



FFTs in Graphics and Vision

Rotational and Reflective
Symmetry Detection



Outline

Representation Theory

Symmetry Detection (1D)

Symmetry Detection (2D)



Representation Theory

Recall:

A group is a set of elements G with a binary operation (often denoted “.”) such that for all $f, g, h \in G$, the following properties are satisfied:

- Closure:

$$g \cdot h \in G$$

- Associativity:

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$

- Identity: There exists an identity element $1 \in G$ s.t.:

$$1 \cdot g = g \cdot 1 = g$$

- Inverse: Every element g has an inverse g^{-1} s.t.:

$$g \cdot g^{-1} = g^{-1} \cdot g = 1$$



Representation Theory

Observation 1:

Given a group $G = \{g_1, \dots, g_n\}$, for any $g \in G$, the (set-theoretic) map that multiplies the elements of G on the left by g is invertible.

(The inverse is the map multiplying the elements of G on the left by g^{-1} .)



Representation Theory

Observation 1:

In particular, this implies that the set $\{g \cdot g_1, \dots, g \cdot g_n\}$ is just a re-ordering of the set $\{g_1, \dots, g_n\}$.

Or more simply, $gG = G$.



Representation Theory

Observation 1:

In particular, this implies that the set $\{g \cdot g_1, \dots, g \cdot g_n\}$ is just a re-ordering of the set $\{g_1, \dots, g_n\}$.

Or more simply, $gG = G$.

Similarly, the set $\{(g_1)^{-1}, \dots, (g_n)^{-1}\}$ is just a re-ordering of the set $\{g_1, \dots, g_n\}$.

Or more simply, $G^{-1} = G$.



Representation Theory

Recall:

A Hermitian inner product is a map from $V \times V$ into the complex numbers that is:

1. Linear: For all $u, v, w \in V$ and any real scalar λ

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

2. Conjugate Symmetric: For all $u, v \in V$

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

3. Positive Definite: For all $v \in V$:

$$\langle v, v \rangle \geq 0$$

$$\langle v, v \rangle = 0 \iff v = 0$$



Representation Theory

Observation 2:

Given a Hermitian inner-product space V , and given a set of vectors $\{v_1, \dots, v_n\} \subset V$, the vector minimizing the sum of squared distances is the average of $\{v_1, \dots, v_n\}$:

$$\frac{1}{n} \sum_{k=1}^n v_k = \arg \min_{v \in V} \left(\sum_{k=1}^n \|v - v_k\|^2 \right)$$



Representation Theory

Recall:

A unitary representation of a group G on a Hermitian inner-product space V is a map ρ that sends every element in G to an orthogonal transformation on V , satisfying:

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h)$$

for all $g, h \in G$.



Representation Theory

Definition:

We say that a vector $v \in V$ is invariant under the action of G if G sends v back to itself:

$$\rho_g(v) = v$$

for all $g \in G$.



Representation Theory

Notation:

We denote by V_G the set of vectors in V that are invariant under the action of G :

$$V_G = \{v \in V \mid \rho_g(v) = v, \forall g \in G\}$$

Representation Theory



Observation 3:

Note that the set V_G is a vector sub-space of V .



Representation Theory

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If $v, w \in V_G$, then for any $g \in G$, we have:

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Representation Theory

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$$\rho_g(v) = v \quad \text{and} \quad \rho_g(w) = w$$

But then we have:

$$\begin{aligned} \rho_g(v + w) &= \rho_g(v) + \rho_g(w) \\ &= v + w \end{aligned}$$



Representation Theory

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But then we have:

$$\begin{aligned} \rho_g(v + w) &= \rho_g(v) + \rho_g(w) \\ &= v + w \end{aligned}$$

So $v + w \in V_G$ as well.



Representation Theory

Observation 4:

Given a finite group G and given vector $v \in V$, the vector obtained by averaging over G :

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is invariant under the action of G .



Representation Theory

Observation 4:

To see this, let h be any element in G .

