Informed Search

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Heuristic

From Wikipedia:

any approach to problem solving, learning, or discovery
that employs a practical method
not guaranteed to be optimal or perfect
but sufficient for the immediate goals

Outline

- Best-first search
- A^* search
- Heuristic algorithms
  - hill-climbing
  - simulated annealing
  - genetic algorithms (briefly)
  - local search in continuous spaces (very briefly)

best-first search

Review: Tree Search

```
function TREE-SEARCH( problem, fringe) returns a solution, or failure
   fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
   loop do
      if fringe is empty then return failure
      node ← REMOVE-FRONT(fringe)
      if GOAL-TEST[problem] applied to STATE(node) succeeds return node
      fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

- Search space is in form of a tree
- Strategy is defined by picking the **order of node expansion**

Best-First Search

- Idea: use an evaluation function for each node
  - estimate of "desirability"

⇒ Expand most desirable unexpanded node

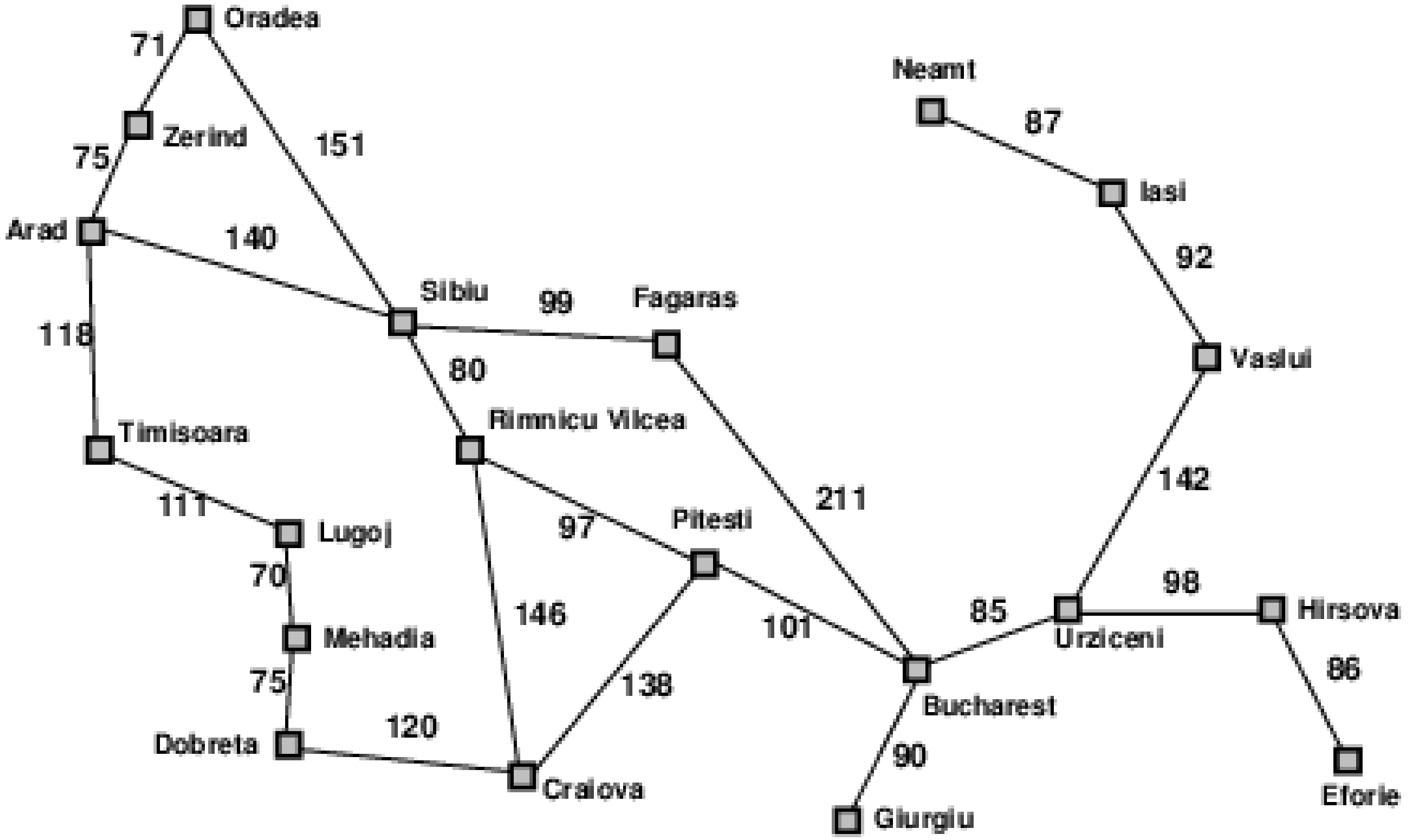
- Implementation:
  fringe is a queue sorted in decreasing order of desirability

- Special cases
  - greedy search
  - A^* search

Romania

Legend:
International border
Motorway
Motorway under construction
Main road
Secondary road
Railroad
Airport
Port
Cities and towns (nr. of inhabitants):
> 1.000.000
300.000 - 1.000.000
100.000 - 300.000
