FINITE ELEMENTS FOR THE BELTRAMI OPERATOR ON ARBITRARY SURFACES

Gerhard Dziuk
Institut für Angewandte Mathematik,
Wegelerstraße 6, D-5300 Bonn, FRG

Abstract: We develop a Finite Element Method for elliptic differential equations on arbitrary two-dimensional surfaces. Global para metrizations are avoided. We prove asymptotic error estimates. Numerical examples are calculated.

Keywords: Finite Elements, Beltrami Operator, Elliptic Equations on Surfaces

Classification Numbers: 65 N 30, 35 A 40

§ 1. INTRODUCTION

Our aim is to develop a Finite Element Method for elliptic differential equations on arbitrary two-dimensional surfaces - not necessarily embedded - in $\mathbb{R}^3$. We shall avoid global parametrizations, and think of surfaces just given by splines. The most important point in our method is that we write down the Laplace-Beltrami operator in terms of the tangential gradient. In order to present the idea we shall confine ourselves to the most simple equation

$$-\Delta_S u = f \text{ on } S .$$

$-\Delta_S$ is the Laplace-Beltrami operator on S . Let us assume for the moment that $\partial S = \emptyset$. We approximate the surface S by a polyhedron S_h and solve

$$-\Delta_{S_h} u_h = f \text{ on } S_h$$

weakly. We use linear elements on the surface S_h , i. e. u_h is a linear polynomial on each triangle of S_h and globally continuous. The Laplace-Beltrami operator on S_h is defined by

$$\int_{S_h} \nabla_{S_h} u \, \nabla_{S_h} \varphi = \int_{S_h} f \varphi$$

for all φ in the Sobolev space $H^1(S_h)$ where

$$\nabla_{S_h} u = \nabla u - (\nabla u \cdot n_h) n_h$$

is the tangential gradient on S_h, ∇ is the three-dimensional gradient and n_h is the normal vector to S_h.

Practically this means that $\nabla_{S_h} u_h$ is constant on each triangle of S_h if u_h is linear. If we take $\varphi_{h1},\dots,\varphi_{hN}$ (N = number of vertices $x_{(k)}$ of S_h) to be those piecewise linear functions on S_h which are globally continous and $\varphi_{hj}(x_{(k)}) = \delta_{jk}$ then

$$u_h(x) = \sum_{j=1}^{N} u_j \, \varphi_{hj}(x)$$

and we have to solve the linear system

$$\sum_{j=1}^{N} u_j \int_{S_h} \nabla_{S_h} \varphi_{hj} \, \nabla_{S_h} \varphi_{hk} = \int_{S_h} f \varphi_{hk}$$

$(k = 1,\dots,N)$. Thus the numerical scheme is just the same as in a plane two-dimensional problem. The only difference is that in our case the computer has to memorize three-dimensional nodes instead of two-dimensional ones. Since the triangles of S_h can be parametrized via the unit triangle in $\mathbb{R}^2$ the method is fairly easy.

We prove that the order of convergence is the same as in plane problems.

Let us mention that error-estimates on surfaces have been proved by J.C.Nedelec in [N] for the boundary element method.

In [BF] and in [S] the authors construct spherical Finite Elements in order to solve problems on $S = S^2$.

§ 2. CONTINUOUS PROBLEM

We consider a compact $C^{k,\alpha}$-hypersurface S $(k \in \mathbb{N} \cup \{0\}\ ,\ 0 \leq \alpha \leq 1)$ in $\mathbb{R}^3$. For simplicity we assume that S can be represented globally by some oriented distance function d which is defined on some open subset U of $\mathbb{R}^3$.

$$S = \{x \in U \mid d(x) = 0\}$$

d is in $C^{k,\alpha}(U)$, $\nabla d \neq 0$. Almost everywhere the normal to S in the direction of growing d is given by

$$n = \frac{\nabla d}{|\nabla d|} .$$

Without loss of generality we assume that $|\nabla d| = 1$. The tangential gradient is

$$\nabla_S u = \nabla u - (n \cdot \nabla u)\, n$$

for some function defined on U . $\nabla_S u$ depends on $u_{|S}$ only.

$$-\Delta_S = -\nabla_S \cdot \nabla_S$$

is the Laplace-Beltrami-operator on S , if $S \in C^2$.
For smooth S we may assume that there is a strip
$U = \{x \in \mathbb{R}^3 | \mathrm{dist}(x,S) < \delta\}$ about S where the decomposition

$$x = a(x) + d(x)\, n(x)$$

is unique. $a(x) \in S$, $n(x)$ is normal to S at $a(x)$, $|n(x)| = 1$, and $|d(x)| = \mathrm{dist}(x,S)$.
This implies that we may extend a function u defined on S uniquely to this strip by

$$\bar{u}(x) = u(x-d(x)n(x)) \qquad (x \in U) .$$

Let us remark that for $S \in C^2$, δ is bounded by the sectional curvature of S .
Let us define the Sobolevspaces we shall use throughout the paper.

$$H^{\ell}(S) = \{u \in L^2(S) \mid u \text{ possesses weak tangential derivatives up to order } \ell \text{ which are in } L^2(S)\}$$

