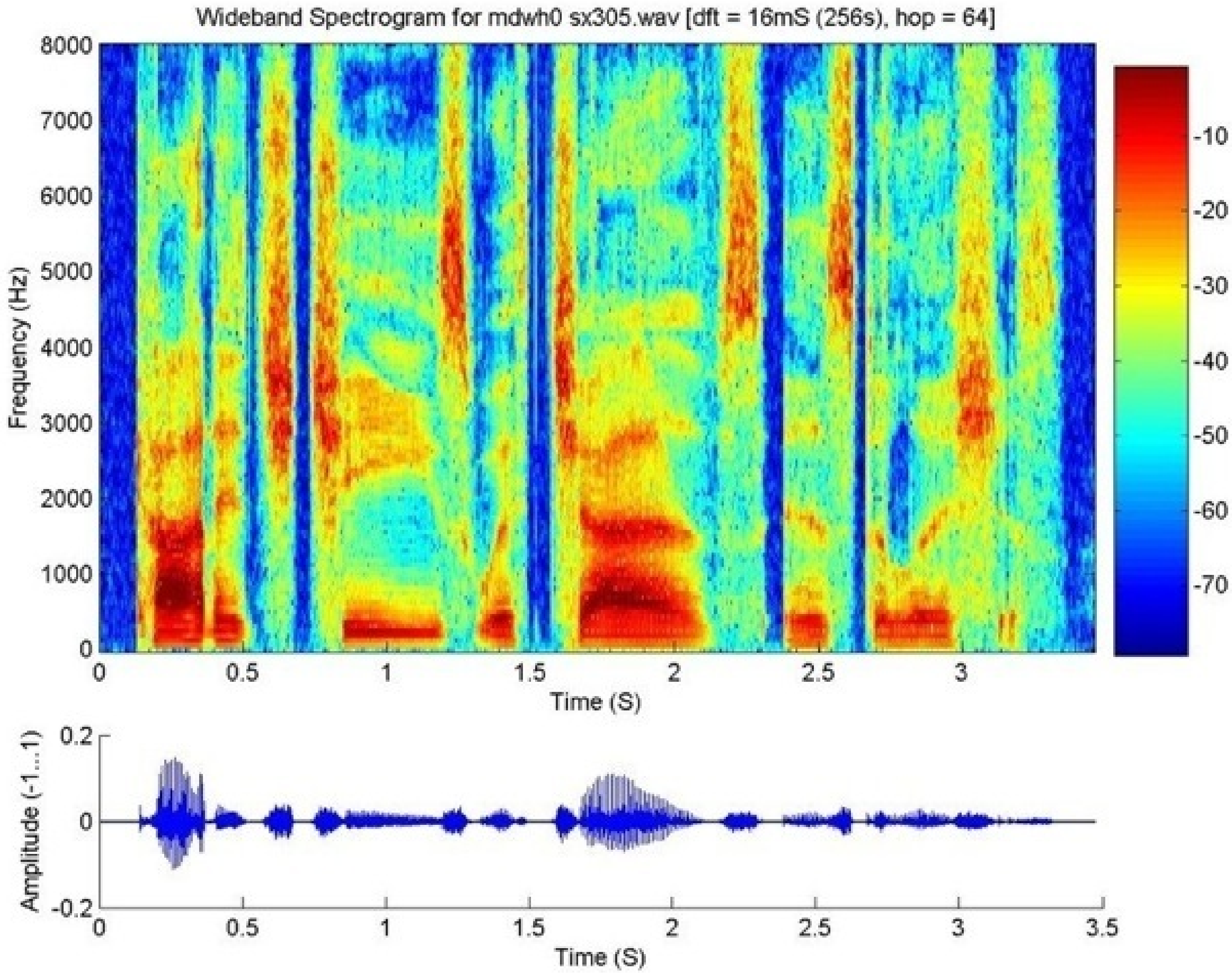
Speech Sounds

- Raw signal is the microphone displacement as a function of time; processed into overlapping 30ms frames, each described by features