50.000 - 100.000
20.000 - 50.000
< 20.000
BUCHAREST National capital
Major urban concentration (over 500.000 inhabitants)
Scale of altitudes and depths:
3000 m
2500 m
2000 m
1500 m
1000 m
500 m
250 m
100 m
50 m
0 m
-500 m
-1000 m
-1500 m
-2000 m
Scale: 1:1.800.000
0 (km) 100
0 (mi) 60
STEREO 70 projection, S-42 (Romania) datum
UKRAINE
HUNGARY
SERBIA
BULGARIA
MOLDOVA
BLACK SEA
BUCHAREST
CLUJ-NAPOCA
TIMIŞOARA
IAŞI
CONSTANŢA
CRAIOVA
BRAŞOV
GALAŢI
PLOIEŞTI
ORADEA
BRĂILA
ARAD
PITEŞTI
SIBIU
BACĂU
TÂRGU MUREŞ
BAIA MARE
BUZĂU
BOTOŞANI
SATU MARE
CHIŞINĂU
BELGRADE
DEBRECEN
SZEGED
RUSE
TRANSYLVANIAN PLATEAU
WALLACHIAN PLAIN
DOBRUJAN PLATEAU
DANUBE DELTA

Romania with Step Costs in km



Straight–line distance to Bucharest	
Arad	366
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Craiova	160
Dobreta	242
Eforie	161
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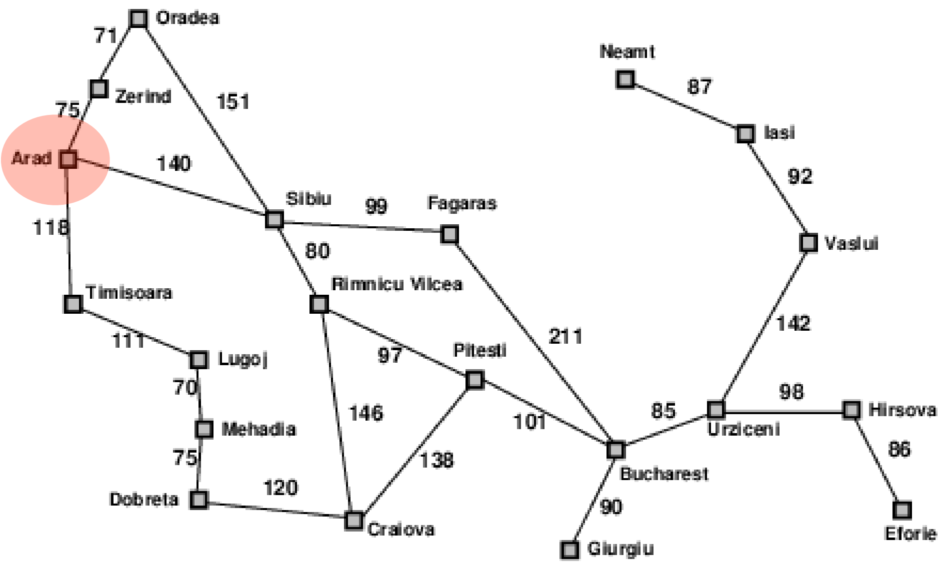
Greedy Search

- State evaluation function $h(n)$ (**heuristic**)
  = estimate of cost from n to the closest goal

- E.g., $h_{\mathrm{SLD}}(n)$ = straight-line distance from n to Bucharest

- Greedy search expands the node that **appears** to be closest to goal

Greedy Search Example



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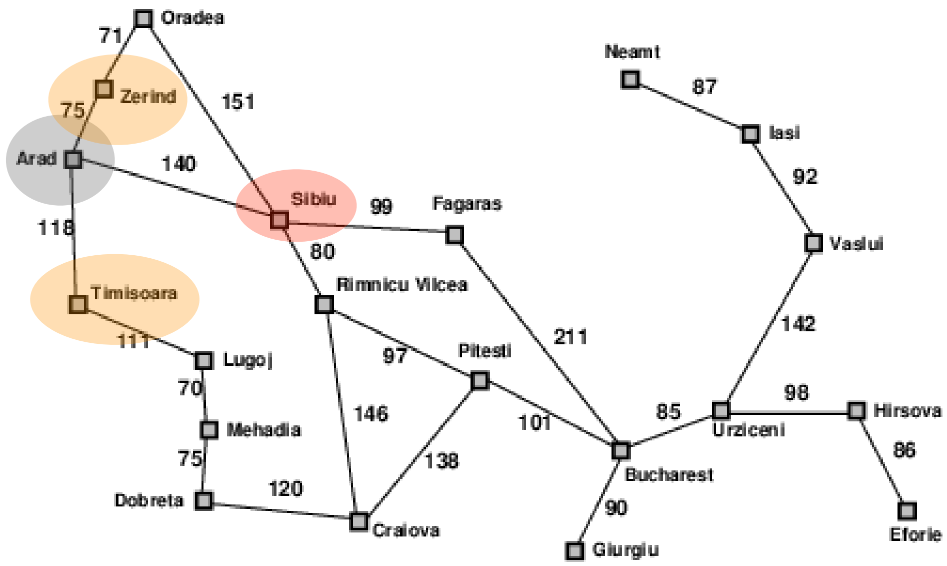

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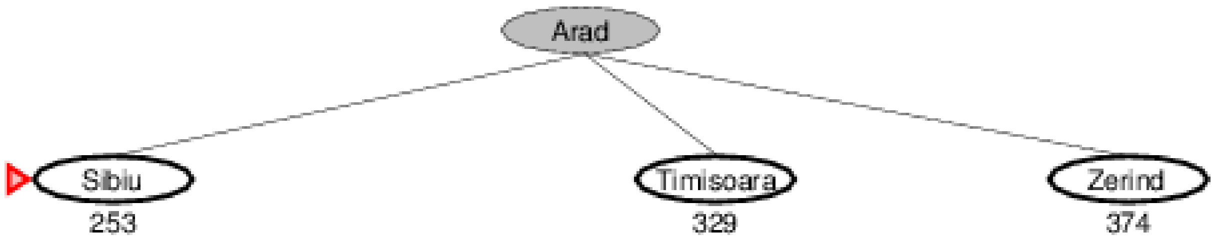

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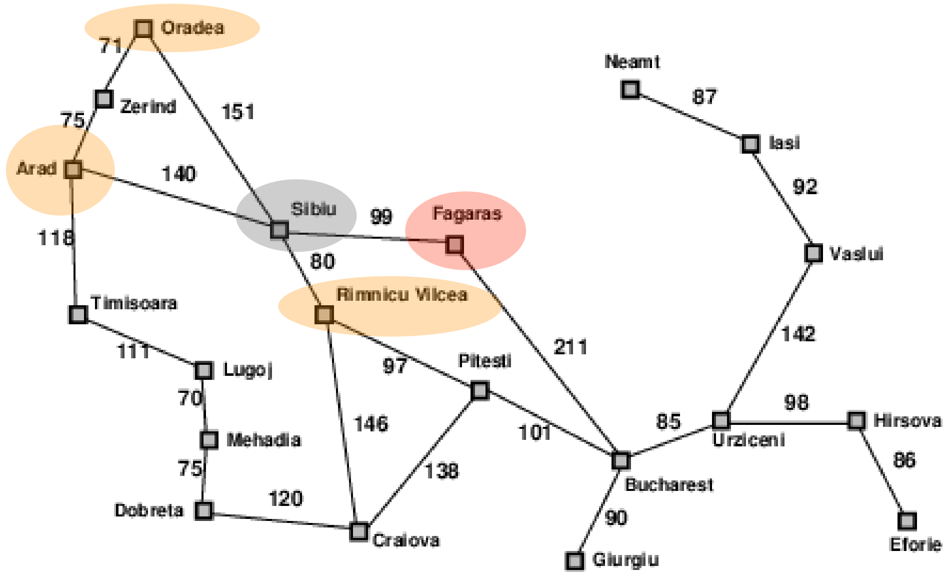