We would like to show that h maps the average back to itself:

$$\rho_h \left(\text{Average}(v, G) \right) = \text{Average}(v, G)$$



Representation Theory

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We would like to show that h maps the average back to itself:

$$\rho_h \left(\text{Average}(v, G) \right) = \text{Average}(v, G)$$

Expanding the left hand side we get:

$$\rho_h \left(\text{Average}(v, G) \right) = \rho_h \left(\frac{1}{|G|} \sum_{g \in G} \rho_g(v) \right)$$



Representation Theory

Observation 4:

$$\rho_h \left(\text{Average}(v, G) \right) = \rho_h \left(\frac{1}{|G|} \sum_{g \in G} \rho_g(v) \right)$$

By the linearity of the representation, we get:

$$\rho_h \left(\text{Average}(v, G) \right) = \frac{1}{|G|} \sum_{g \in G} \rho_h \left(\rho_g(v) \right)$$



Representation Theory

Observation 4:

$$\rho_h \text{Average}(v, G) \stackrel{\cong}{=} \frac{1}{|G|} \sum_{g \in G} \rho_{h \cdot g}(v)$$

Since the representation preserves the group structure, we get:

$$\rho_h \text{Average}(v, G) \stackrel{\cong}{=} \frac{1}{|G|} \sum_{g \in G} \rho_{h \cdot g}(v)$$



Representation Theory

Observation 4:

$$\rho_h \text{ Average}(v, G) \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} \rho_{h \cdot g}(v)$$

But this can be re-written as a summation over the set hG :

$$\rho_h \text{ Average}(v, G) \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{g \in hG} \rho_g(v)$$



Representation Theory

Observation 4:

$$\rho_h \text{Average}(v, G) \stackrel{\cong}{=} \frac{1}{|G|} \sum_{g \in hG} \rho_g(v)$$

And since $hG=G$, this implies that:

$$\begin{aligned} \rho_h \text{Average}(v, G) &\stackrel{\cong}{=} \frac{1}{|G|} \sum_{g \in G} \rho_g(v) \\ &= \text{Average}(v, G) \end{aligned}$$



Representation Theory

Observation 5:

Given a finite group G and given a vector $v \in V$, the average of v over G is the closest G -invariant vector to v :

$$\text{Average}(v, G) = \arg \min_{v_0 \in V_G} \|v_0 - v\|^2$$



Representation Theory

Observation 5:

$$\text{Average}(v, G) = \arg \min_{v_0 \in V_G} \|v_0 - v\|^2$$

Since v_0 is invariant under the action of G , we can write out the squared distances as:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|\rho_g(v_0) - v\|^2$$



Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|\rho_g(v_0) - v\|^2$$

Since the representation is unitary, we can re-write this as:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - (\rho_g)^{-1}(v)\|^2$$



Representation Theory

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$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - (\rho_g)^{-1}(v)\|^2$$

Since the representation preserves the group structure, we get:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_{g^{-1}}(v)\|^2$$



Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_{g^{-1}}(v)\|^2$$

Re-writing this as a summation over G^{-1} , we get:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G^{-1}} \|v_0 - \rho_g(v)\|^2$$



Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G^{-1}} \|v_0 - \rho_g(v)\|^2$$

And finally, using the fact that the set G^{-1} is just a re-ordering of the set G , we get:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g(v)\|^2$$



Representation Theory

Observation 5:

$$\|v_0 - v\|^2 = \frac{1}{|G|} \sum_{g \in G} \|v_0 - \rho_g(v)\|^2$$

Thus, v_0 is the G -invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\rho_{g_1}(v), \dots, \rho_{g_n}(v)$$



Representation Theory

Observation 5:

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Thus, v_0 is the G -invariant vector minimizing the squared distance to v if and only if it minimizes the sum of squared distances to the vectors:

$$\rho_{g_1}(v), \dots, \rho_{g_n}(v)$$

So v_0 must be the average of these vectors:

$$v_0 = \frac{1}{|G|} \sum_{g \in G} \rho_g(v) = \text{Average}(v, G)$$



Representation Theory

Note:

Since the average map:

$$\text{Average}(v, G) = \frac{1}{|G|} \sum_{g \in G} \rho_g(v)$$

is a linear map returning the closest G -invariant vector to v , the average map is just the projection map from V to V_G .



