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Asymmetric cubature formulas for polynomial integration in the triangle and square

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Abstract

We present five new cubature formula in the triangle and square for exact integration of polynomials. The points were computed numerically with a cardinal function algorithm which does not impose any symmetry requirements on the points. Cubature formula are presented which integrate degrees 10, 11 and 12 in the triangle and degrees 10 and 12 in the square. They have positive weights, contain no points outside the domain, and have fewer points than previously known results.

MSC: 65D32

Keywords: Multivariate integration; Quadrature; Cubature; Fekete points; Triangle; Polynomial approximation

1. Introduction

We consider a set of N points $\{z_1, z_2, \ldots, z_N\}$ and associated weights $\{w_1, w_2, \ldots, w_N\}$ to be a cubature formula of degree d if the cubature approximation for a domain Ω,

$$\int_\Omega g \simeq \sum_{j=1}^{N} w_j g(z_j),$$

is exact for all polynomials g up to degree d. Among all cubature formulas of degree d, the optimal ones are those with the fewest possible points N. The cubature problem has been extensively studied and has a long history of both theoretical and numerical development. For a recent review, see [1,7,4,2]. An on-line database containing many of the best known cubature formulas is described in [3]. These formulas are especially useful for higher order finite element methods, and as such many of these formulas were republished in the book [11], where they are available on the included CD-ROM.

One successful approach for numerically finding cubature formulas dates to [8]. A generalized version was used recently in [14]. Newton's method is used to solve the nonlinear system of algebraic equations for the cubature weights and locations of the points. Symmetry is used to reduce the complexity of the problem. If the cubature points are invariant

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under the action of a group G, then the number of equations can be reduced to the dimension of the polynomial subspace invariant under G.

Recently, a cardinal function algorithm has been developed which can provide additional reduction in the complexity of the cubature problem [12]. It is motivated by a similar cardinal function Fekete point algorithm [13]. The key idea is to look for cubature formula that have the same number of points as the dimension of a lower dimensional polynomial space. One can then construct a cardinal function basis for this lower dimensional space, make use of the interpolatory cubature weights (a multi-variate generalization of the Newton–Cotes weights) and derive a surprisingly simple expression analytically relating the variation in the cubature weights to the variation of the cubature points. The net result is a 33% reduction in the number of equations and unknowns, while still retaining analytic expressions for the gradients necessary to apply steepest descent or Newton iterations.

Symmetry can still be used to further reduce the complexity of the problem if needed. However here we have focused on fully asymmetric cubature formulas as these have been less studied. We have obtained several new cubature formula which improve on previous results. From a numerical point of view the gain is quite small—one or two fewer points for a given degree of exact integration. More important is that these results further minimize the gap between theory and known cubature formulas. For the degrees presented here (10, 11 and 12), the current best theoretical lower bounds on the number of cubature points have not been achieved and it is unknown if they can be achieved.

2. Notation

Let $\xi = (\xi_1, \xi_2)$ be an arbitrary point in $\Re^2$. We will work in the right triangle, $\xi_1 \geqslant 0$, $\xi_2 \geqslant 0$ and $\xi_1 + \xi_2 \leqslant 1$, or the square $|\xi_1| \leqslant 1$ and $|\xi_2| \leqslant 1$. Let $\mathscr{P}_d$ be the finite dimensional vector space of polynomials of at most degree d,

$$\mathscr{P}_d = \text{span}\{\xi_1^n \xi_2^m, m + n \leqslant d\}.$$

We note that

$$\dim \mathscr{P}_d = \tfrac{1}{2}(d+1)(d+2).$$

The monomials $\xi_1^n \xi_2^m$ are notoriously ill-conditioned, so it is necessary to describe $\mathscr{P}_d$ with a more reasonable basis. In the triangle there are several suitable choices of orthogonal basis functions. We use the normalized Proriol polynomials $\{g_{m,n}\}$ [9,6,5]. The indexes m and n specify the top degree in each coordinate. Here we convert this traditional double index (m, n) into a single index by $i = (m+n+1)(m+n+2)/2 - m$, so that $\mathscr{P}_d = \text{span}\{g_i, i = 1, \ldots, \dim \mathscr{P}_d\}$. Recurrence relations to evaluate these polynomials and their derivatives are given in [10]. In the square, we use the well-known tensor product basis $g_{m,n}(\xi) = P_m(\xi_1)P_n(\xi_2)$, where $P_m(x)$ is the normalized Legendre polynomial in $[-1, 1]$ of degree m.