- Frame features are typically formants—peaks in the power spectrum

Speech Spectrogram



Phone Models

- Frame features in $P(features|phone)$ summarized by
  – an integer in $[0 \ldots 255]$ (using vector quantization); or
  – the parameters of a mixture of Gaussians

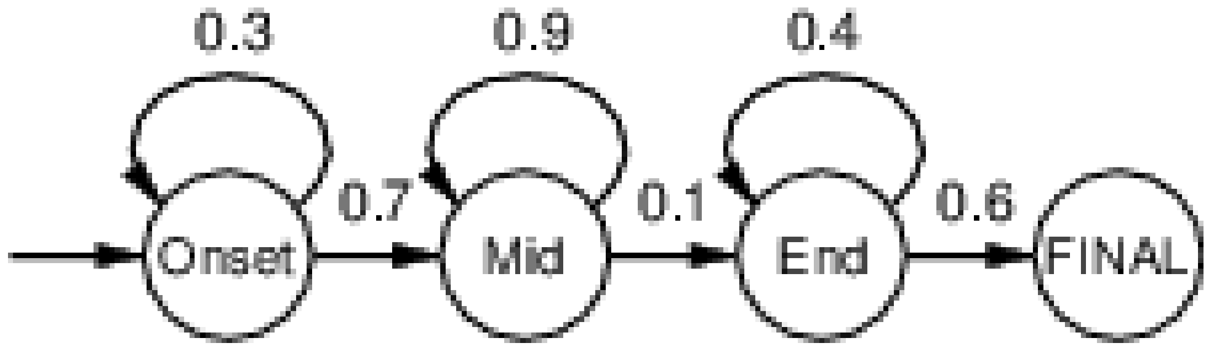
- Three-state phones: each phone has three phases (Onset, Mid, End)
  E.g., [t] has silent Onset, explosive Mid, hissing End
  $\Rightarrow P(features|phone, phase)$

- Triphone context: each phone becomes n^2 distinct phones, depending on the phones to its left and right
  E.g., [t] in "star" is written [t(s,aa)] (different from "tar"!)

- Triphones useful for handling coarticulation effects: the articulators have inertia and cannot switch instantaneously between positions
  E.g., [t] in "eighth" has tongue against front teeth

Phone Model Example

Phone HMM for [m]:

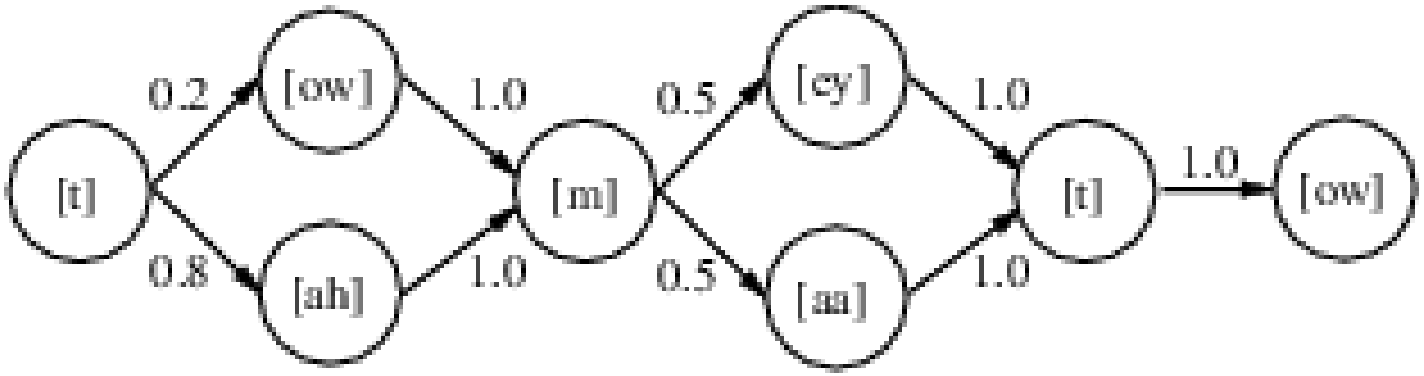


Output probabilities for the phone HMM:

Onset:	Mid:	End:
C1: 0.5	C3: 0.2	C4: 0.1
C2: 0.2	C4: 0.7	C6: 0.5
C3: 0.3	C5: 0.1	C7: 0.4

Word Pronunciation Models

- Each word is described as a distribution over phone sequences
- Distribution represented as an HMM transition model



$$P([towmeytow]|\text{“tomato”}) = P([towmaatow]|\text{“tomato”}) = 0.1$$
$$P([tahmeytow]|\text{“tomato”}) = P([tahmaatow]|\text{“tomato”}) = 0.4$$

- Structure is created manually, transition probabilities learned from data

Recognition of Isolated Words

- Phone models + word models fix likelihood $P(e_{1:t}|word)$ for isolated word

$$P(word|e_{1:t}) = \alpha P(e_{1:t}|word)P(word)$$

- Prior probability $P(word)$ obtained simply by counting word frequencies

  $P(e_{1:t}|word)$ can be computed recursively: define

$$\ell_{1:t} = \mathbf{P}(\mathbf{X}_t, \mathbf{e}_{1:t})$$

  and use the recursive update

$$\ell_{1:t+1} = \text{FORWARD}(\ell_{1:t}, \mathbf{e}_{t+1})$$

  and then $P(e_{1:t}|word) = \sum_{\mathbf{x}_t} \ell_{1:t}(\mathbf{x}_t)$

- Isolated-word dictation systems with training reach 95–99% accuracy

Continuous Speech

- Not just a sequence of isolated-word recognition problems!
  - adjacent words highly correlated
  - sequence of most likely words ≠ most likely sequence of words
  - segmentation: there are few gaps in speech
  - cross-word coarticulation—e.g., “next thing”

- Complications
  - mismatch between speaker in training and test
  - noise
  - crosstalk
  - bad microphone position

- Continuous speech systems manage over 90% accuracy on a good day

Language Model

- Prior probability of a word sequence is given by chain rule:

$$P(w_1 \cdots w_n) = \prod_{i=1}^{n} P(w_i | w_1 \cdots w_{i-1})$$

- Bigram model:

$$P(w_i | w_1 \cdots w_{i-1}) \approx P(w_i | w_{i-1})$$

- Train by counting all word pairs in a large text corpus

- More sophisticated models (trigrams, grammars, etc.) help a little bit

Combined HMM

- States of the combined language+word+phone model are labelled by
  the word we're in + the phone in that word + the phone state in that phone