Here we need $S \in C^{k,\alpha}$ with $k + \alpha \geq 1$ and $\ell \leq k + \alpha$ if $k + \alpha \in \mathbb{N}$, $\ell < k + \alpha$ if $k + \alpha \notin \mathbb{N}$.

$$\mathring{H}^{\ell}(S) = \overline{C_0^{\ell}(S)}^{\|\cdot\|_{H^{\ell}(S)}} ,$$

$$\|u\|_{H^{\ell}(S)} = \left[\sum_{|\nu|=0}^{\ell} |u|^2_{H^{|\nu|}(S)} \right]^{1/2}$$

$$|u|_{H^{\ell}(S)} = \left[\sum_{|\mu|=\ell} \|D_S^{\mu} u\|^2_{L^2(S)} \right]^{1/2}$$

It will be important that for $S \in C^{0,1}$ the spaces $H^1(s)$ and $\mathring{H}^1(S)$ are well defined. For more details see for example [W] p. 92.
Let us now formulate the basic existence and regularity results (see for example [A] p. 104). If not stated otherwise we shall assume that S and ∂S are of class C^3 .

1. THEOREM:

a) $\partial S \neq \emptyset$. For every $f \in L^2(S)$ there exists a unique weak solution $u \in \mathring{H}^1(S)$ of the problem

$$-\Delta_S u = f \text{ on } S , \; u = 0 \text{ on } \partial S ,$$

i.e. for every $\varphi \in \mathring{H}^1(S)$

(1) $$\int_S \nabla_S u \nabla_S \varphi = \int_S f \varphi ,$$

and

$$\|u\|_{H^2(S)} \leq c \|f\|_{L^2(S)} .$$

b) $\partial S = \emptyset$. For every $f \in L^2(S)$ with $\int_S f \, do = 0$ there exists a weak solution $u \in H^1(S)$ of

$$-\Delta_S u = f \quad \text{on} \quad S ,$$

i.e. (1) holds for all $\varphi \in H^1(S)$, and u is unique up to a constant , and

$$\|u\|_{H^2(S)} \leq c \left[\|f\|_{L^2(S)} + \|u\|_{L^2(S)} \right] .$$

§ 3. DISCRETE PROBLEM

We shall approximate the smooth surface S by a surface S_h which globally is of class $C^{0,1}$. For example S_h is a polyhedron consisting of triangles T_h of size proportional to h^2 with corners on S . The conclusions of 1. Theorem hold as long as H^2 is not involved. Let X_h be a finite-dimensional subspace of $H^1(S_h)$.

2. THEOREM:

a) $\partial S_h \neq \emptyset$. For every $f_h \in L^2(S_h)$ there exists a unique weak solution $u_h \in X_h \cap \overset{\circ}{H}{}^1(S_h)$ of

$$-\Delta_{S_h} u_h = f_h \quad \text{on} \quad S_h , \quad u_h = 0 \quad \text{on} \quad \partial S_h .$$

b) $\partial S_h = \emptyset$. For every $f_h \in L^2(S_h)$ with $\int_{S_h} f_h \, do_h = 0$ there exists a weak solution $u_h \in X_h$ of

$$-\Delta_{S_h} u_h = f_h \quad \text{on} \quad S_h$$

which is unique up to a constant.

The proof is a simple application of the usual Hilbert space methods.

§ 4. PROJECTION

We shall first have a look at the case where $S \in C^3$ is approximated by a polyhedron $S_h \in C^{o,1}$ which is the union of triangles T_h with diameter $\leq c_1 h$ and inner radius $\geq c_2 h$ and corners on S. Refering to our considerations in 2. we define

$$T = \{a(x) \in S \mid x \in T_h\}$$

(see figure 1.)

Fig. 1

In order to compare the discrete solution on S_h with the continuous solution on S we lift a function v_h defined on S_h onto S by

(2) $$v_h(x) = v(x-d(x)n(x)) \qquad (x \in T_h) .$$

<u>3. LEMMA</u>

$$\frac{1}{c} \|v_h\|_{L^2(T_h)} \leq \|v\|_{L^2(T)} \leq c \|v_h\|_{L^2(T_h)}$$

$$\frac{1}{c} |v_h|_{H^1(T_h)} \leq |v|_{H^1(T)} \leq c |v_h|_{H^1(T_h)}$$

$$|v_h|_{H^2(T_h)} \leq c \left[|v|_{H^2(T)} + h|v|_{H^1(T)} \right]$$

<u>Proof:</u> It is obvious that for $\mu_h = do / do_h$ we have

$$0 < \frac{1}{c} \leq \mu_h \leq c < \infty .$$

Thus

$$\frac{1}{c} \|v\|_{L^2(T)} \leq \|v_h\|_{L^2(T_h)} \leq c \|v\|_{L^2(T)} .$$

On each triangle with normal n_h

$$\nabla_{S_h} v_h = \nabla v_h - (n_h \cdot \nabla v_h) n_h$$
$$= P_h \nabla v_h$$