3. Cubature formula for $\mathscr{P}_d$

For any set of $M = \dim \mathscr{P}_d$ non-degenerate points $\{x_j,\ j = 1, \ldots, M\}$ in the triangle or square, we can always obtain a cubature formula for $\mathscr{P}_d$ by using the interpolatory cubature weights given by solving, for w_j, the $M \times M$ system

$$\sum_{j=1}^{M} w_j g_i(x_j) = \int_\Omega g_i \, \mathrm{d}\xi \quad \forall g_i \in \mathscr{P}_d. \tag{1}$$

By construction, these weights and the points $\{x_j\}$ give a cubature formula which exactly integrates our M basis functions, and thus exactly integrates all their linear combinations and so is a cubature formula for $\mathscr{P}_d$. This is the classical upper bound on the number of cubature points required for a formula of degree d. Our goal is to find a solution to Eq. (1) that requires only $N < M$ points, with N as small as possible.

We use a modification of the cardinal function algorithm from [12]. That algorithm relies on interpolatory cubature formula for a polynomial subspace $\mathscr{P}' \subset \mathscr{P}_d$ with $N = \dim \mathscr{P}'$. Letting $\{g_i'\}$ be a basis for $\mathscr{P}'$, then the interpolatory weights for $\mathscr{P}'$ and a given set of points $\{z_j,\ j = 1, \ldots, N\}$ are uniquely determined by the $N \times N$ system

$$\sum_{j=1}^{N} w_j g_i'(z_j) = \int_\Omega g_i' \, \mathrm{d}\xi \quad \forall g_i' \in \mathscr{P}'. \tag{2}$$

If the points are also a cubature formula of degree d, then these weights must also be the cubature weights for $\mathscr{P}_d$, since any cubature formula for $\mathscr{P}_d$ must exactly integrate $\mathscr{P}'$. Thus the weights can be considered known and to find a cubature formula of degree d the algorithm must only search for suitable points $\{z_j\}$. Furthermore, since $\mathscr{P}'$ is integrated exactly by these weights, this search needs only consider integrating a basis for the smaller space given by $\mathscr{P}_d \backslash \mathscr{P}'$.

The end result is that to find a cubature formula for $\mathscr{P}_d$, one must solve the system $\mathbf{F}=0$ where $\mathbf{F}=\{F_m : g_m \in \mathscr{P}_d \backslash \mathscr{P}'\}$,

$$F_m = \sum_j w_j g_m(z_j),$$

$\mathbf{F}$ is considered as a function only of $\{z_j\}$. We note that integrals of the basis functions do not appear in the equation $\mathbf{F}=0$ because of the fact that $\int g_i = 0 \forall g_i \in \mathscr{P}_d \backslash \mathscr{P}'$. If a cardinal function bases is used for $\mathscr{P}'$, the gradients of $\mathbf{F}$ needed for Newton or gradient-based methods are known analytically [12].