Outline

Representation Theory

Symmetry Detection (1D)

Symmetry Detection (2D)



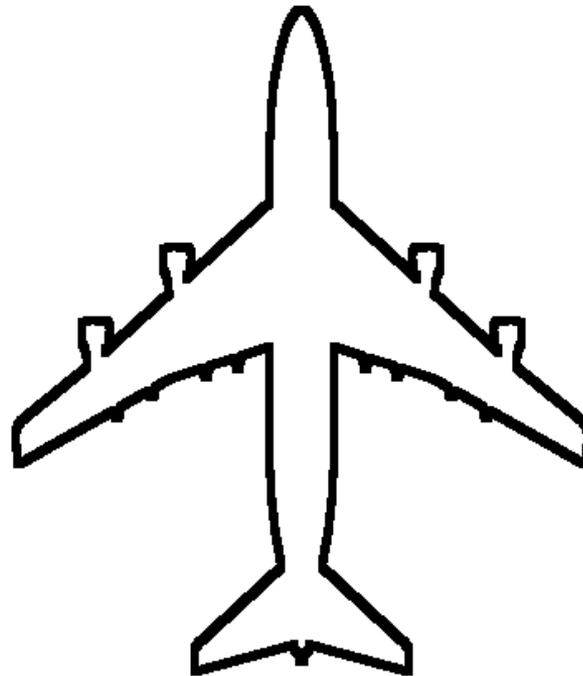
Symmetry Detection (1D)

What kind of reflective/rotational symmetry does the shape have?



Symmetry Detection (1D)

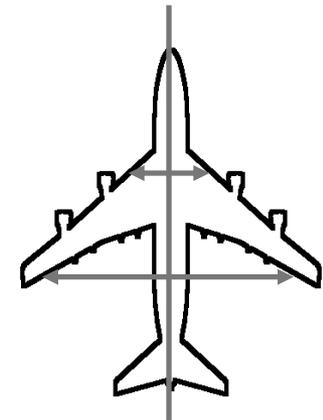
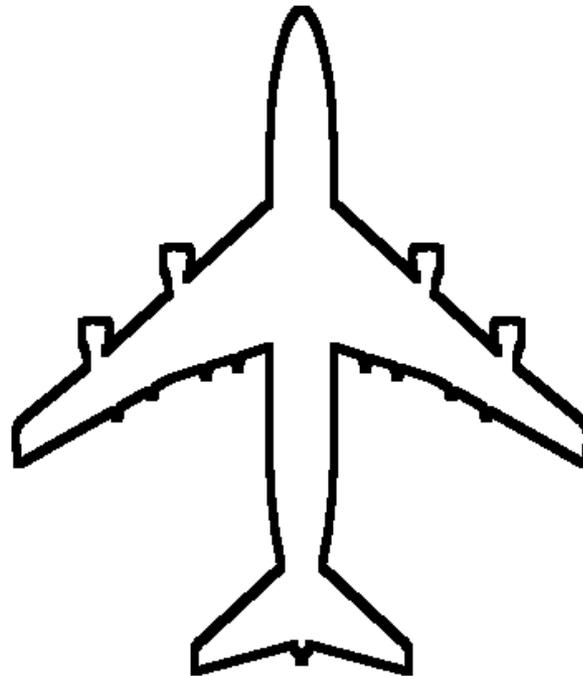
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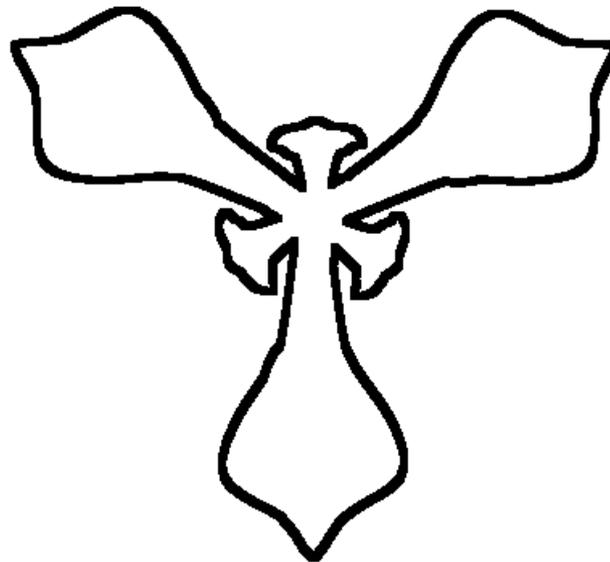


Reflective



Symmetry Detection (1D)