with $P_{hik} = \delta_{ik} - n_{hi} n_{hk} \quad (i,k = 1,2,3)$, and

$$\nabla v_h = (P - dH) \nabla v$$

where $P_{ik} = \delta_{ik} - n_i n_k$ and $H_{ik} = d_{x_i x_k} = n_{i x_k} = n_{k x_i}$. But since $PH = HP = H$ we get

$$\nabla_{S_h} v_h = P_h(I - dH)\, P\nabla v = P_h(I - dH)\, \nabla_S v\ .$$

Because of $|n - n_h| \le ch$ and $|d| \le ch^2$, for $h \le h_o$:

$$|\nabla_S v \cdot n_h| \le \tfrac{1}{2} |\nabla_S v| ,$$

$$\frac{1}{c} |\nabla_S v| \le |\nabla_{S_h} v_h| \le c\, |\nabla_S v|$$

on T , i.e.

$$\frac{1}{c} |v|_{H^1(T)} \le |v_h|_{H^1(T_h)} \le c\, |v|_{H^1(T)}$$

A short calculation delivers

$$|D_{S_h i} D_{S_h k} v_h| \le c \left[\sum_{|\mu|=2} |D_S^\mu v| + (|n_i - (n\cdot n_h)\, n_{hi}| + |d|)\, |\nabla_S v| \right]$$
$$\le c \left[\sum_{|\mu|=2} |D^\mu v| + h|\nabla_S v| \right]$$

which proves the Lemma.

§ 5. ENERGY ESTIMATE

We now are ready to prove the energy estimate. Let us first of all treat the case $\partial S = \emptyset$, $\partial S_h = \emptyset$. So we have got $u \in H^2(S)$, $u_h \in X_h \subset H^1(S_h)$, $f \in L^2(S)$, $\int_S f = 0$, $f_h \in L^2(S_h)$, $\int_{S_h} f_h = 0$ with

(3) $$\int_S \nabla_S u\, \nabla_S\, \varphi do = \int_S f\, \varphi do \qquad (\varphi \in H^1(S))$$

and

(4) $$\int_{S_h} \nabla_{S_h} u_h\, \nabla_{S_h} \varphi_h\, do_h = \int_{S_h} f_h\, \varphi_h\, do_h \qquad (\varphi_h \in X_h)$$

According to (2) we define

$$u_h(x) = U_h(x - d(x)n(x)) \qquad (x \in S_h)$$

and

$$\varphi_h(x) = \Phi_h(x - d(x)n(x)) \qquad (x \in S_h).$$

With these transformations we get from (4)

$$\int_S P_h\, (I - dH)\, \nabla_S\, U_h\, P_h\, (I - dH)\, \nabla_S\, \Phi_h \frac{1}{\mu_h}\, do = \int_S F_h\, \Phi_h\, do$$

where F_h is the transformed f_h times $1/\mu_h$. If we define $A_h = \frac{1}{\mu_h} P(I - dH)\, P_h(I - dH)P$, since P is a projection this reads

$$\int_S \nabla_S\, U_h\, \nabla_S\, \Phi_h\, do = \int_S F_h\, \Phi_h\, do + \int_S (A_h - I)\, \nabla_S\, U_h\, \nabla_S\, \Phi_h\, do$$

This together with (3) gives us

$$\int_S \nabla_S(u - U_h)\, \nabla_S \phi_h \, do = \int_S (I - A_h)\, \nabla_S U_h \, \nabla_S \phi_h \, do + \int_S (f - F_h)\, \phi_h \, do$$

for all projections $\phi_h \in H^1(S)$ of testfunctions $\varphi_h \in X_h$. So,

$$\begin{aligned} \|\nabla_S(u - U_h)\|^2_{L^2(S)} &= \int_S \nabla_S(u - U_h)\, \nabla_S(u - \phi_h)\, do \\ &+ \int_S (A_h - I)\, \nabla_S U_h \, \nabla_S(U_h - \phi_h)\, do \\ &- \int_S (f - F_h)\, (U_h - \phi_h)\, do \\ &\leq \|\nabla_S(u - U_h)\|_{L^2(S)} \, \|\nabla_S(u - \phi_h)\|_{L^2(S)} \\ &+ \|(A_h - I)P\|_{L^\infty(S)} \, \|\nabla_S U_h\|_{L^2(S)} \, \|\nabla_S(U_h - \phi_h)\| \\ &+ \|f - F_h\|_{L^2(S)} \, \|U_h - \phi_h\|_{L^2(S)} . \end{aligned}$$

Here it is important for later use that we have to estimate $\|(A_h - I)P\|_{L^\infty(S)}$ instead of $\|(A_h - I)\|_{L^\infty(S)}$.