The results obtained in [12] considered only subspaces of the form $\mathscr{P}' = \mathscr{P}_e$ with $e < d$, and thus were limited to cubature formulas with points $N = \frac{1}{2}(e+1)(e+2)$. To relax this restriction we introduce a simple procedure to construct subspaces of dimension N for arbitrary values of N. We start by taking the largest e such that

$$\dim \mathscr{P}_e \leqslant N < \dim \mathscr{P}_{e+1}.$$

We then augment $\mathscr{P}_e$ with M basis functions, where $N = \dim \mathscr{P}_{e+1} + M$. We restrict ourselves to choosing the M basis functions from the set $g_i \in \mathscr{P}_{e+1} \backslash \mathscr{P}_e$. Denoting the span of these augmenting functions by $\mathscr{A}$, we can write

$$\mathscr{P}' = \mathscr{P}_e \oplus \mathscr{A}. \tag{3}$$

There are many possible choices of $\mathscr{A}$. The only requirement is that Eq. (2) be well conditioned. If a zero is found by the Newton iteration procedure, then it will represent a cubature formula for $\mathscr{P}_d$ no matter which subspace $\mathscr{P}'$ was used. In practice, for a set of points which are used to initialize the Newton iteration, we compute all possible $\mathscr{P}'$ of the form in Eq. (3) and take the one which gives the most well-conditioned system for Eq. (2). If during the iteration this system becomes ill-conditioned as the points change, we repeat the procedure and compute a new $\mathscr{P}'$.

All other aspects of the algorithm, including the procedure used to construct initial conditions for the iterations are identical to [12].

4. Results

Our results are summarized in Table 1. All the cubature points have positive weights and no points lie outside the triangle, although neither of these properties is in any way guaranteed by the cardinal function algorithm. The errors presented in the table is the max norm of the cubature error over all the ortho-normal basis functions:

$$\max_{g_i \in \mathscr{P}_d} \left| \sum_i w_i g_i(z_i) - \int g_i \, \mathrm{d}\xi \right|$$

with normalization $\int g_i^2 \, \mathrm{d}\xi = A$, where A is the area of the triangle or square. None of the cubature sets are invariant under the symmetry group of rotations and reflections of the domain.

Table 1
Cubature points computed with the cardinal function algorithm

Domain	Number of points	Degree of exact integration	Error
Triangle	24	10	9.3×10^{-16}
Triangle	27	11	6.2×10^{-15}
Triangle	32	12	2.0×10^{-15}
Square	22	10	2.0×10^{-15}
Square	31	12	1.6×10^{-15}