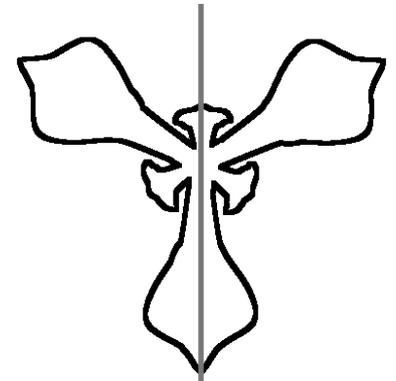
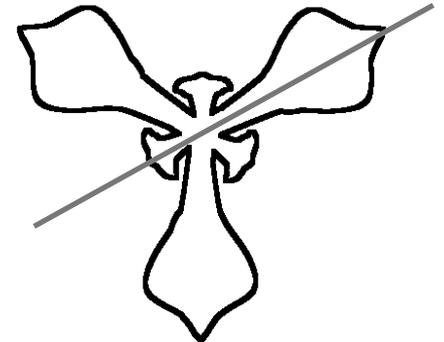
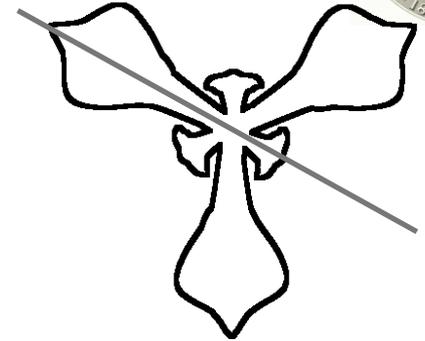
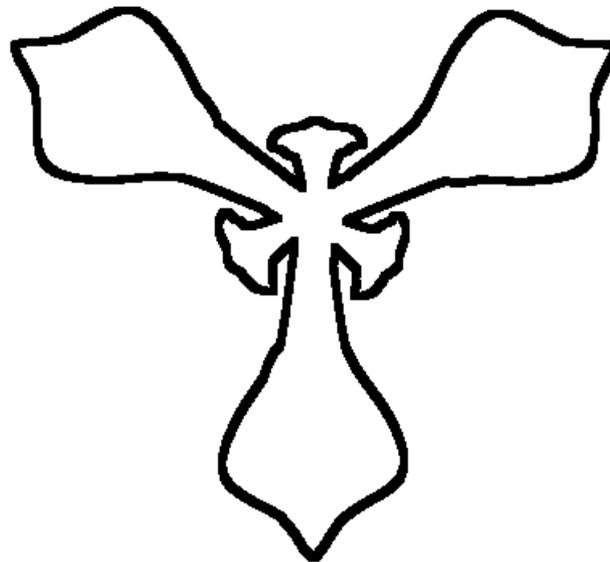
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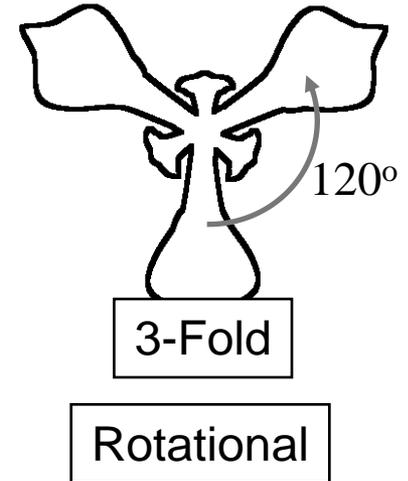
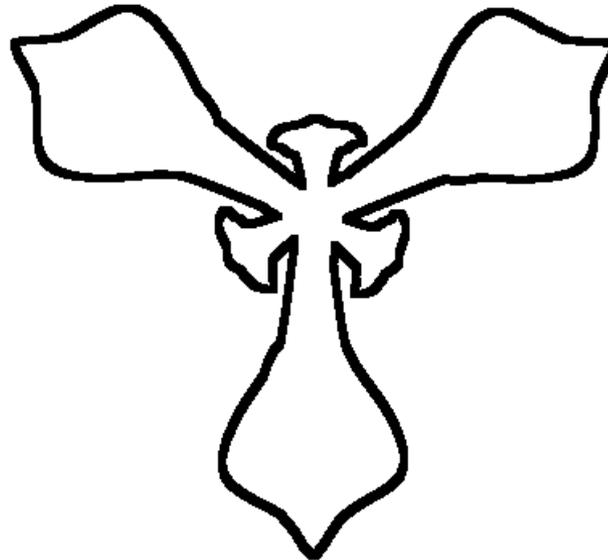