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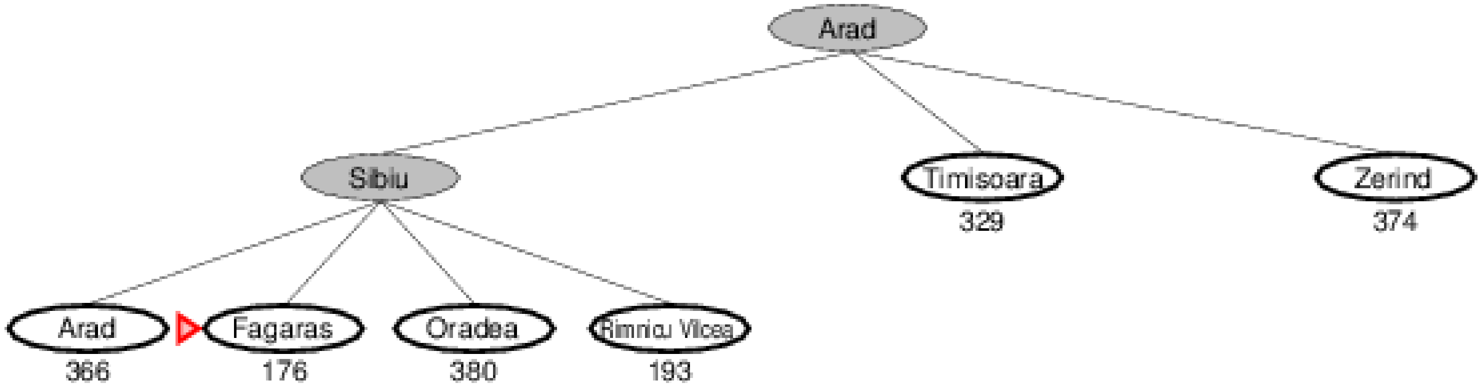

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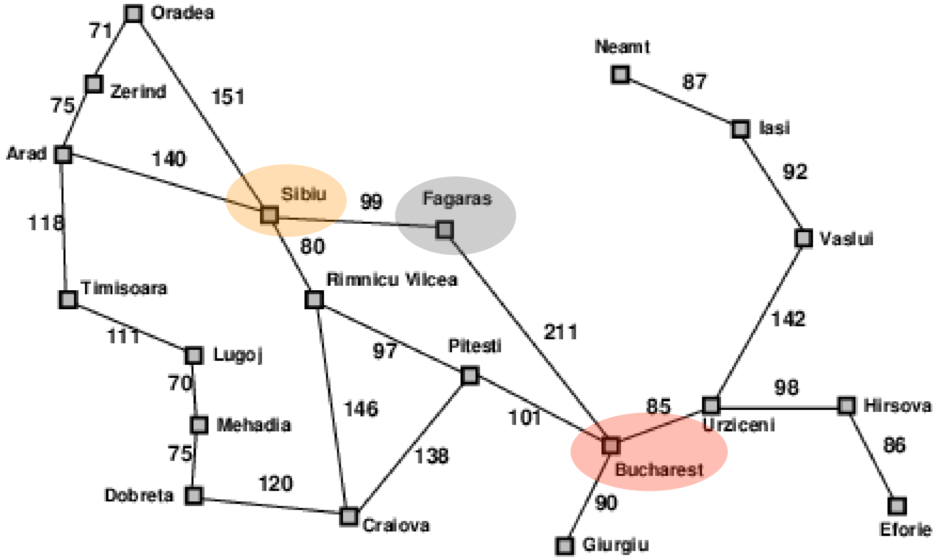

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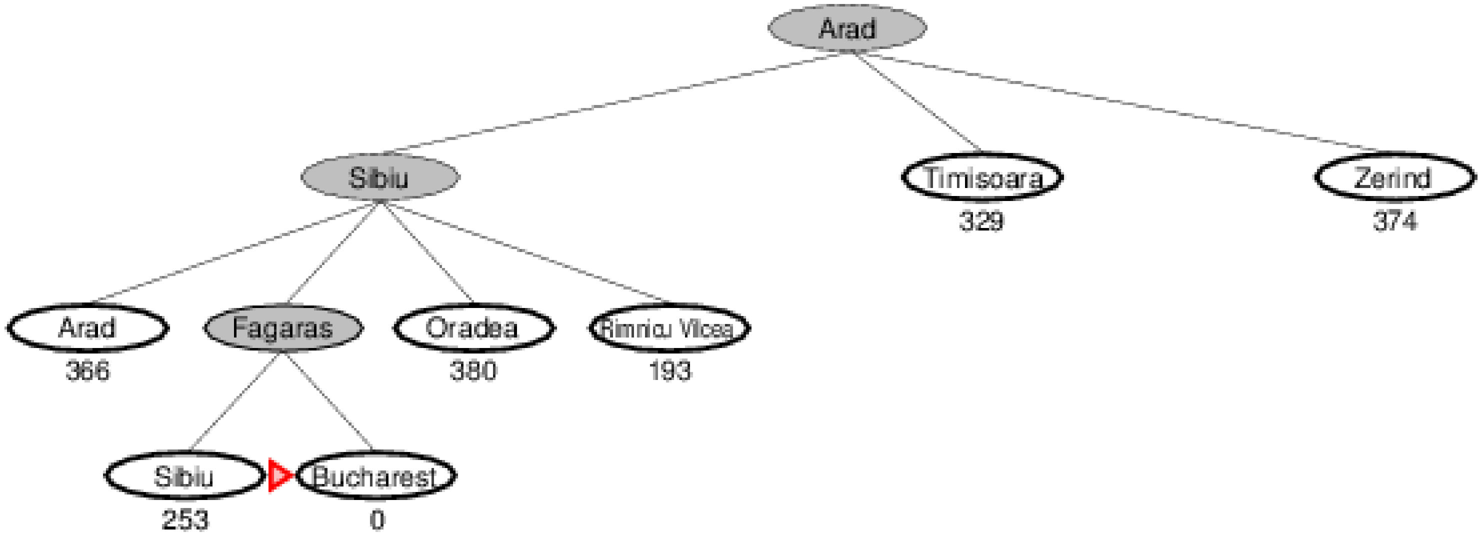
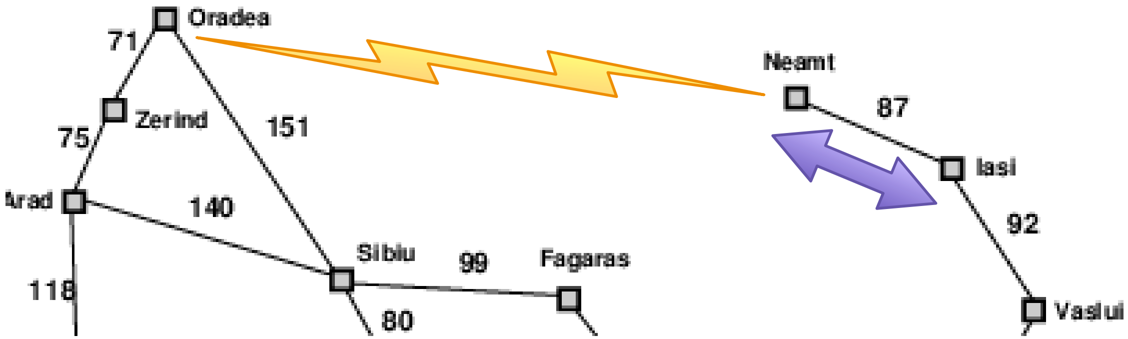


Properties of Greedy Search

- Complete? No, can get stuck in loops, e.g., with Oradea as goal,
  Iasi → Neamt → Iasi → Neamt →



  Complete in finite space with repeated-state checking

- Time? $O(b^m)$, but a good heuristic can give dramatic improvement

- Space? $O(b^m)$—keeps all nodes in memory

- Optimal? No

a* search

A* Search

- Idea: avoid expanding paths that are already expensive

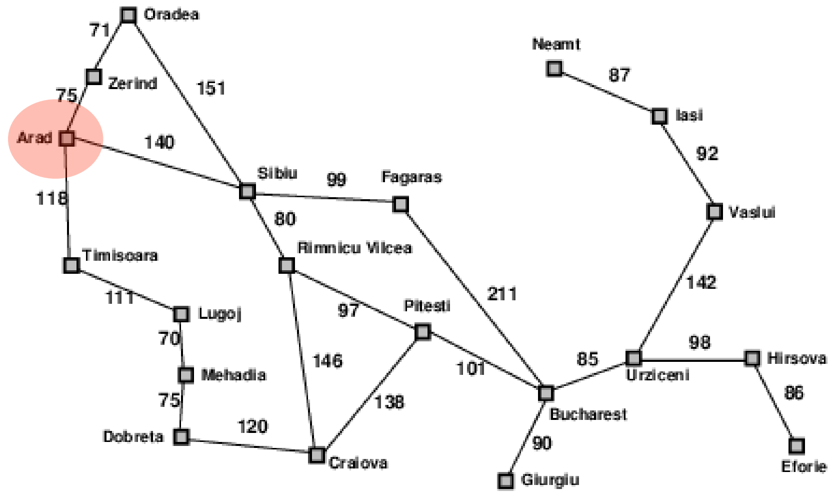
- State evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach n
  - $h(n)$ = estimated cost to goal from n
  - $f(n)$ = estimated total cost of path through n to goal

- A* search uses an admissible heuristic
  - i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the **true** cost from n
  - also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G

- E.g., $h_{\text{SLD}}(n)$ never overestimates the actual road distance

- Theorem: A* search is optimal

A* Search Example



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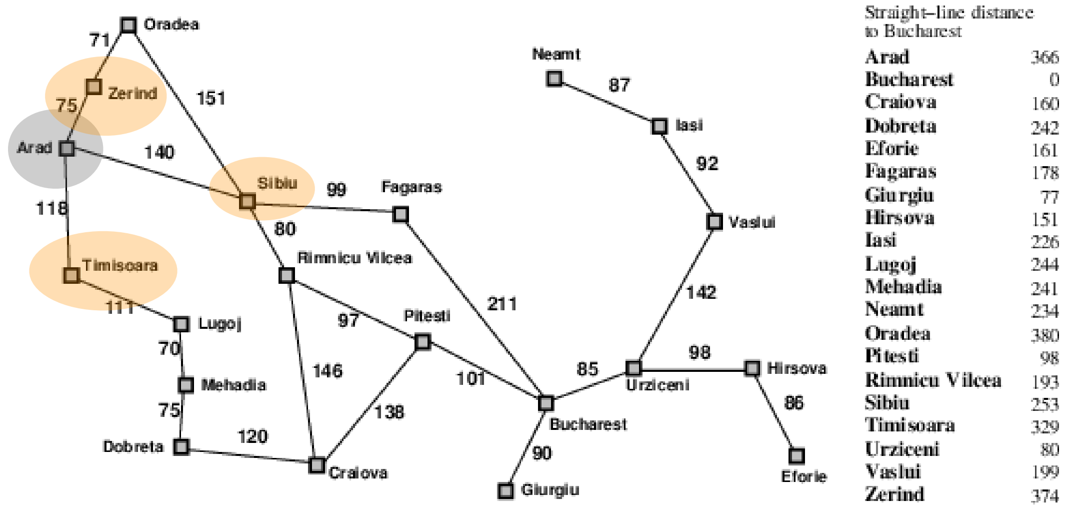

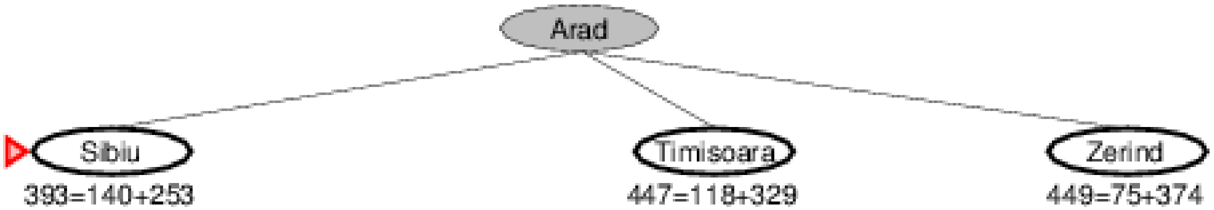

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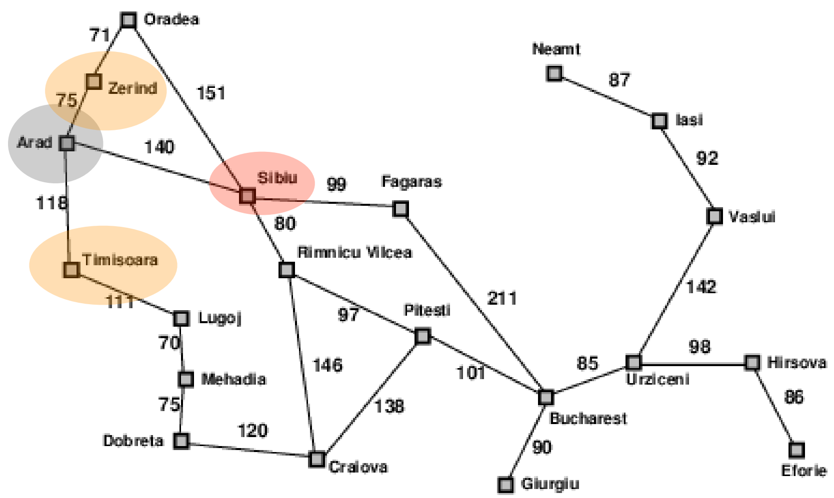