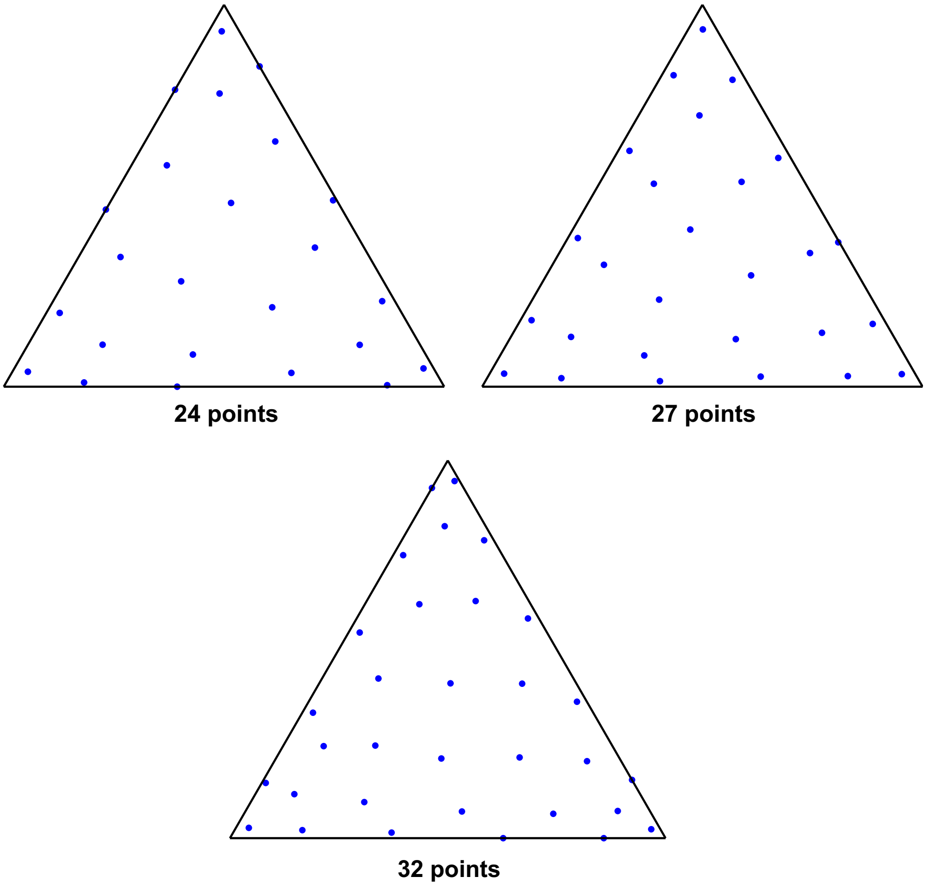

Fig. 1. Cubature points for the triangle which exactly integrate polynomials of degree 10, 11 and 12.

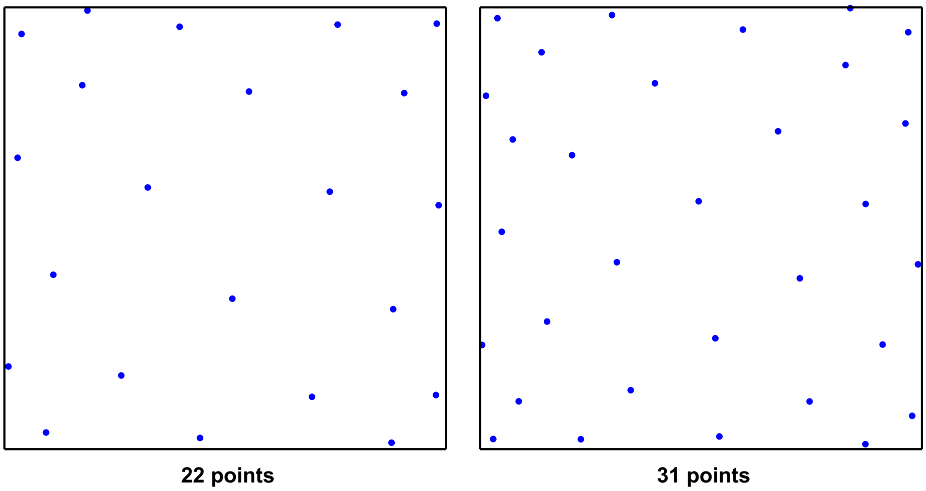

Fig. 2. Cubature points for the square which exactly integrate polynomials of degree 10 (left) and 12 (right).

All formulas improve upon the best previously published results, as taken from the extensive database described in [3]. In particular, for the triangle, if one only considers formula with positive weights and no points outside the domain, these three results have one less point than the previous results. For the square, the best cubature formula for degree 10 required 24 points, but it was also a cubature formula for degree 11. Similarly for degree 12, the best previous formula required 33 points, but it was also a cubature formula of degree 13.

Plots for all of the cubature points are given in Figs. 1 and 2. For the plots, the right triangle has been mapped linearly to the equilateral triangle in order to make the asymmetry in the points more visible. The coordinates of the cubature points are given in Appendix A.

5. Summary

We have presented a modified cardinal function algorithm for computing cubature points. The modified algorithm removes the restriction that the number of cubature points must be of the form $n(n+1)/2$. The algorithm was then applied to the square and the triangle, where five new cubature formulas were found. The formulas have one or two fewer points then the best previously known formulas with positive weights and no points outside the domain.

Acknowledgments

We thank R. Cools for suggesting we look for asymmetric cubature formulas to fill in some of the gaps between the theory and the known formulas for the triangle and square.

Appendix A. Tables of cubature points

We now list the coordinates of the cubature points described in Table 1. For each line, we give the two coordinates of each point followed by the associated cubature weight. In the triangle, we give the x- and y-coordinates in the unit right triangle, which are equivalent to the first to two barycentric coordinates. In the square the coordiantes are in a square of length 2 centered at the origin.