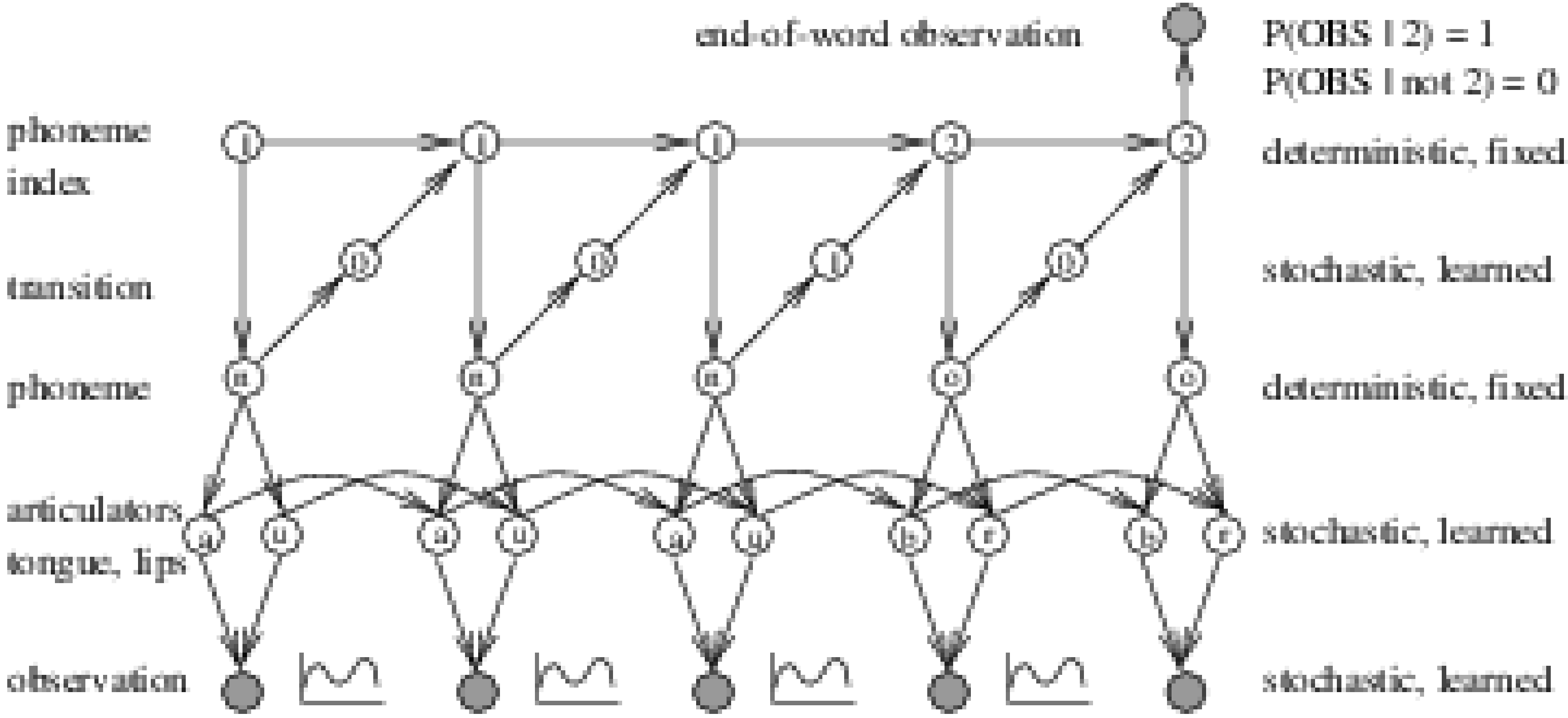
- Viterbi algorithm finds the most likely **phone state** sequence

- Does segmentation by considering all possible word sequences and boundaries

- Doesn't always give the most likely word sequence because
  each word sequence is the sum over many state sequences