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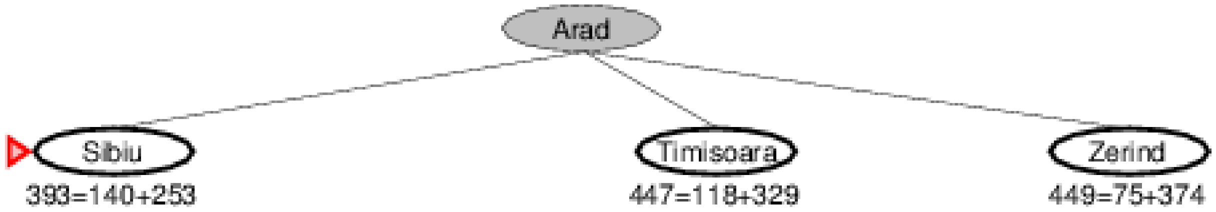

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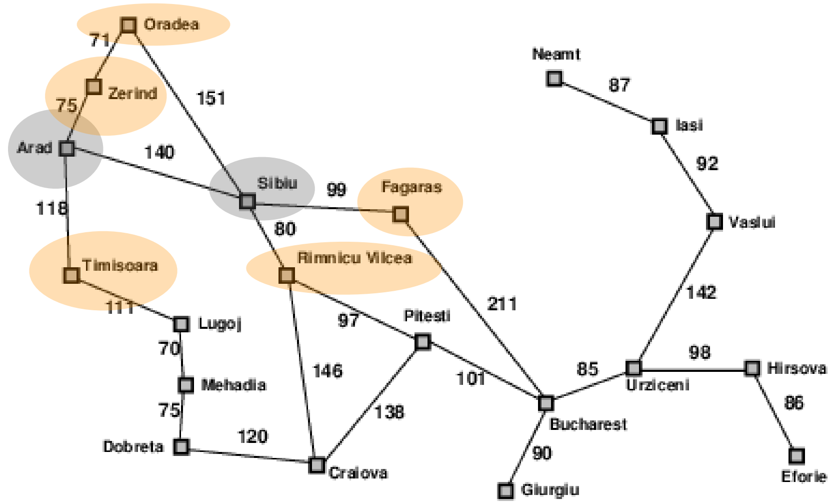

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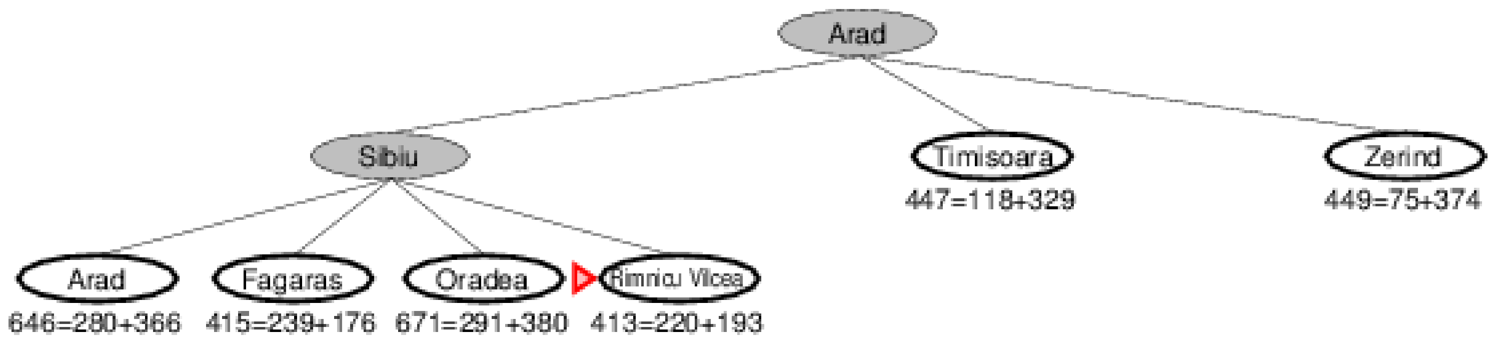

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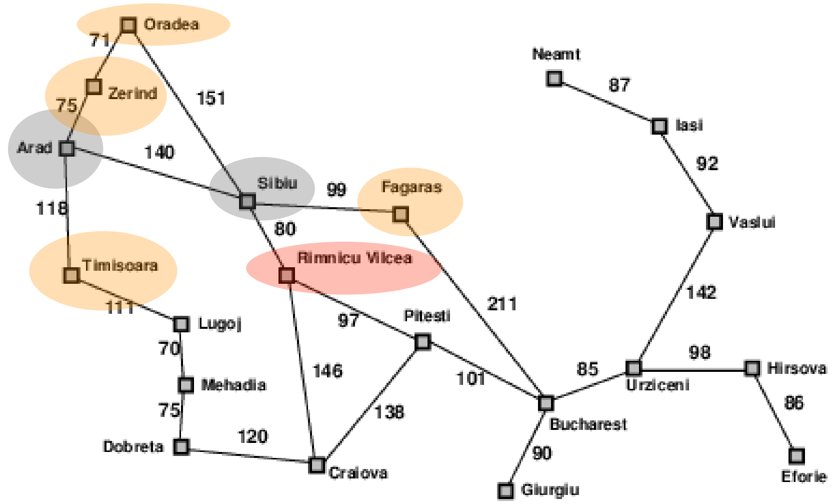

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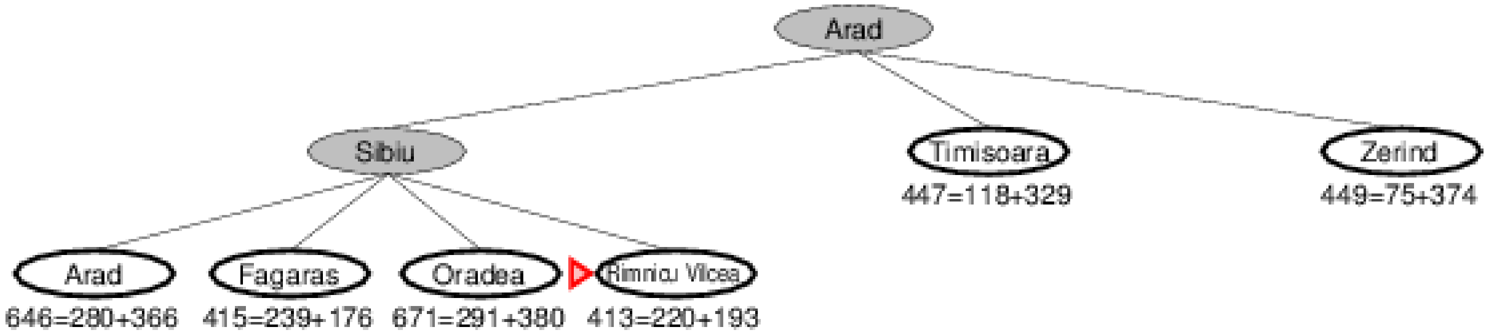

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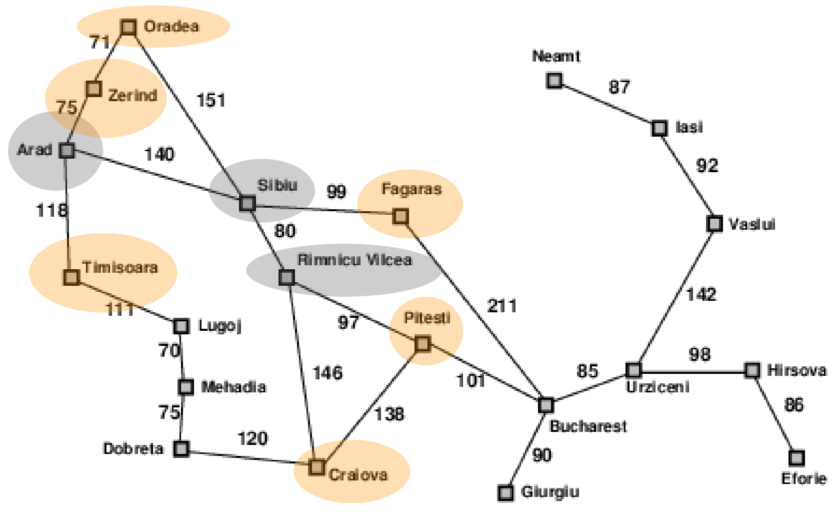

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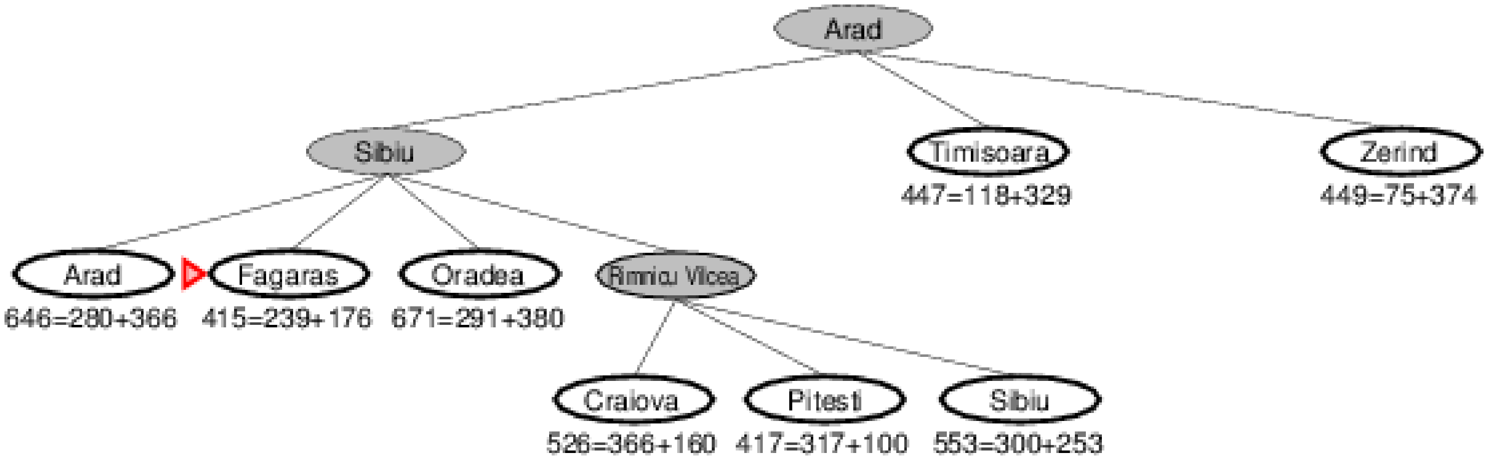

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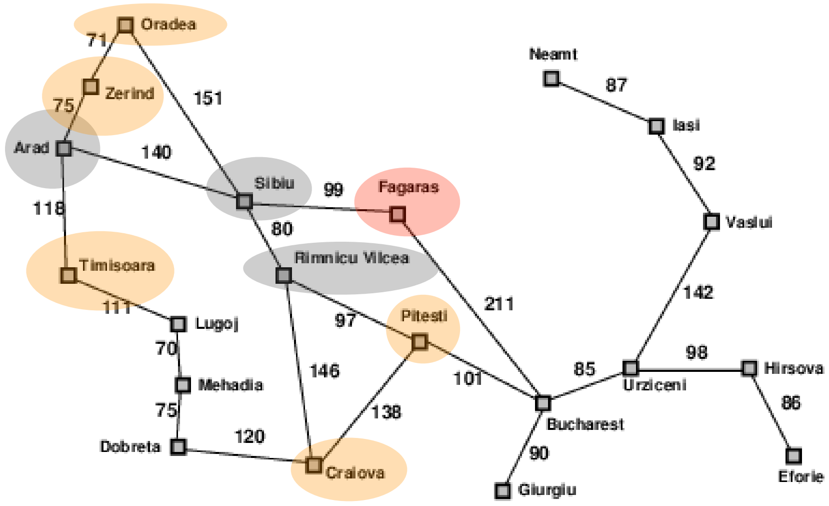

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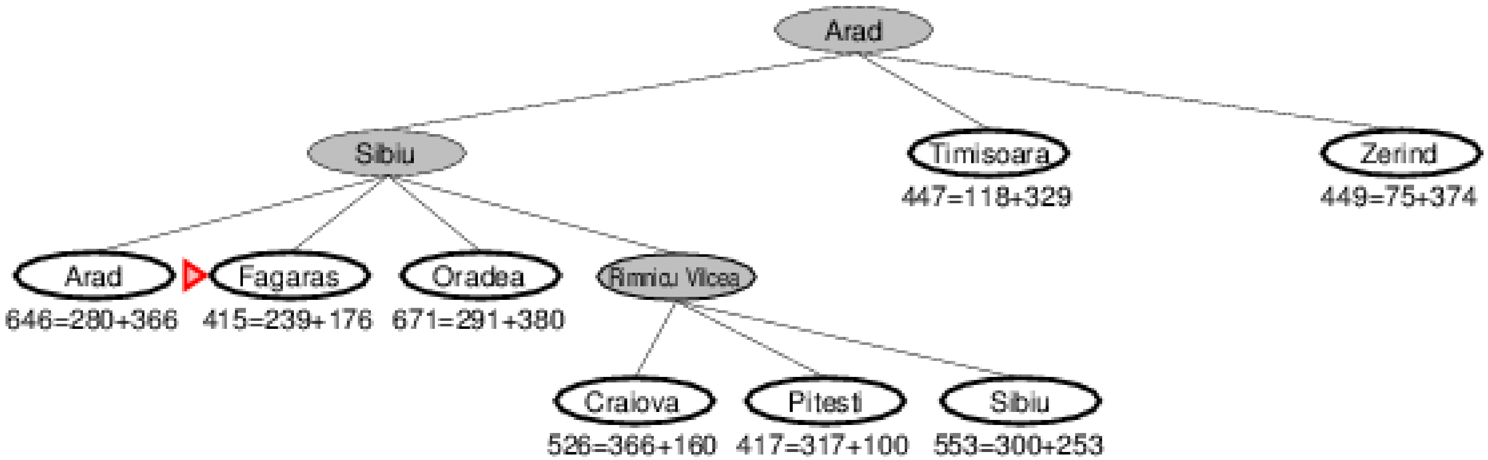

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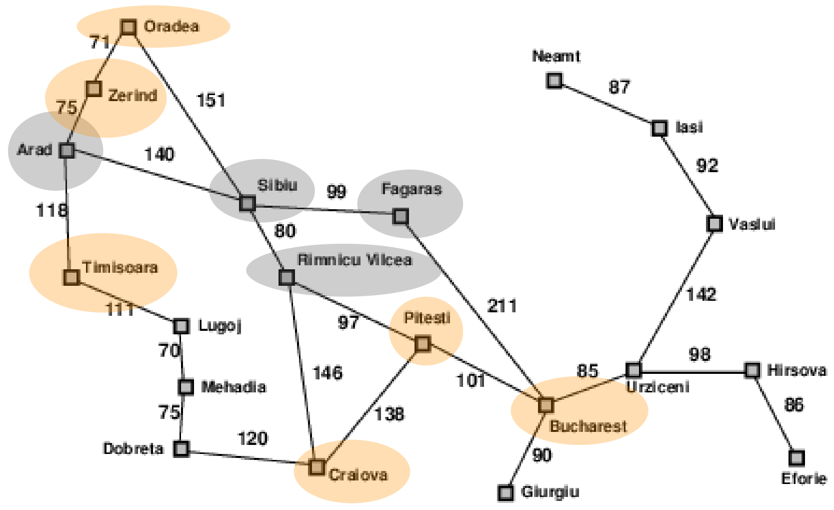

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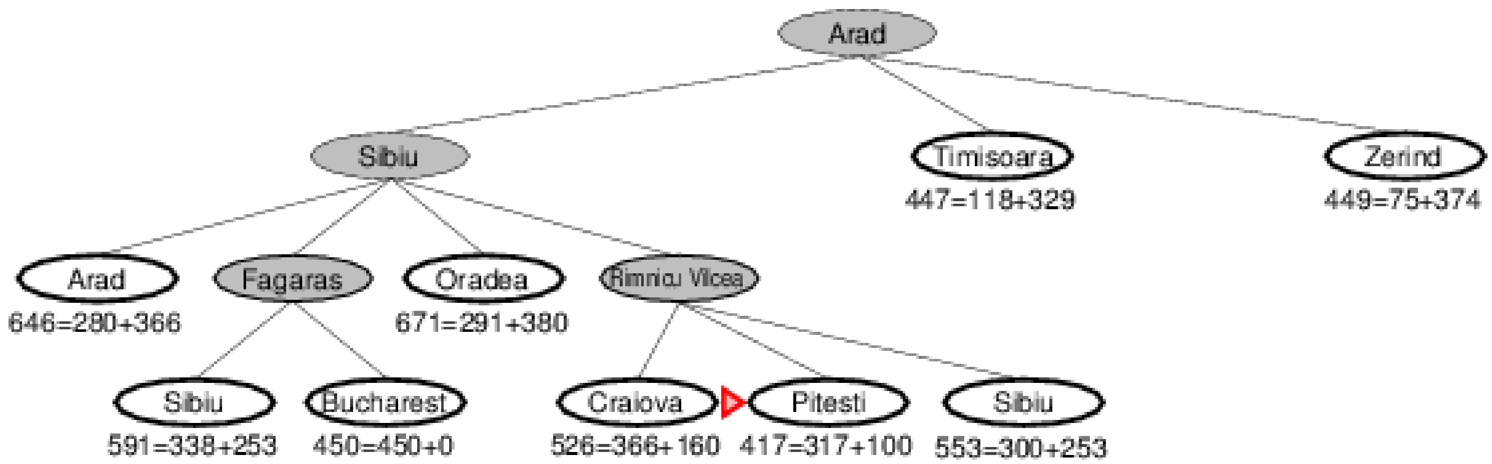

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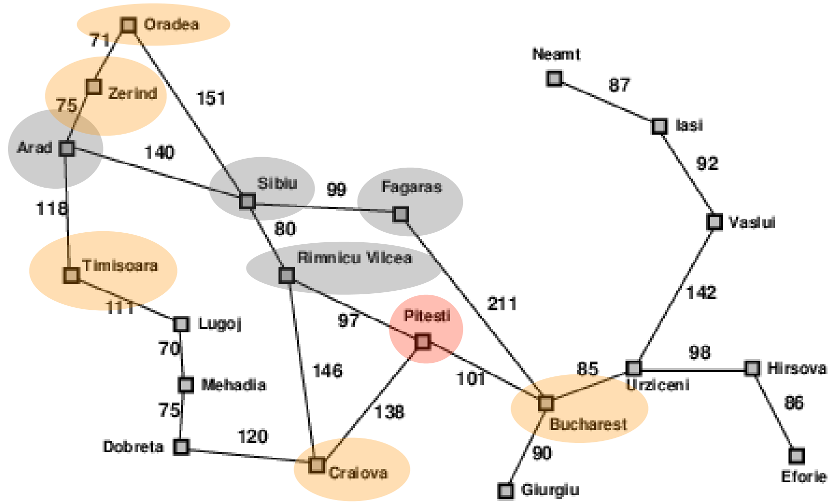

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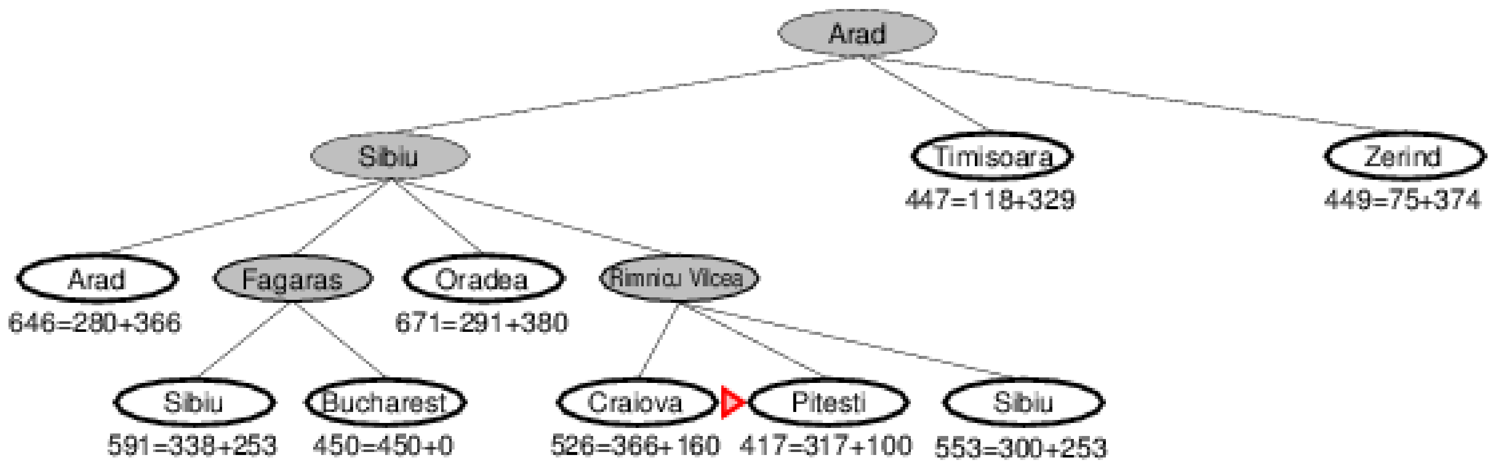

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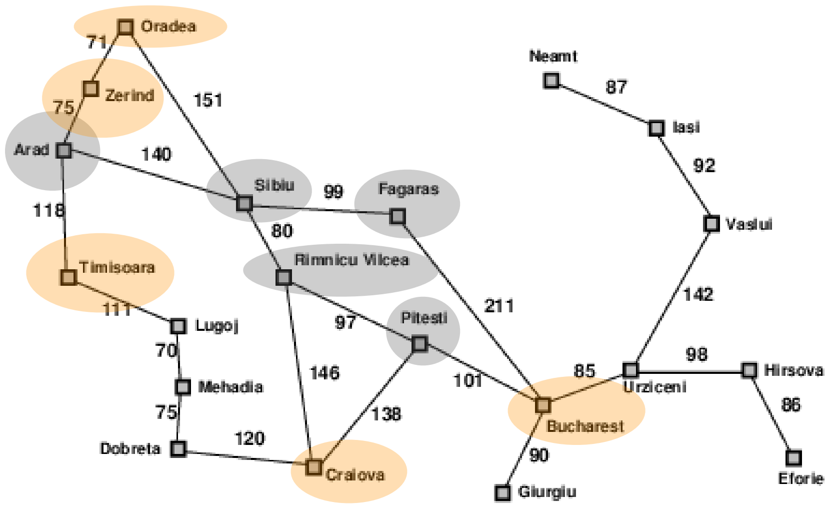

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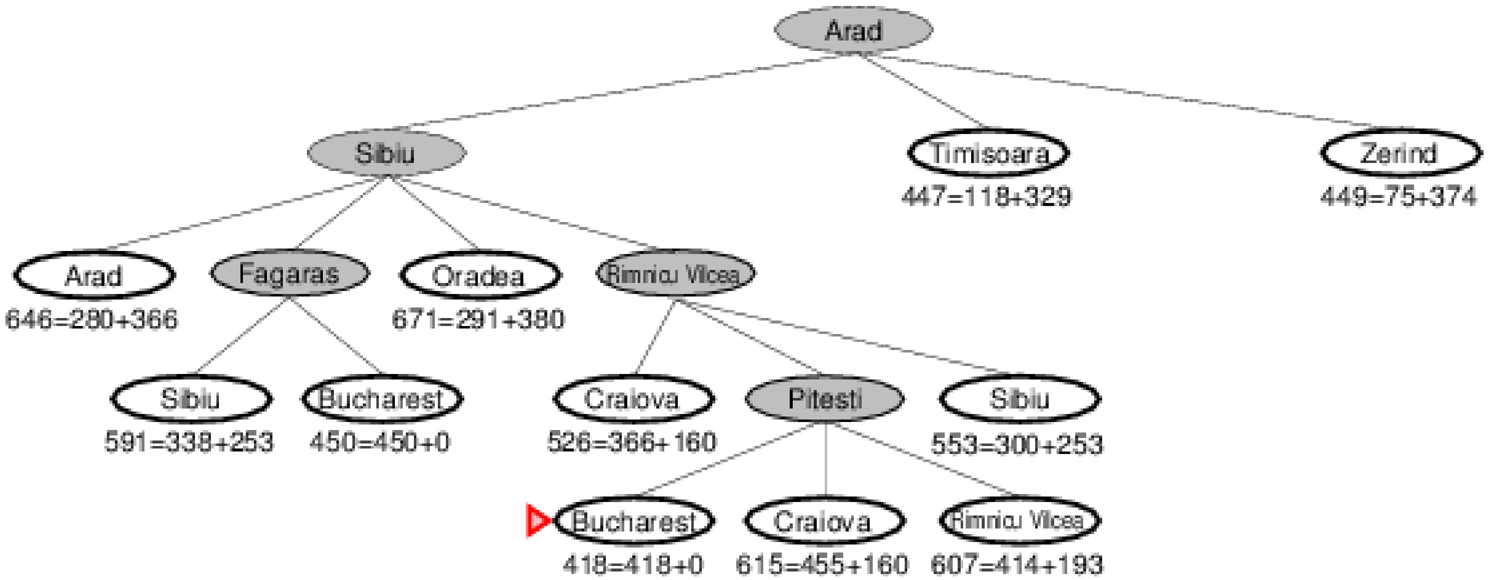