Without loss of generality we assume that

$$\int_S U_h - \phi_h \, do = 0$$

and so, by Poincaré's inequality on S and elementary operations we achieve

$$\|\nabla_S(u - U_h)\|_{L^2(S)} \leq c \left[\|\nabla_S(u - \phi_h)\|_{L^2(S)} + \|(A_h - I)P\|_{L^\infty(S)} \, \|\nabla_S U_h\|_{L^2(S)} + \|f - F_h\|_{L^2(S)} \right]$$

Now we observe that

$$|1 - \mu_h| \leq ch^2 \ , \ |d| \leq ch^2$$

and

$$|(A_h - I)P| \leq ch^2 .$$

We shall prove the last inequality only.

$$\begin{aligned} (A_h - I)\, P &= \left(\frac{1}{\mu_h} P(I - dH)P_h(I - dH)P - I\right)P \\ &= (P(I - dh)P_h(I - dH)P - I)P + 0(h^2) \\ &= PP_hP - P + 0(h^2) \end{aligned}$$

$$|(A_h - I)P| \leq |n \wedge n_h|^2 + ch^2 \leq ch^2 .$$

In addition to that we have as an a priori bound from the discrete problem

$$\|\nabla_S U_h\|_{L^2(S)} \leq c \, \|f_h\|_{L^2(S_h)} .$$

This altogether implies the inequality

(5) $$\|\nabla_S(u - U_h)\|_{L^2(S)} \leq c \left[\inf \|\nabla_S(u - \Phi_h)\|_{L^2(S)} + \|f_h\|_{L^2(S_h)} h^2 + \|f - F_h\|_{L^2(S)} \right].$$

Now we have to make clear which f_h we shall chose with respect to f. The most simple choice would be to take

$$f_h(x) = f(x - d(x)n(x))\mu_h(x) \qquad (x \in S_h) ,$$

but numerically it is not easy to compute μ_h. So, let us take

(6) $$f_h = \tilde{f}_h - \int_{S_h} \tilde{f}_h \, do_h$$

where $\tilde{f}_h$ is the lifted f. But then it is clear that in (5)

$$\|f_h\|_{L^2(S_h)} \leq c\|f\|_{L^2(S)}$$

and

$$\|f - F_h\|_{L^2(S)} \leq ch^2 \|f\|_{L^2(S)} .$$

Up to now we have proved:

4. LEMMA:

Let $\partial S = \emptyset$, $S \in C^3$. If u is a continuous solution as in 1. Theorem b and u_h is a discrete solution as in 2. Theorem b with respect to f_h defined in (6), then

(7) $$|u - U_h|_{H^1(S)} \leq c \left[\inf_{\Phi_h \in Y_h} |u - \Phi_h|_{H^1(S)} + h^2\|f\|_{L^2(S)} \right]$$

where U_h is defined by

$$u_h(x) = U_h(x - d(x)n(x)) \qquad (x \in S_h)$$

and $$Y_h = \{\Phi_h(x - d(x)n(x)) = \varphi_h(x) \quad (x \in S_h) , \varphi_h \in X_h\} .$$

This means that we have estimated the consistency error which stems from the approximation of S by S_h. It remains to define an interpolation operator from $H^2(S)$ to Y_h.

5. LEMMA:

For $S \in C^3$, $\partial S = \emptyset$ let

$$X_h = \{\varphi_h : S_h \longrightarrow \mathbb{R} \mid \varphi_{h|T_h} \text{ linear polynomial}, \ \varphi_h \in C^0(S_h)\}$$

and Y_h the transformed space as in 4. Lemma. Then for given

$u \in H^2(S)$ there exists a unique $I_h u \in Y_h$ such that

$$|u - I_h u|_{H^1(S)} \leq ch \left[|u|_{H^2(S)} + h|u|_{H^1(S)} \right] .$$

Proof: According to Sobolev's theorem u is in $C^0(S)$, and so the linear interpolation $\tilde{I}_h u \in X_h$ is well defined by

$$\tilde{I}_h u (a_j) = u(a_j)$$

where a_j $(j=1,\ldots,N)$ are the nodes of S_h . It is well known, [C], that for $\tilde{u} = u$ lifted onto S_h :

$$|\tilde{u} - \tilde{I}_h \tilde{u}|_{H^1(T_h)} \leq ch |\tilde{u}|_{H^2(T_h)} .$$

But with 3. Lemma this implies

$$\begin{aligned} |u - I_h u|_{H^1(T)} &\leq c |\tilde{u} - \tilde{I}_h \tilde{u}|_{H^1(T_h)} \\ &\leq ch |\tilde{u}|_{H^2(T_h)} \\ &\leq ch \left[|u|_{H^2(T)} + h |u|_{H^1(T)} \right] . \end{aligned}$$

Let us summarize the results.

6. LEMMA:

Let the situation be as in 4. Lemma. Then

(8) $$|u - U_h|_{H^1(S)} \leq ch \|f\|_{L^2(S)}$$

if X_h is as in 5. Lemma.

§ 6. L^2-ESTIMATE AND RESULT

We employ the Aubin-Nitsche-trick in order to get quadratic asymptotic convergence in the $L^2(S)$-norm. Let us confine ourselves to surfaces S without boundary.