Reflective



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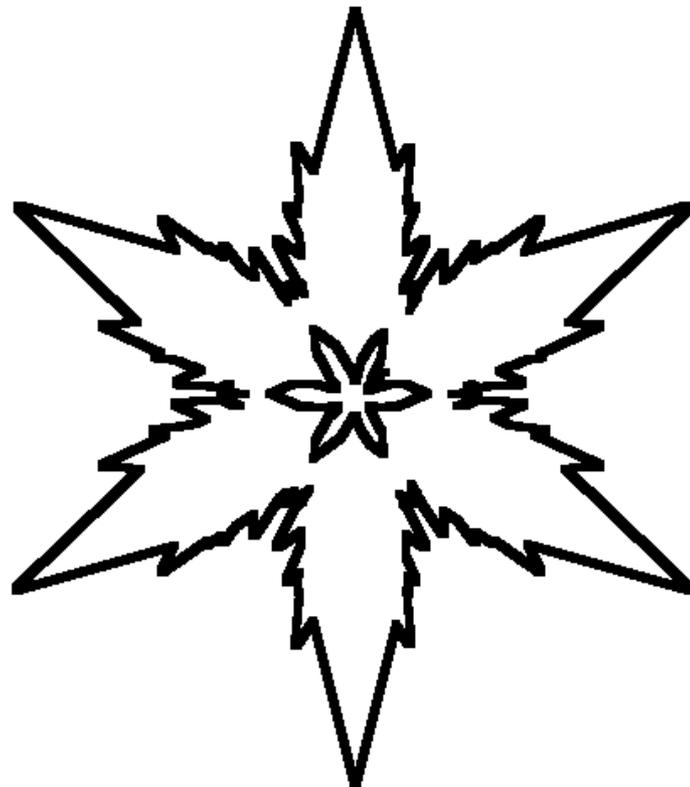
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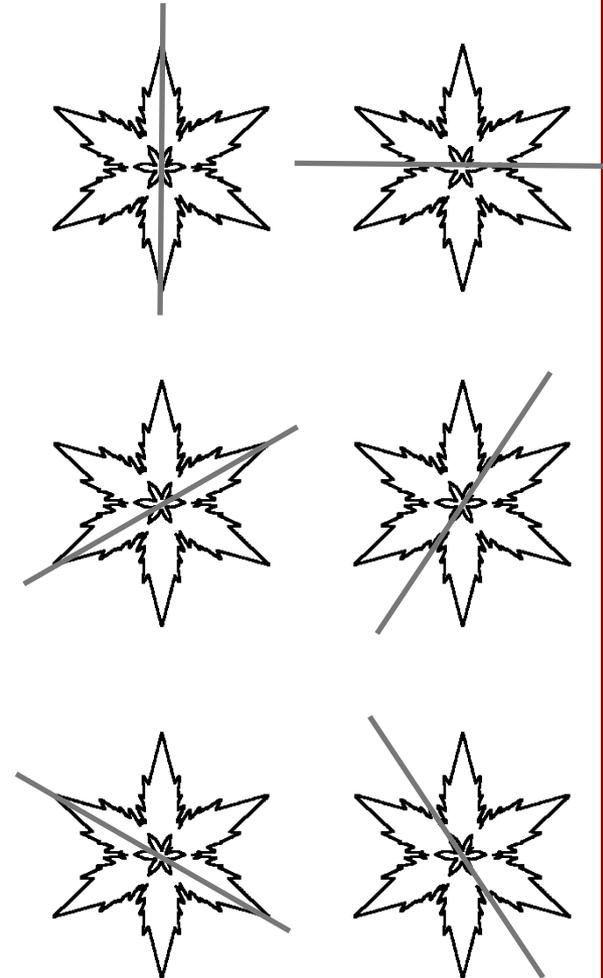