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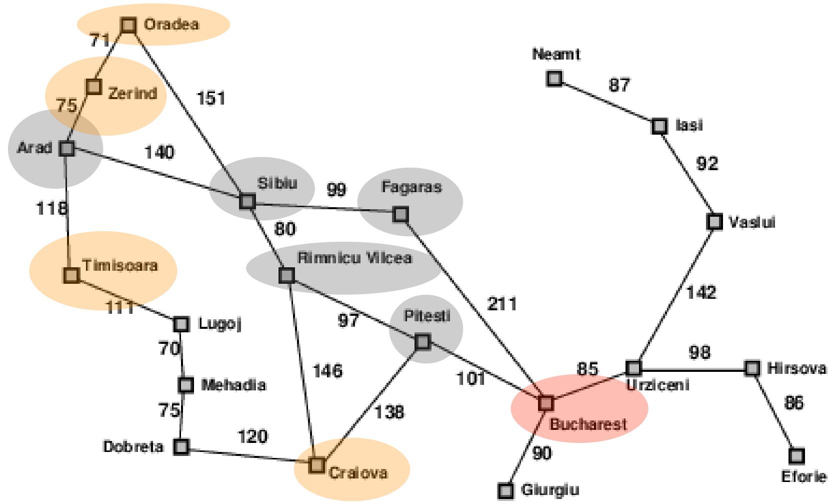

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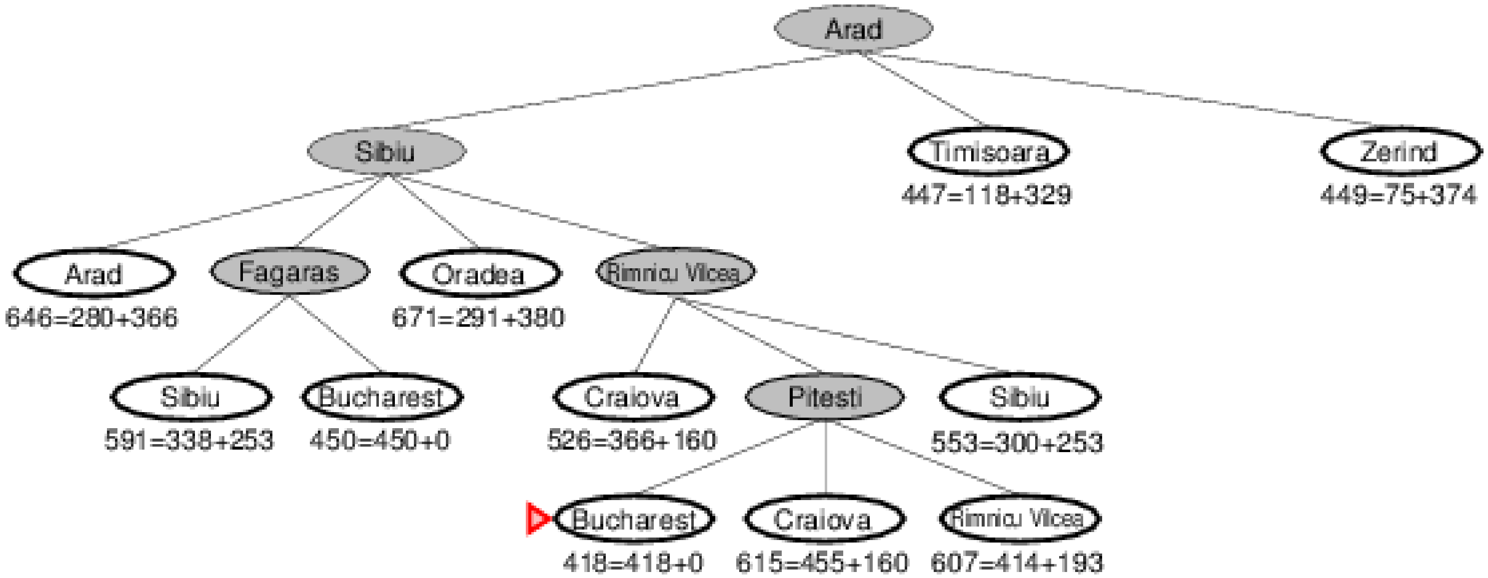
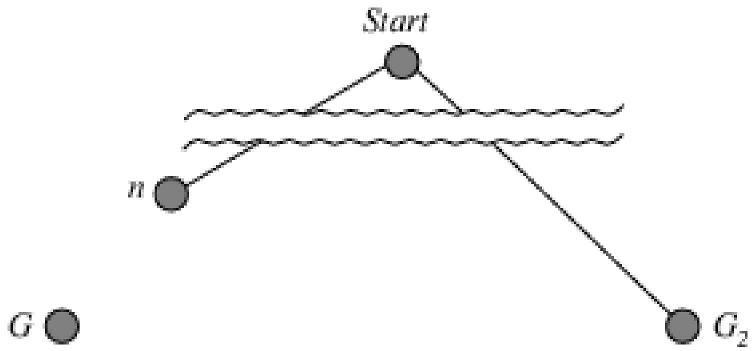


Optimality of A* (Standard Proof)

- Suppose some suboptimal goal G_2 has been generated and is in the queue
- Let n be an unexpanded node on a shortest path to an optimal goal G_1

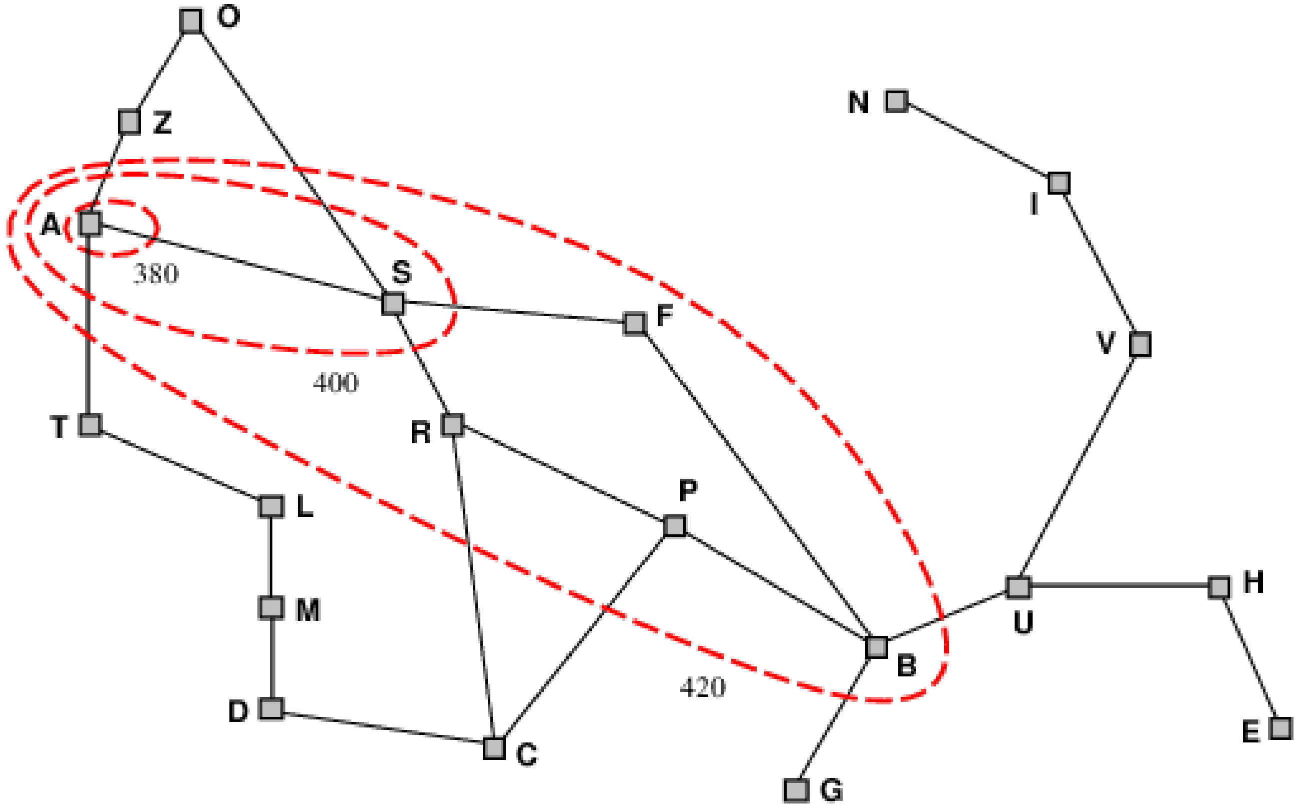


$$\begin{aligned} f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\ &> g(G_1) && \text{since } G_2 \text{ is suboptimal} \\ &\geq f(n) && \text{since } h \text{ is admissible} \end{aligned}$$

- Since $f(G_2) > f(n)$, A* will never terminate at G_2

Optimality of A* (More Useful)

- Lemma: A* expands nodes in order of increasing f value*
- Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)
- Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$



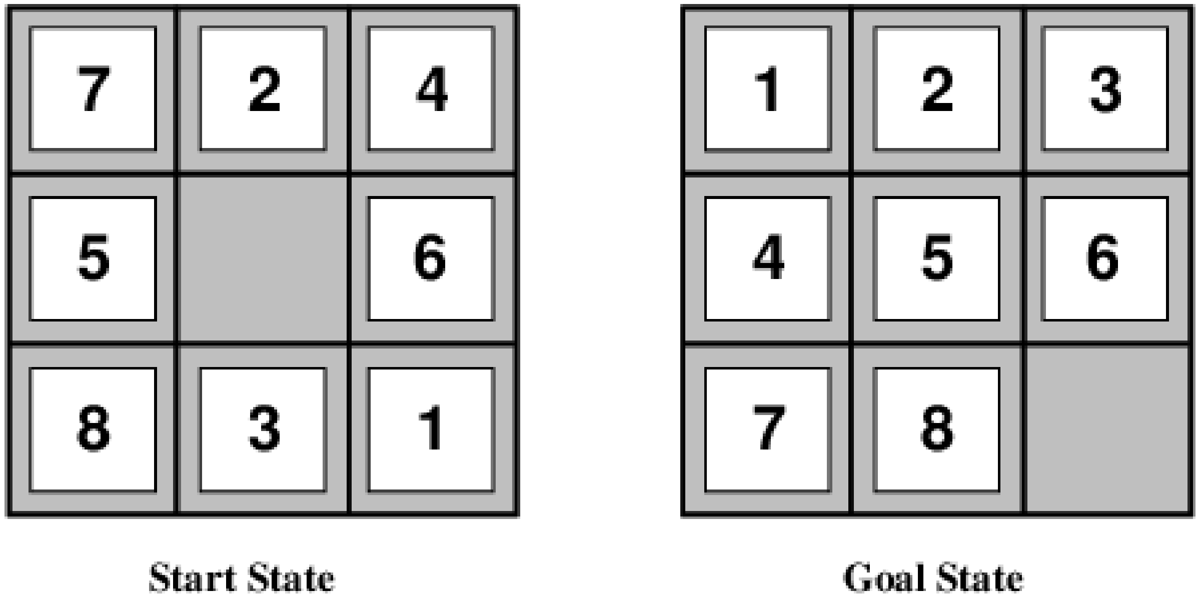
Properties of A*

- Complete? Yes, unless there are infinitely many nodes with $f \leq f(G)$
- Time? Exponential in [relative error in $h \times$ length of solution]
- Space? Keeps all nodes in memory
- Optimal? Yes—cannot expand f_{i+1} until f_i is finished

  A* expands all nodes with $f(n) < C^*$
  A* expands some nodes with $f(n) = C^*$
  A* expands no nodes with $f(n) > C^*$

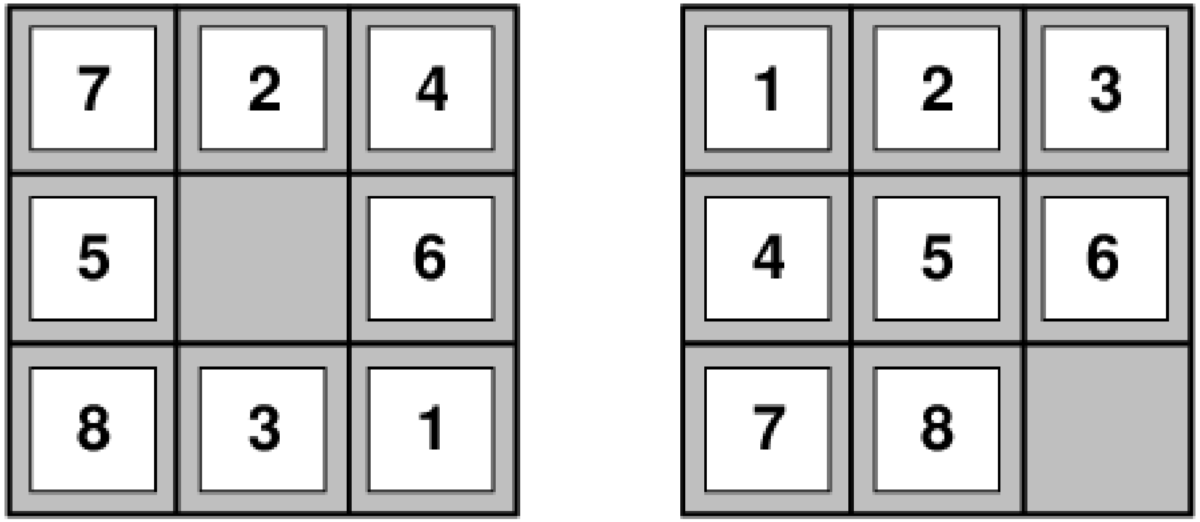
Admissible Heuristics

- E.g., for the 8-puzzle



Admissible Heuristics

- E.g., for the 8-puzzle
  - $h_1(n)$ = number of misplaced tiles