7. LEMMA:

Let $\partial S = \emptyset$. Then

$$\|u - U_h\|_{L^2(S)/\mathbb{R}} \leq ch^2$$

Proof: We solve the problem

$$-\Delta_S v = u - U_h - m \quad \text{on} \quad S , \quad \int_S v do = 0 ,$$

where

$$m = \int_S u - U_h \, do .$$

Due to 1. Theorem there exists a unique solution $v \in H^2(S)$ and

$$\|v\|_{H^2(S)} \leq c \, \|u - U_h\|_{L^2(S)/\mathbb{R}} .$$

$$\begin{aligned} \|u - U_h\|^2_{L^2(S)/\mathbb{R}} &= \int_S \nabla_S (u - U_h) \, \nabla_S \, v \, do \\ &= \int_S \nabla_S (u - U_h) \, \nabla_S (v - I_h v) \, do \\ &\quad + \int_S (I - A_h) \, P\nabla_S \, U_h \, \nabla_S \, I_h \, v \, do \\ &\leq \|\nabla_S (u - U_h)\|_{L^2(S)} \, \|\nabla_S (v - I_h v)\|_{L^2(S)} \\ &\quad + ch^2 \, \|f\|_{L^2(S)} \, \|u - U_h\|_{L^2(S)/\mathbb{R}} \\ &\leq ch^2 \, \|u - U_h\|_{L^2(S)/\mathbb{R}} . \end{aligned}$$

We remark that we never seriously used that S had no boundary. This means that the energy estimate (8) remains valid for $\partial S \neq 0$ as long as $u \in H^2(S)$ although $\partial S \in C^{0,1}$ only. S is the projection of a polyhedron S_h and thus has piecewise smooth boundary. We assume in the case $\partial S \neq \emptyset$ that for every $f \in L^2(S)$ the weak solution is in $H^2(S)$ and the corresponding a priori bound holds. So we can summarize our results.

8. THEOREM

Let $\partial S = \emptyset$, $S \in C^3$. If u is a continuous solution as in 1. Theorem b and u_h is a discrete solution with respect to f_h defined in (6), then for U_h as in 4. Lemma

$$\|u - U_h\|_{L^2(S)/\mathbb{R}} + h \, |u - U_h|_{H^1(S)} \leq ch^2 .$$

If $S \in C^3$, $\partial S \neq \emptyset$ is the projection of a polyhedron such that for every $f \in L^2(S)$ the bound $\|u\|_{H^2(S)} \leq c \, \|f\|_{L^2(S)}$ holds, then

$$\|u - U_h\|_{L^2(S)} + h \, |u - U_h|_{H^1(S)} \leq ch^2 .$$

§ 7. NUMERICAL EXAMPLE

To illustrate the method and to assess the sharpness of the convergence rate given in the preceeding section, we present numerical results for a simple test problem. The surface S is taken to be

$$S = \{x \in \mathbb{R}^3 \mid (x_1 - x_3^2)^2 + x_2^2 + x_3^2 = 1\}$$

and we consider the problem $-\Delta_S u = f$ on S whose exact solution is given by $u(x) = x_1 x_2$. Let us remark that the right hand side f is not that simple since

$$f = -\nabla_S \cdot v \ , \ v = \nabla_S u = \nabla u - (\nabla u \cdot n) n$$

$$\nabla_S \cdot v = \nabla \cdot v - \sum_{j=1}^{3} (\nabla v_j \cdot n) n_j$$

where

$$n(x) = (x_1 - x_3^2, x_2, x_3(1 - 2(x_1 - x_3^2))) \ / \ (1 + 4x_3^2(1 - x_1 - x_2^2))^{1/2}$$

We start with a very crude six-node approximation of S . If h_j is the largest diameter of the jth grid we determine the experimental order of convergence by

$$\ln \frac{E(h_j)}{E(h_{j+1})} \ / \ \ln \frac{h_j}{h_{j+1}} \qquad (j=1,\ldots,5)$$

where E is the relative error in the L^2-norm

$$E(h) = \|u - U_h\|_{L^2(S)} \ / \ \|u\|_{L^2(S)} \ .$$

The results are given in table 1.

triangulation level	nodes	triangles	h	relative L^2-error	experimental order of convergence
1	6	8	2.236	0.8120	
					1.06
2	18	32	1.399	0.4930	
					1.93
3	66	128	0.8426	0.1855	
					2.10
4	258	512	0.4613	0.5227 E-1	
					2.04
5	1026	2048	0.2384	0.1356 E-1	
					1.99
6	4098	8192	0.1233	0.3664 E-2	

Table 1 Results for the test problem

In order to give an impression of the discretization we plot the approximation surface S_h and some limes of the discrete solution on S_h in Figure 2