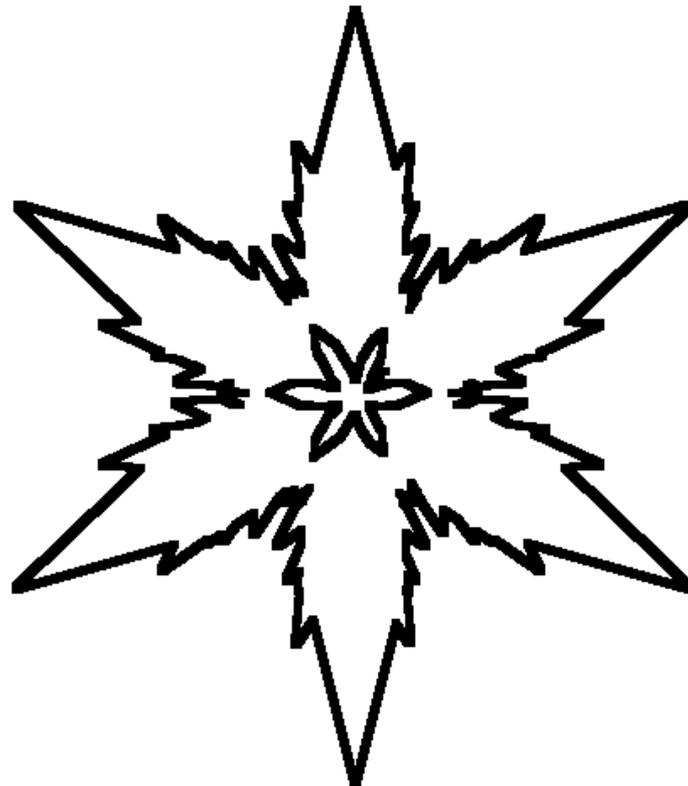
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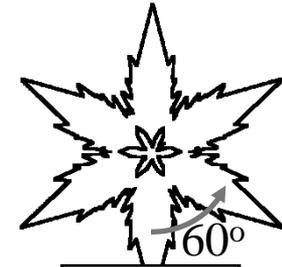
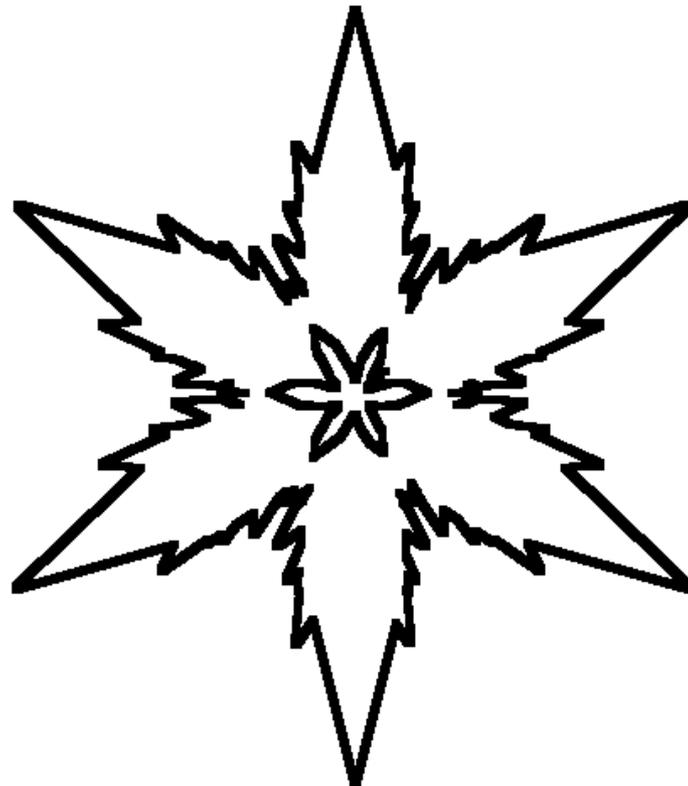