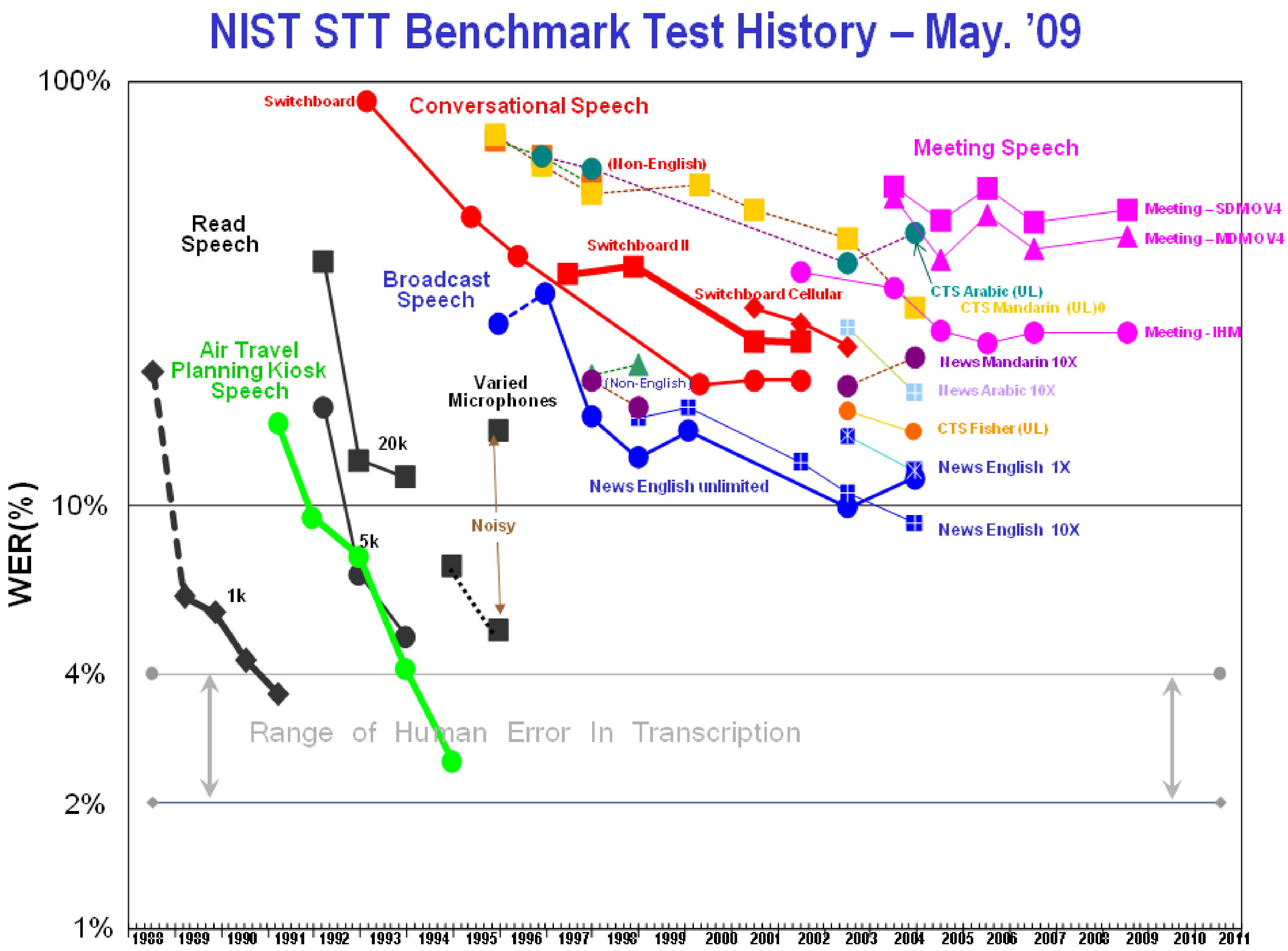
- Jelinek invented A* in 1969 a way to find most likely word sequence
  where "step cost" is $-\log P(w_i|w_{i-1})$

DBNs for Speech Recognition



- Also easy to add variables for, e.g., gender, accent, speed
- Zweig and Russell (1998) show up to 40% error reduction over HMMs