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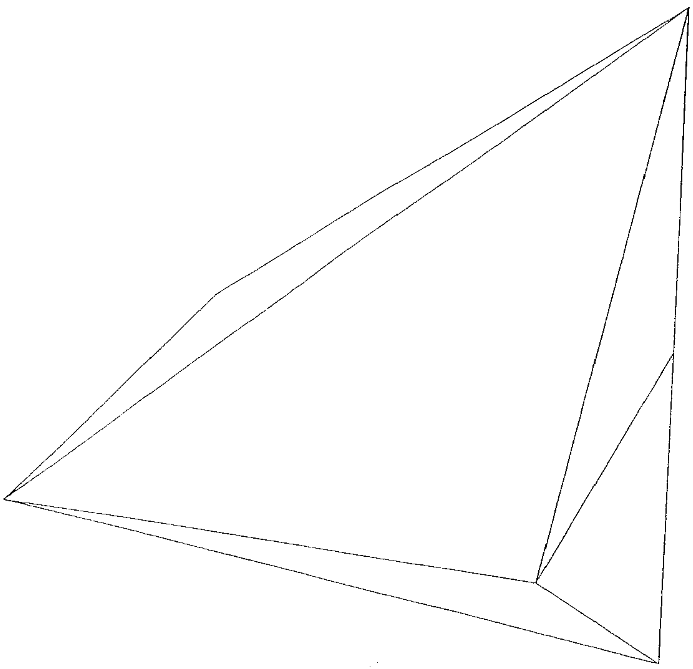