[-1,1]x[-1,1] Integration strength = 10 number of points = 22

4.7324898849276598e-01	1.6557852510038315e-01	3.7171764930896173e-01
-3.5072672608918981e-01	1.8447172062121983e-01	3.8811447402440874e-01
-4.7113921490701688e-01	-6.6664733059821124e-01	2.8395842218278933e-01
3.2110023120386515e-02	-3.1879357593640706e-01	4.0824197726154576e-01
1.0733227865108741e-01	6.1886619139299248e-01	3.1498878221231136e-01
8.1037492260191812e-01	6.1159678303492504e-01	2.0018320620277513e-01
-7.7882541598318511e-01	-2.1052738914821550e-01	2.6587113477126029e-01
-6.4763548426267536e-01	6.4749469817525440e-01	2.3057004553370086e-01
3.9247487539609610e-01	-7.6311149392438349e-01	2.5827939410342804e-01
7.6050655071397388e-01	-3.6631391678067937e-01	2.6042397681916851e-01
-8.1171510601648733e-01	-9.2466842429053542e-01	8.8686202216975499e-02
-1.1429517364223797e-01	-9.4921913140887015e-01	1.1861767207465973e-01
7.5296563247996007e-01	-9.7071837396777472e-01	5.9110195150351146e-02
-6.2402437958984702e-01	9.8538331193146012e-01	3.3387129247073037e-02
-2.0650134619887220e-01	9.1195887103573425e-01	1.3460977386198075e-01
5.0891319042960681e-01	9.2152907557898267e-01	1.3337731192240113e-01
-9.8171192640479688e-01	-6.2586619353239670e-01	6.3269272761110995e-02
-9.4061855719921172e-01	3.1884535968392919e-01	1.1984158532391266e-01
-9.2254816825741193e-01	8.7923480439903223e-01	6.2537941187552140e-02
9.5380192234255112e-01	-7.5512692061435549e-01	7.1805898760516657e-02
9.6634208368735852e-01	1.0431232556636386e-01	9.7940429484131938e-02

9.5774959160007522e-01	9.2621050012583894e-01	3.4467525588983812e-02

[-1,1]x[-1,1] Integration strength = 12 number of points = 31

-3.8131119459148788e-01	-1.5398000784199919e-01	2.3598701292933691e-01
6.4573191109453948e-02	-4.9794871568657245e-01	2.4110813600847844e-01
-1.0834321329194775e-02	1.2224478971657705e-01	2.7048224227639595e-01
-3.1882703020851938e-01	-7.3294898893865745e-01	1.8648166880884143e-01
4.9124051440092803e-01	-7.8409257688727241e-01	1.7241958393889878e-01
-2.0906799447670715e-01	6.5611051497298534e-01	2.4109442092821515e-01
1.8948337189272124e-01	8.9911805536738287e-01	1.4125391453247216e-01
-6.9740094114568829e-01	-4.2248127033807525e-01	1.8292025454973745e-01
-5.8402513589553640e-01	3.3100645378716304e-01	2.2222997581864359e-01
4.4701377237977313e-01	-2.2659664973677845e-01	2.4904537785770031e-01
3.4867912208378016e-01	4.3860404955163790e-01	2.5112362229462498e-01
-8.2562697914640848e-01	-7.8339223275580994e-01	9.2811359998802967e-02
8.2839794669248942e-02	-9.4230798695085749e-01	9.8238337173087234e-02
-4.0351974647584699e-01	9.6482881096343187e-01	6.8583817743889233e-02
6.5401042056406233e-01	7.3857098028675039e-01	1.5506938444449642e-01
-9.0264377285460362e-01	-1.5881852211379360e-02	1.3182320881366066e-01
-8.5295397551927077e-01	4.0163251602231481e-01	6.0068518355283269e-02
8.2216786295901656e-01	-5.2660748692256265e-01	1.5766018519773284e-01
7.4510555770497011e-01	1.0988549815212308e-01	1.9930860602643424e-01
-9.4133986362315170e-01	-9.5414148142964417e-01	2.1719109379092324e-02
-5.4461187387612175e-01	-9.5530036379122441e-01	6.4596098758824216e-02
7.4407322966908984e-01	-9.7744835460673618e-01	4.0584519909105145e-02
-7.2254885481492603e-01	7.9662230649630528e-01	1.2603341007939087e-01
6.7524406569352735e-01	9.9626110485909680e-01	3.0006912476756006e-02
9.3807676975195520e-01	8.8729244026658205e-01	4.4974753329261395e-02
-9.9171408764710778e-01	-5.2853945477454323e-01	3.8721244577288268e-02
-9.7391149358748841e-01	5.9955340548238656e-01	4.7718056922379071e-02
-9.2231295237113653e-01	9.5082592343728289e-01	3.0056189691189335e-02
9.5557708019583965e-01	-8.4903353418547223e-01	4.4499282846528827e-02
9.8216591295043421e-01	-1.6369711363774472e-01	5.6566093223890265e-02
9.2557212335129857e-01	4.7425880434330858e-01	9.6814701109561335e-02