  - $h_2(n)$ = total Manhattan distance
    (i.e., no. of squares from desired location of each tile)



Start State Goal State

- $h_1(S) =?$
- $h_2(S) =?$

Admissible Heuristics

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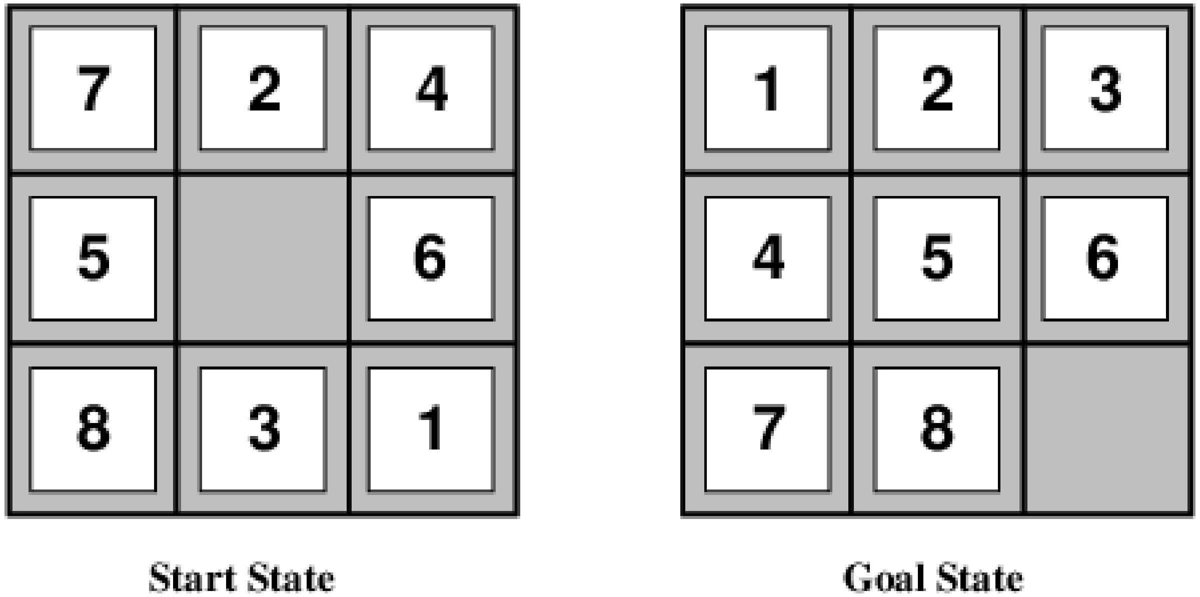


- $h_1(S) =?$ 6
- $h_2(S) =?$ 4+0+3+3+1+0+2+1 = 14

Dominance

- If $h_2(n) \geq h_1(n)$ for all n (both admissible)
  $\rightarrow$ h_2 dominates h_1 and is better for search

- Typical search costs:

$d = 14$	IDS = 3,473,941 nodes
	$A^*(h_1)$ = 539 nodes
	$A^*(h_2)$ = 113 nodes
$d = 24$	IDS ≈ 54,000,000,000 nodes
	$A^*(h_1)$ = 39,135 nodes
	$A^*(h_2)$ = 1,641 nodes

- Given any admissible heuristics h_a, h_b,

$$h(n) = \max(h_a(n), h_b(n))$$

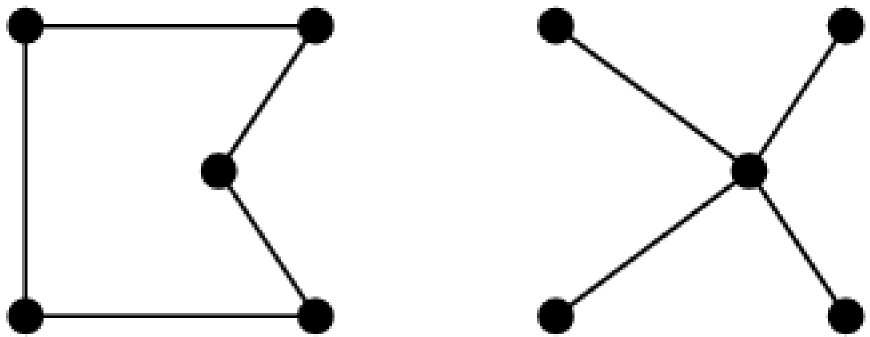
  is also admissible and dominates h_a, h_b

Relaxed Problems

- Admissible heuristics can be derived from the **exact** solution cost of a **relaxed** version of the problem
- If the rules of the 8-puzzle are relaxed so that a tile can move **anywhere** $\Rightarrow h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to **any adjacent square** $\Rightarrow h_2(n)$ gives the shortest solution
- Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed Problems

- Well-known example: travelling salesperson problem (TSP)
- Find the shortest tour visiting all cities exactly once



- Minimum spanning tree
  - can be computed in $O(n^2)$
  - is a lower bound on the shortest (open) tour

Summary: A*