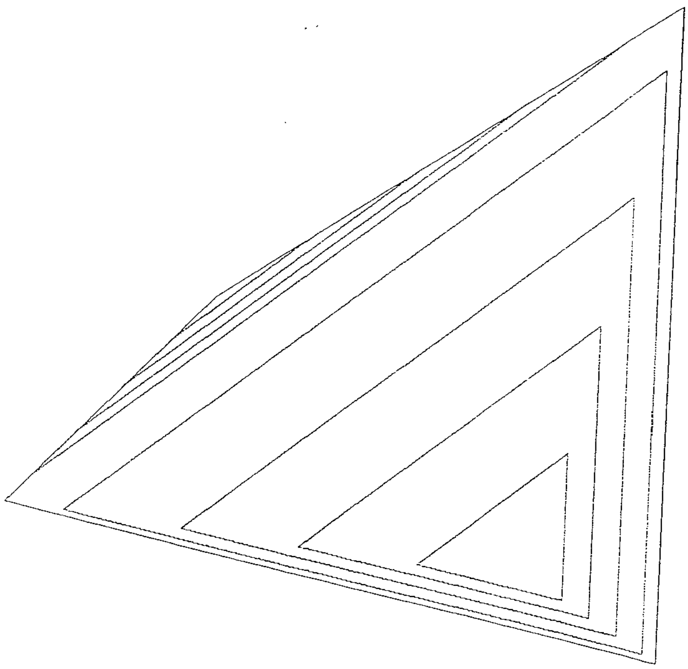

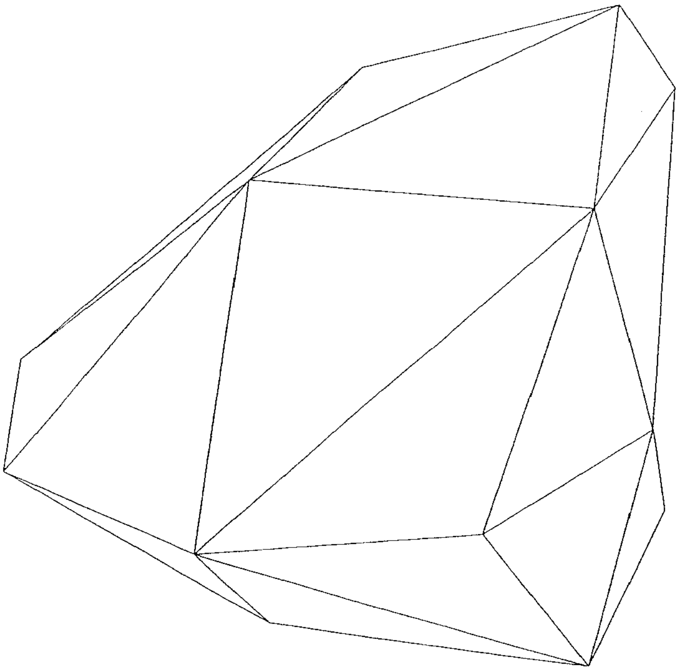

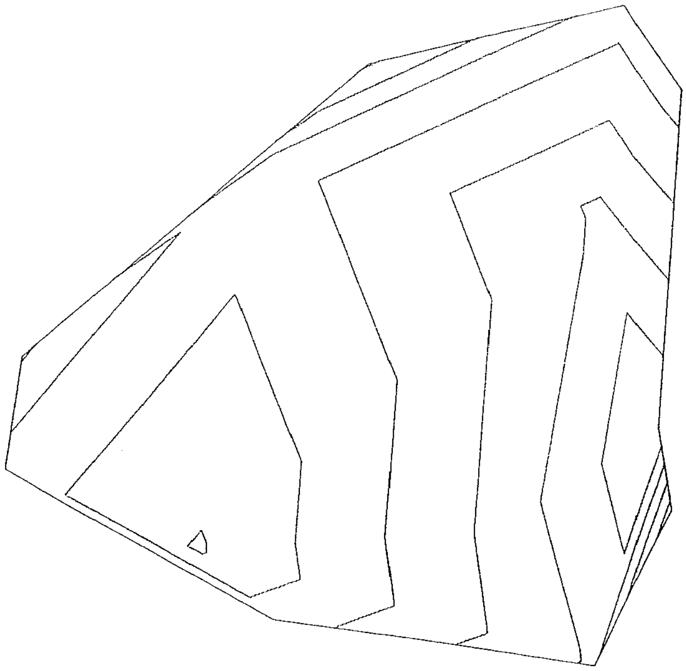

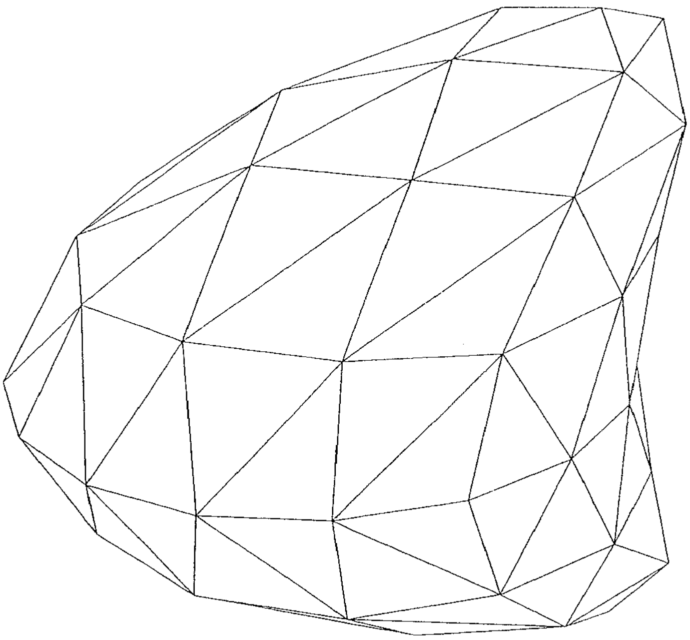

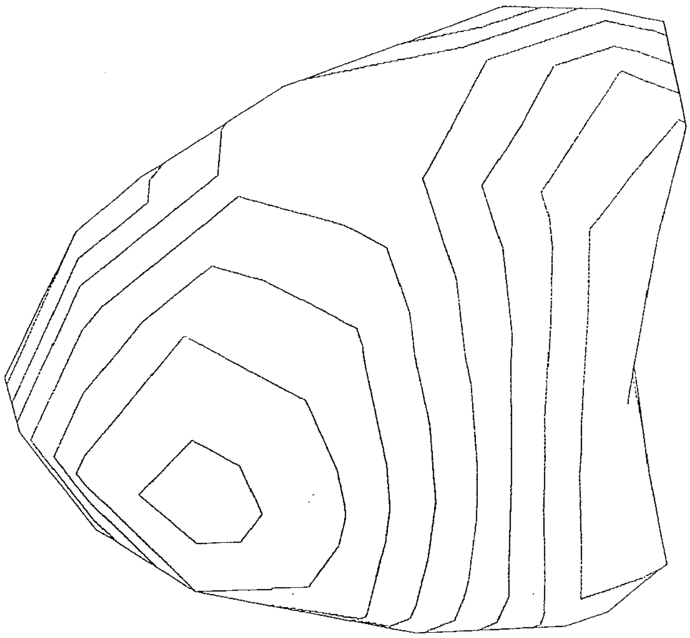