Unit Right Triangle Integration strength = 10 number of points = 24

5.0550507373529086e-01	2.0776116575484826e-01	1.7344807725532943e-01
2.7542385024412980e-01	4.8123289062464247e-01	1.9053311454269983e-01
2.6481531651496770e-01	2.7586334089315967e-01	1.6882888511942015e-01
7.5329402776254240e-01	1.0954959855585467e-01	1.0546076281767805e-01
5.2433682558924433e-01	3.6419744430339263e-01	1.4815929467355968e-01
2.9530445535851102e-01	6.4203365318662664e-01	1.0983120878770872e-01
1.0614642990289996e-01	7.6777680170023954e-01	1.0507331820482332e-01
6.3491832379200652e-01	3.6036266787907723e-02	8.5924658784158670e-02
3.8729657913960353e-01	8.4198522115543739e-02	1.2537585060182724e-01
1.6929927488966462e-01	1.0999439055630450e-01	1.1594828119739846e-01
8.0491894656105567e-02	5.7966325105486349e-01	1.3237226895051976e-01
9.5379208487721689e-02	3.3947290311800554e-01	1.2348449173239080e-01
9.2899486985787905e-01	4.7768381772022417e-02	2.9216658446243379e-02
7.4726591728868819e-01	2.2376358774275851e-01	6.4605204046914597e-02
5.0365825075943971e-01	4.8798437805397499e-01	3.9118824435043810e-02
1.6134650499890957e-01	8.3865349500109043e-01	2.2133893564494179e-02
2.9553592846822851e-02	9.3049846900263089e-01	3.0406188052025412e-02
8.6854386943076545e-01	3.8102570854643414e-03	2.1333382551825181e-02