Progress


NIST STT Benchmark Test History – May. '09
100%
10%
4%
2%
1%
WER(%)
Switchboard
Conversational Speech
(Non-English)
Meeting Speech
Meeting – SDM OV4
Meeting – MDM OV4
Read Speech
Switchboard II
Broadcast Speech
Switchboard Cellular
CTS Arabic (UL)
CTS Mandarin (UL)
Meeting - IHM
Air Travel Planning Kiosk Speech
Varied Microphones
(Non-English)
News Mandarin 10X
News Arabic 10X
CTS Fisher (UL)
20k
News English 1X
News English unlimited
Noisy
5k
News English 10X
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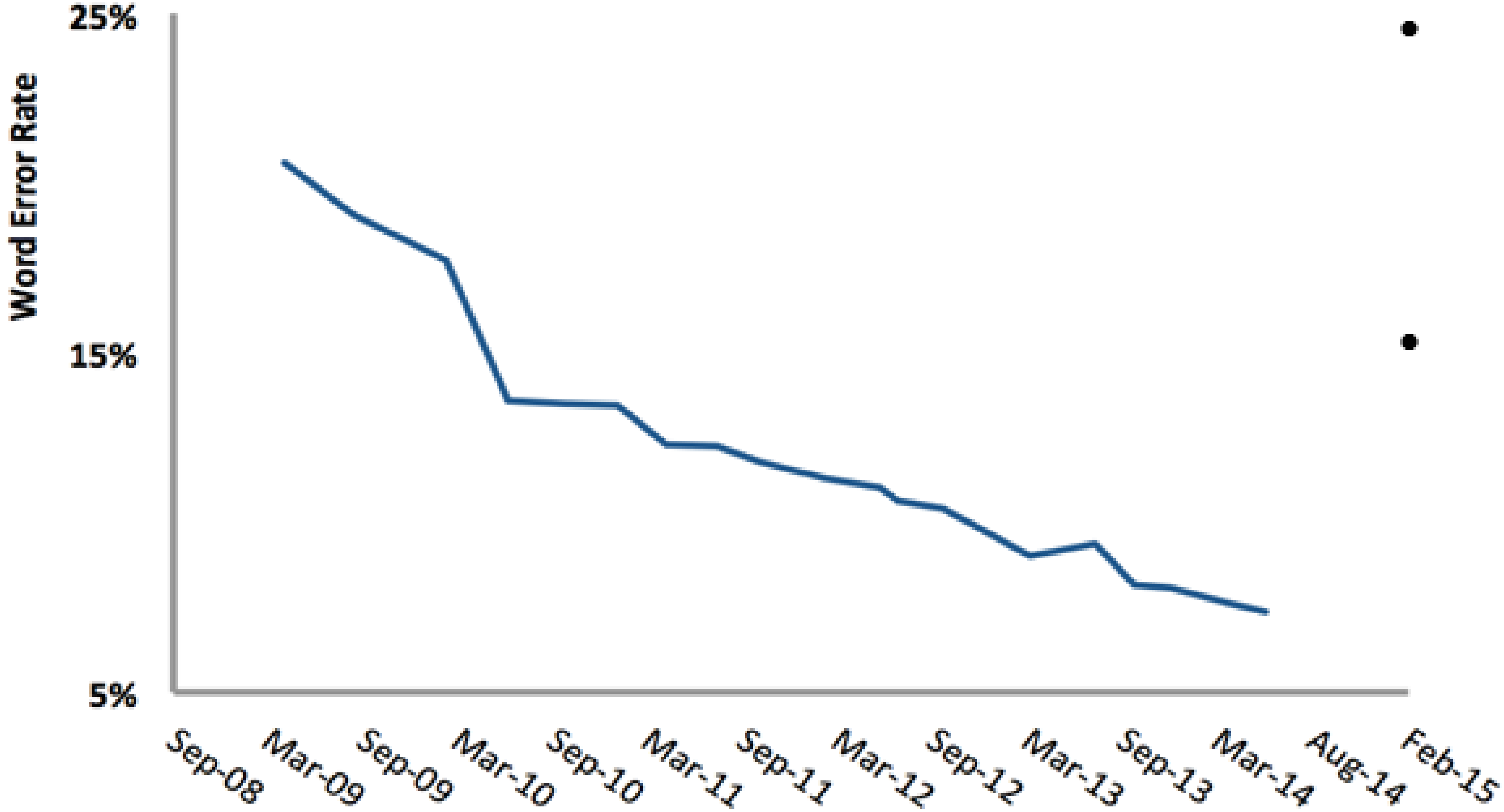
Range of Human Error In Transcription
1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011

Progress


25%
Word Error Rate
15%
5%
Sep-08
Mar-09
Sep-09
Mar-10
Sep-10
Mar-11
Sep-11
Mar-12
Sep-12
Mar-13
Sep-13
Mar-14
Aug-14
Feb-15

Summary

- Temporal models use state and sensor variables replicated over time
- Markov assumptions and stationarity assumption, so we need
  - transition model $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$
  - sensor model $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$
- Tasks are filtering, smoothing, most likely sequence;
  all done recursively with constant cost per time step
- Hidden Markov models have a single discrete state variable; used for speech recognition
- Dynamic Bayes nets subsume HMMs
- Speech recognition