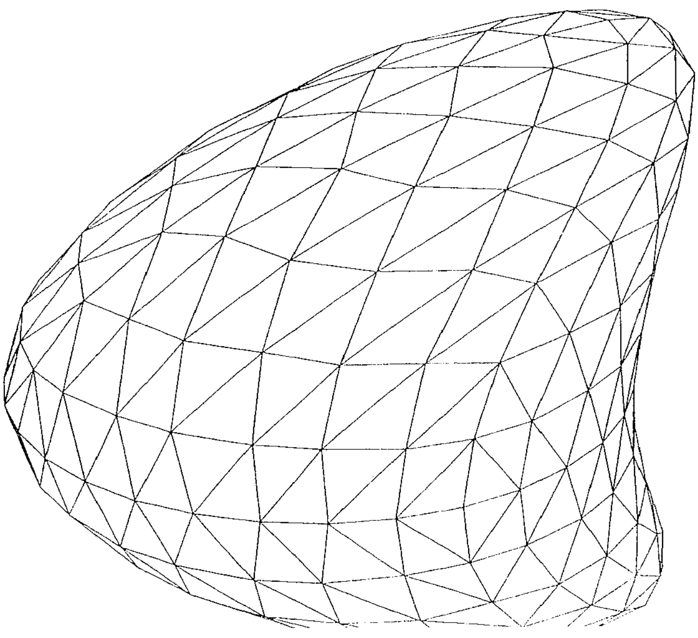

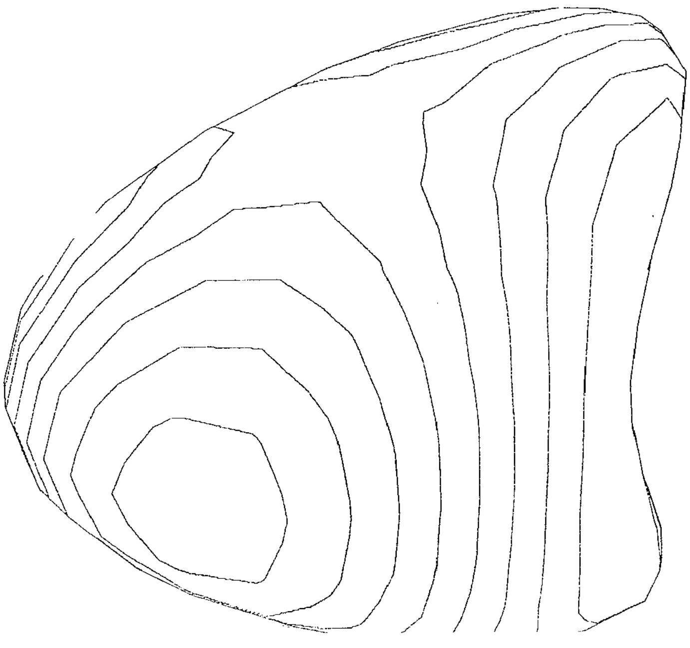

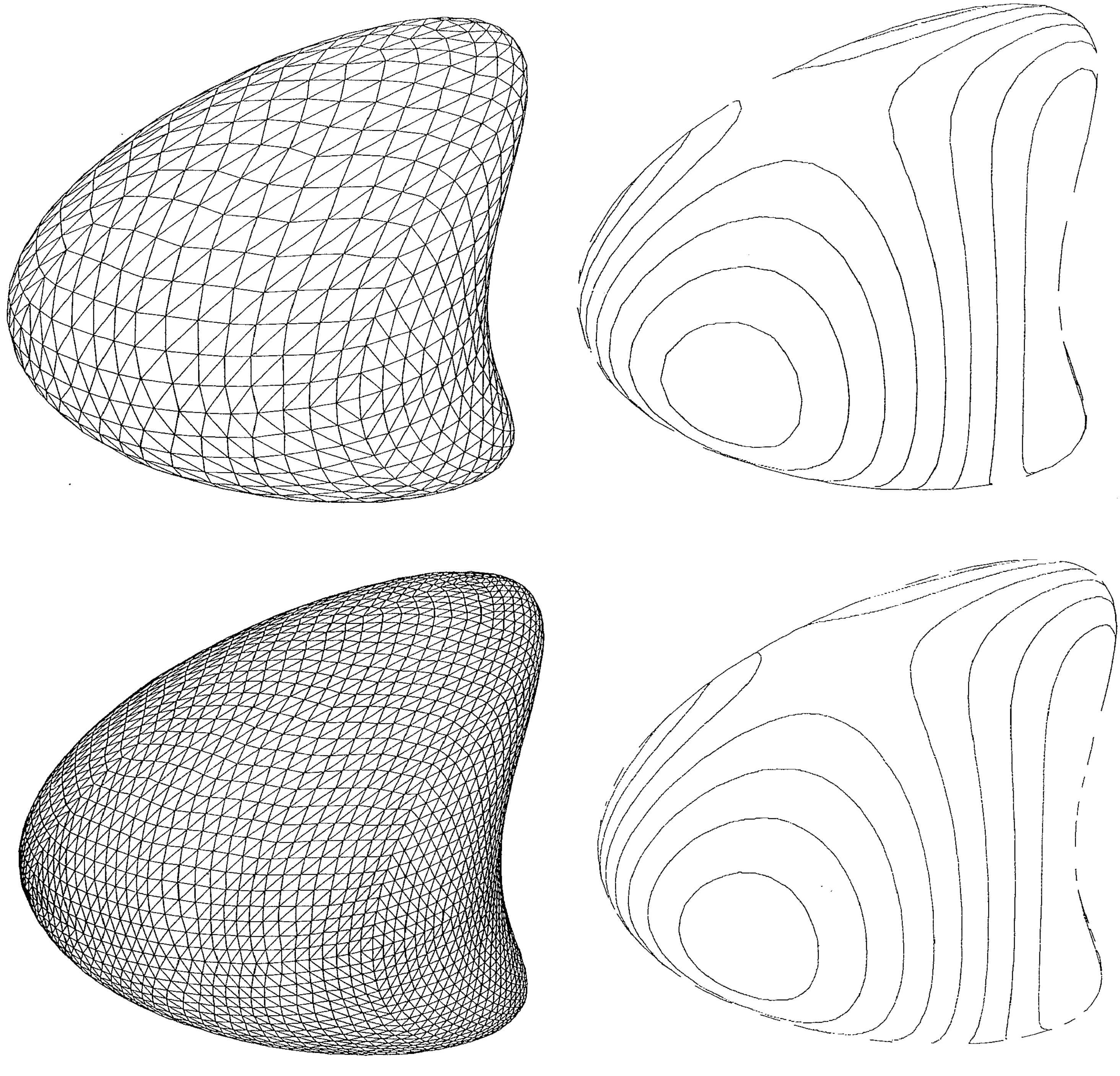

Fig. 2 . S_h and level lines of u_h on S_h for triangulations 1-6 .

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