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3.9366774470722010e-01     0.0000000000000000e+00     2.3800609628471206e-02
1.7690730625559031e-01     1.0939142057119933e-02     2.9693247293360987e-02
3.5319656252586096e-02     3.9099745550423282e-02     3.5311689185924387e-02
0.0000000000000000e+00     7.7757518429429107e-01     2.6798161571713618e-02
0.0000000000000000e+00     4.6374383867430541e-01     3.0312523835131357e-02
3.0573404093099332e-02     1.9305903224251936e-01     6.2829404721337689e-02
Unit Right Triangle Integration strength = 11 number of points = 27
4.6494564773693992e-01     2.9133859436942361e-01     1.3648275991498204e-01
3.2081957909482994e-01     5.3634228112084714e-01     1.2438630022250971e-01
5.1353143433447235e-01     1.2454405910544103e-01     1.1329177024539897e-01
2.8790310224819649e-01     2.2789955884347501e-01     1.3228489176992250e-01
2.6677168071577745e-01     4.1132499178904658e-01     1.1722353681481934e-01
1.1698976413323442e-01     3.1909737814681871e-01     1.0998202543484477e-01
8.1626233715968810e-01     2.7719522918618567e-02     4.7284119131529377e-02
5.6938486195327997e-01     3.4992914334288650e-01     1.0994399601768742e-01
3.7272769861629096e-01     5.9895439629934211e-01     6.5193746289815974e-02
2.6807150626772580e-02     8.1562969693268217e-01     4.6224760707242137e-02
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9.3614893514675623e-01     3.2899822292186298e-02     2.6211869704176473e-02
8.0454974747615537e-01     1.6429286715713465e-01     5.5191800300359820e-02
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1.6655614492060572e-01     8.0364834053903877e-01     5.3513272326506316e-02
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3.9659731669586495e-01     1.4354532010930898e-02     3.6717768780272685e-02
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3.2916403878999745e-02     3.4222771841359190e-02     2.9096217361124159e-02
2.5660186833052434e-02     6.1758873171277151e-01     5.7443554735054178e-02
1.2417148586801485e-01     5.3141960154079959e-01     1.0824295295050959e-01
2.5252704638304480e-02     1.7400571673032256e-01     4.8140601001216463e-02
Unit Right Triangle Integration strength = 12 number of points = 32
3.7986021093401956e-01     2.1078525939140391e-01     1.1887566790227083e-01
3.0141709320909305e-01     4.0978657777002531e-01     1.5044412520664885e-01
5.5802528953120256e-01     2.1377743253005960e-01     1.2632909284531338e-01
1.2512299505810387e-01     6.1938125736255578e-01     1.0192984975357525e-01
2.1117939909804934e-01     2.4498296509349016e-01     9.4999150650614317e-02
8.5431474947580432e-01     7.1871496101589105e-02     4.4981492398316447e-02
7.1788185898052326e-01     2.0376848107772977e-01     7.9147211585943858e-02
4.6631787462323071e-01     4.0896380449124475e-01     1.1997941465421234e-01
2.5015500335339214e-01     6.2768261568031403e-01     1.0670416609764186e-01
7.9955384841381316e-02     8.2600331401756000e-01     6.1058344824144795e-02
7.1008125956836521e-01     6.4413220382260550e-02     8.2563774790925248e-02
4.9732063377796598e-01     7.0566724344036824e-02     9.6297610073814668e-02
2.6077068256562896e-01     9.5428585810584610e-02     9.1875684331583440e-02
8.9602705800587434e-02     1.1638649906727733e-01     6.1150555208077911e-02
2.3088148766115757e-02     7.4918973979067949e-01     4.3370170834023010e-02
1.2953296900433620e-01     4.2260565743346001e-01     1.0829374522633514e-01
9.3448087604440955e-02     2.4345813394879973e-01     5.5887468639759713e-02
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9.5526919357006035e-01	2.3551733249578710e-02	1.3351800054734712e-02
8.4593539837314391e-01	1.5406460162685609e-01	1.5428984747249670e-02
6.1600929617267497e-01	3.6118159118967208e-01	5.0198346855370224e-02
3.9316510319604808e-01	5.8168921474014745e-01	5.6291117210426664e-02
1.8920633061715936e-01	7.8860171922313160e-01	4.1240008239364231e-02
4.3010560106405471e-02	9.4547507322097091e-01	1.4239502872161450e-02
8.5815888421533082e-01	0.0000000000000000e+00	1.3691069308687381e-02
6.2731531923241179e-01	0.0000000000000000e+00	1.9309417484872689e-02
3.6384660446077510e-01	1.4566514788346974e-02	3.7090960843213061e-02
1.5557066896897953e-01	2.1152223383121949e-02	3.6967371622461546e-02
2.9754117496841759e-02	2.7110971356255786e-02	2.1018653471205032e-02
0.0000000000000000e+00	9.2734897448394982e-01	9.7760996293200769e-03
2.5716283623693881e-02	5.4444667627192522e-01	5.6339308919459923e-02
2.4506286636990005e-02	3.3212908394764507e-01	4.9808146403015403e-02
9.2296909059649268e-03	1.4604496167217568e-01	2.1361687315256585e-02

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