- Heuristic functions estimate costs of shortest paths
- Good heuristics can dramatically reduce search cost
- Greedy best-first search expands lowest h
  - incomplete and not always optimal
- A* search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)
- Admissible heuristics can be derived from exact solution of relaxed problems

iterative improvement algorithms

Iterative Improvement Algorithms

- In many optimization problems, **path** is irrelevant;
  the goal state itself is the solution

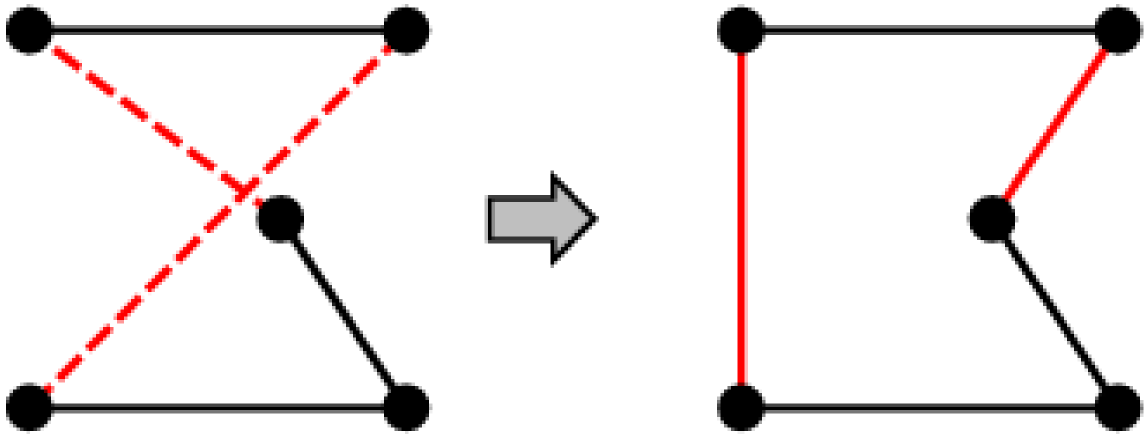
- Then state space = set of "complete" configurations
  - find **optimal** configuration, e.g., TSP
  - find configuration satisfying constraints, e.g., timetable

- In such cases, can use iterative improvement algorithms
  → keep a single "current" state, try to improve it

- Constant space, suitable for online as well as offline search

Example: Travelling Salesperson Problem

- Start with any complete tour, perform pairwise exchanges

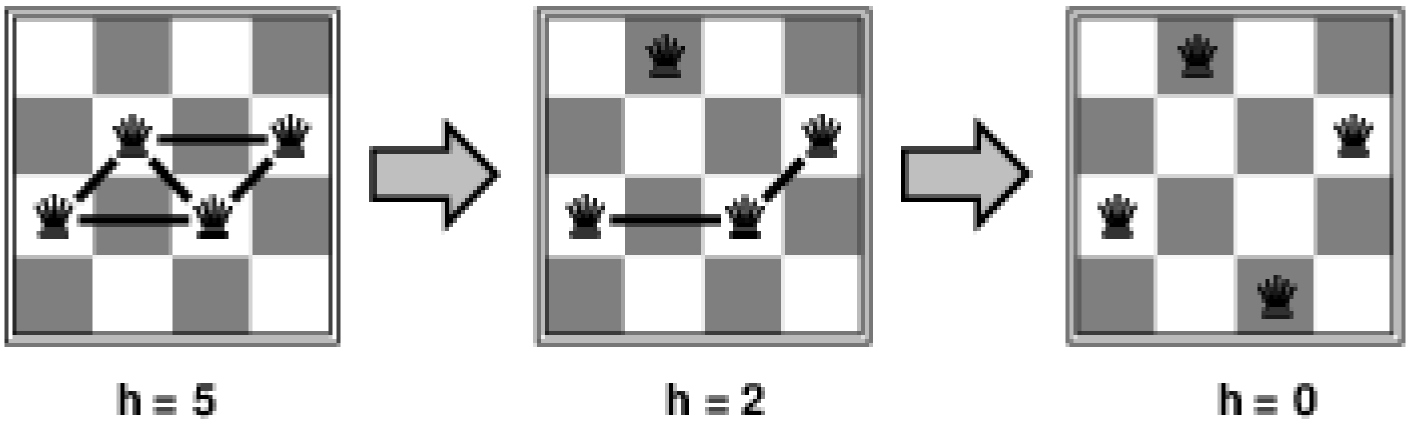


- Variants of this approach get within 1% of optimal quickly with 1000s of cities

Example: n-Queens

- Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal

- Move a queen to reduce number of conflicts



- Almost always solves n-queens problems almost instantaneously for very large n, e.g., $n = 1$ million

Hill-Climbing

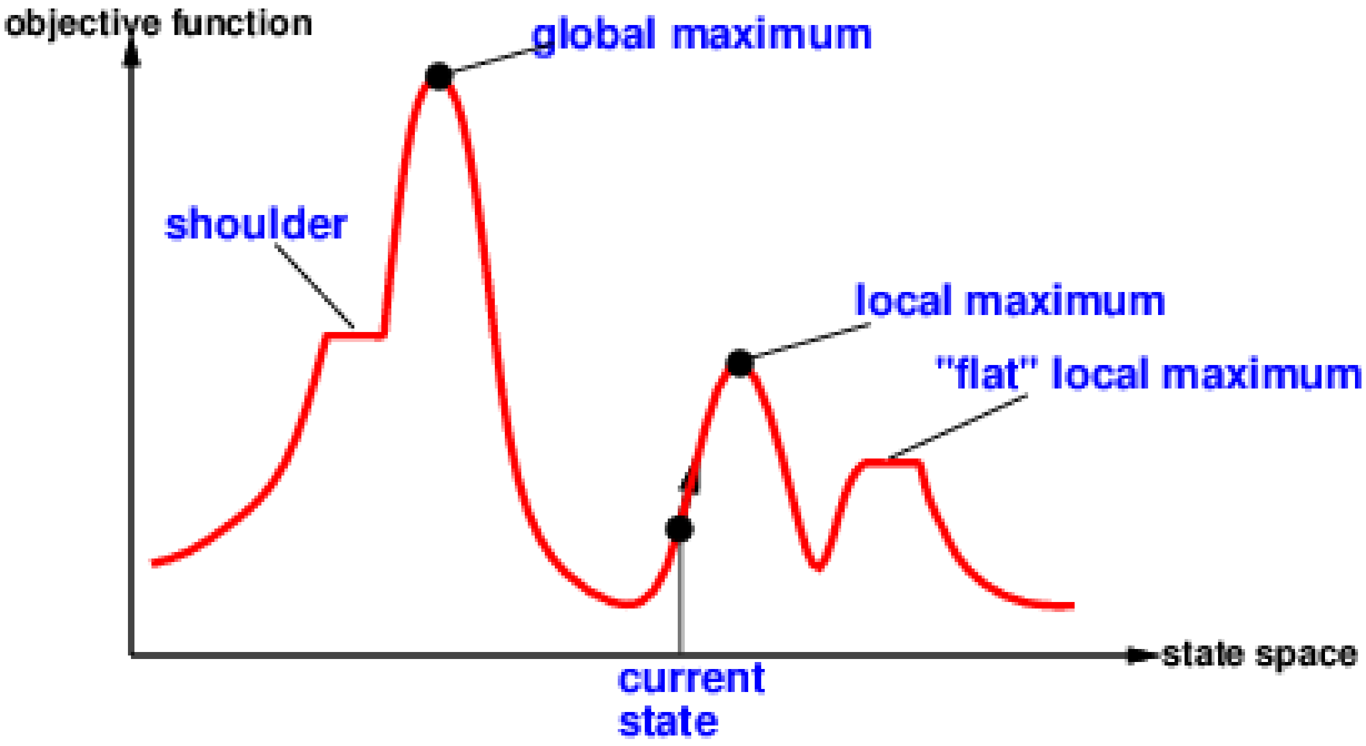
- For instance Gradient Ascent (or Descent)
- "Like climbing Everest in thick fog with amnesia"

```
function HILL-CLIMBING( problem) returns a state that is a local maximum
   inputs: problem, a problem
   local variables: current, a node
                    neighbor, a node

   current ← MAKE-NODE(INITIAL-STATE[problem])
   loop do
      neighbor ← a highest-valued successor of current
      if VALUE[neighbor] ≤ VALUE[current] then return STATE[current]
      current ← neighbor
   end
```

Hill-Climbing

- Useful to consider state space landscape



- Random-restart hill climbing overcomes local maxima—trivially complete
- Random sideways moves ☺ escape from shoulders ☹ loop on flat maxima

Simulated Annealing

- Idea: escape local maxima by allowing some "bad" moves
- **But gradually decrease their size and frequency**

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
   inputs: problem, a problem
           schedule, a mapping from time to "temperature"
   local variables: current, a node
                    next, a node
                    T, a "temperature" controlling prob. of downward steps

   current ← MAKE-NODE(INITIAL-STATE[problem])
   for t ← 1 to ∞ do
      T ← schedule[t]
      if T = 0 then return current
      next ← a randomly selected successor of current
      ΔE ← VALUE[next] – VALUE[current]
      if ΔE > 0 then current ← next
      else current ← next only with probability e^(ΔE/T)
```

Properties of Simulated Annealing

- At fixed "temperature" T, state occupation probability reaches Boltzman distribution

$$p(x) = \alpha e^{\frac{E(x)}{kT}}$$

- T decreased slowly enough $\Longrightarrow$ always reach best state x^* because $e^{\frac{E(x^*)}{kT}} / e^{\frac{E(x)}{kT}} = e^{\frac{E(x^*)-E(x)}{kT}} \gg 1$ for small T

- Is this necessarily an interesting guarantee?

- Devised by Metropolis et al., 1953, for physical process modelling

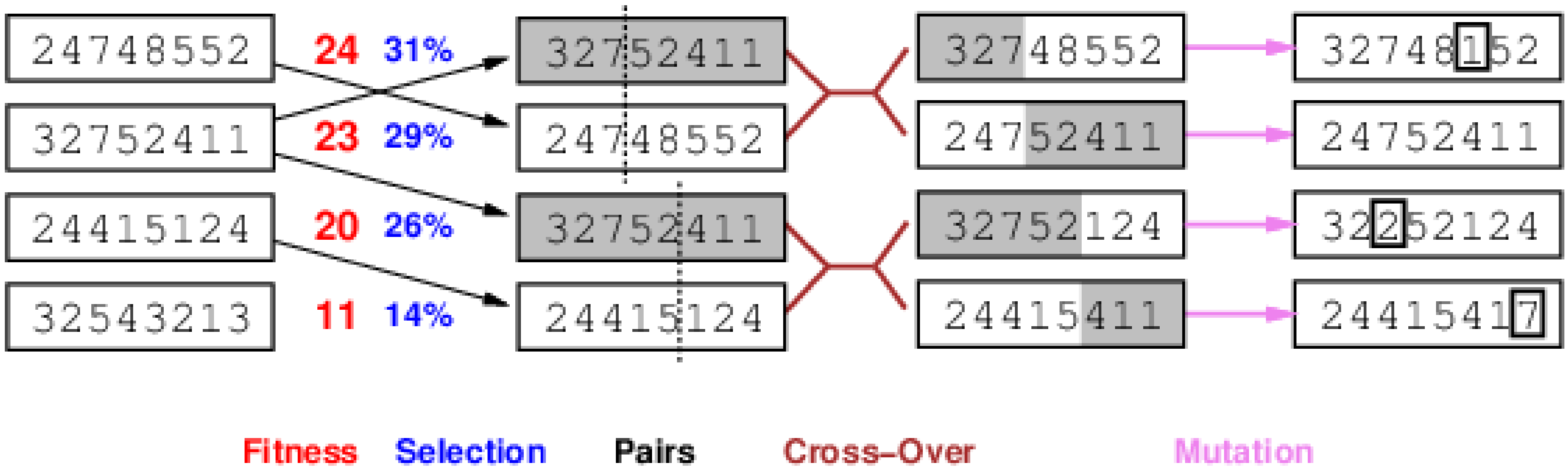
- Widely used in VLSI layout, airline scheduling, etc.

Local Beam Search

- Idea: keep k states instead of 1; choose top k of all their successors
- Not the same as k searches run in parallel!
- Problem: quite often, all k states end up on same local hill
- Idea: choose k successors randomly, biased towards good ones
- Observe the close analogy to natural selection!

Genetic Algorithms

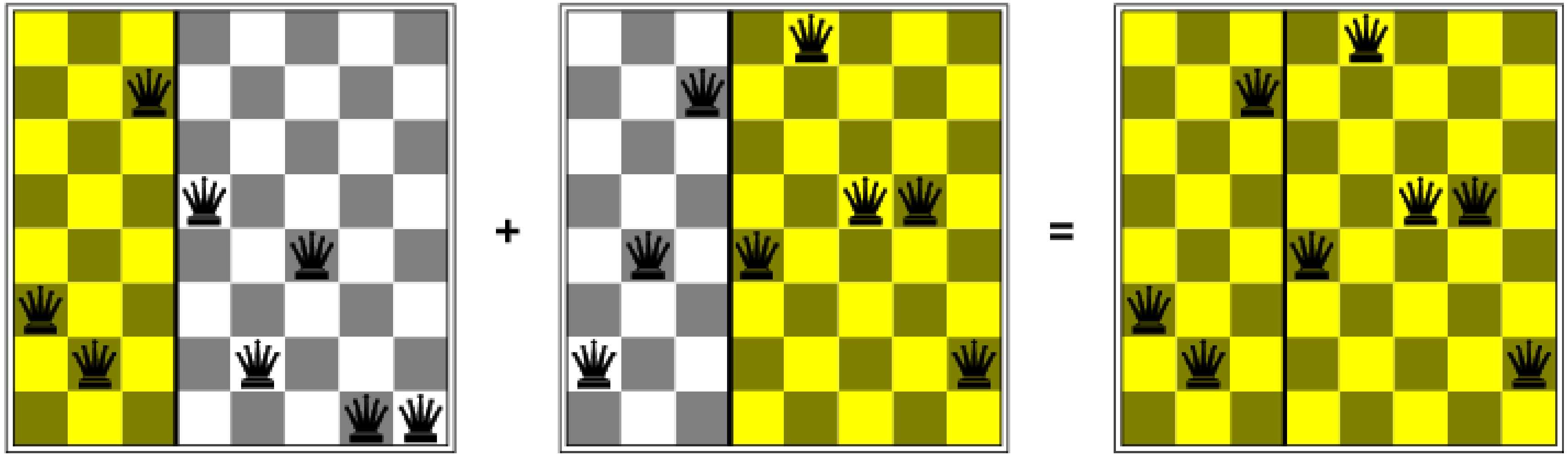
- Stochastic local beam search + generate successors from **pairs** of states



Genetic Algorithms

- GAs require states encoded as strings (GPs use programs)

- Crossover helps **iff substrings are meaningful components**



Continuous State Spaces

- Suppose we want to site three airports in Romania
  - 6-D state space defined by (x_1, y_2), (x_2, y_2), (x_3, y_3)
  - objective function $f(x_1, y_2, x_2, y_2, x_3, y_3) =$
    sum of squared distances from each city to nearest airport

- Discretization methods turn continuous space into discrete space, e.g., empirical gradient considers $\pm\delta$ change in each coordinate

- Gradient methods compute

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial y_3}\right)$$

  to increase/reduce f, e.g., by $\mathbf{x} \leftarrow \mathbf{x} + \alpha \nabla f(\mathbf{x})$

- Sometimes can solve for $\nabla f(\mathbf{x}) = 0$ exactly (e.g., with one city)
  Newton–Raphson (1664, 1690) iterates $\mathbf{x} \leftarrow \mathbf{x} - \mathbf{H}_f^{-1}(\mathbf{x}) \nabla f(\mathbf{x})$
  to solve $\nabla f(\mathbf{x}) = 0$, where $\mathbf{H}_{ij} = \partial^2 f / \partial x_i \partial x_j$

Summary

- Exact search
  - exhaustive exploration of the search space
  - search with heuristics: a*
- Approximate search
  - hill-climbing
  - simulated annealing
  - genetic algorithms (briefly)
  - local search in continuous spaces (very briefly)