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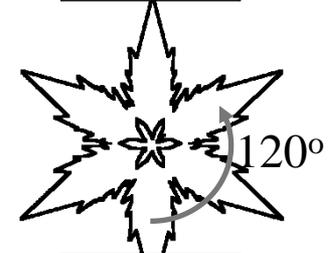


Symmetry Detection (1D)

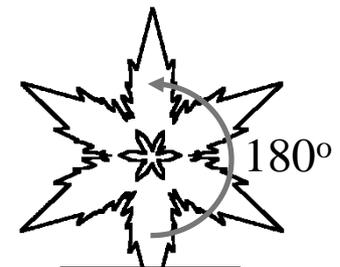
What kind of reflective/rotational symmetry does the shape have?



6-Fold



3-Fold



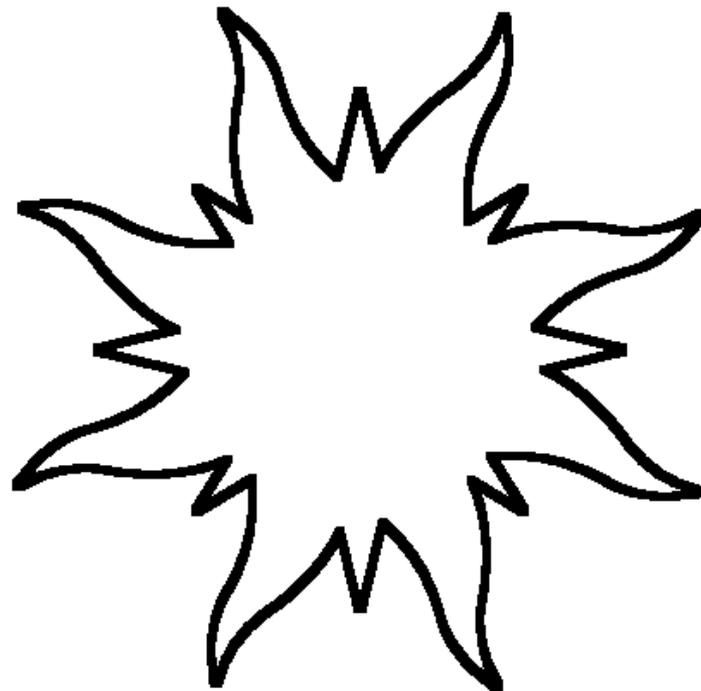
2-Fold

Rotational



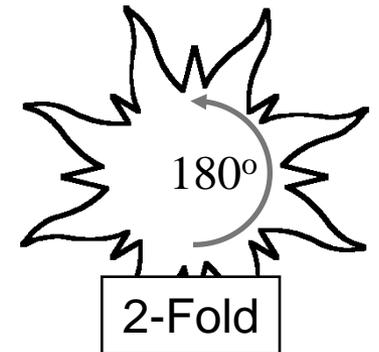
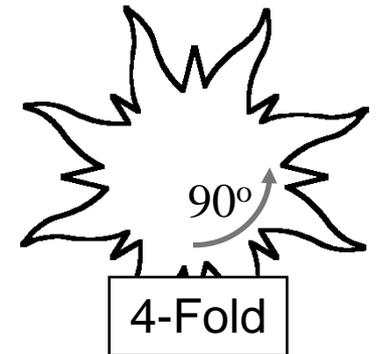
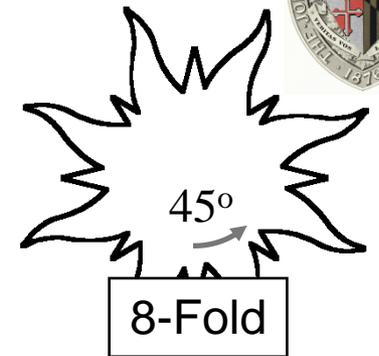
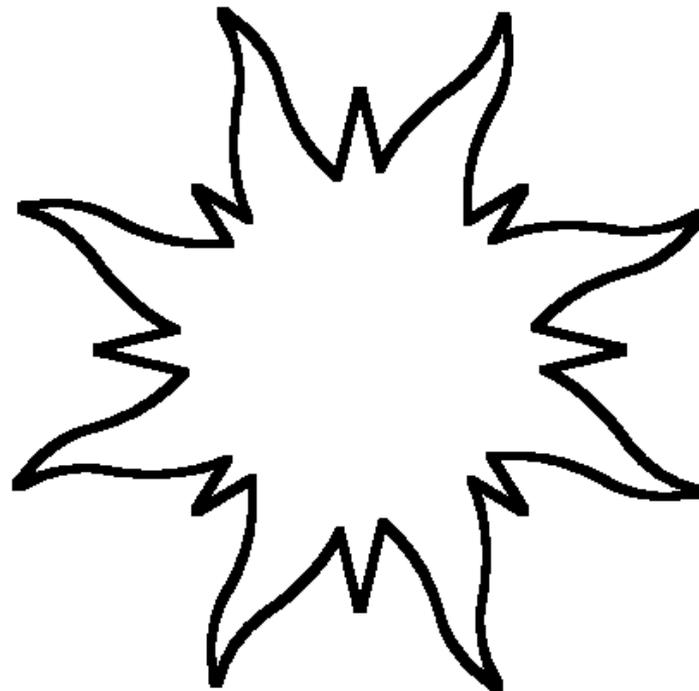
Symmetry Detection (1D)

What kind of reflective/rotational symmetry does the shape have?



Symmetry Detection (1D)

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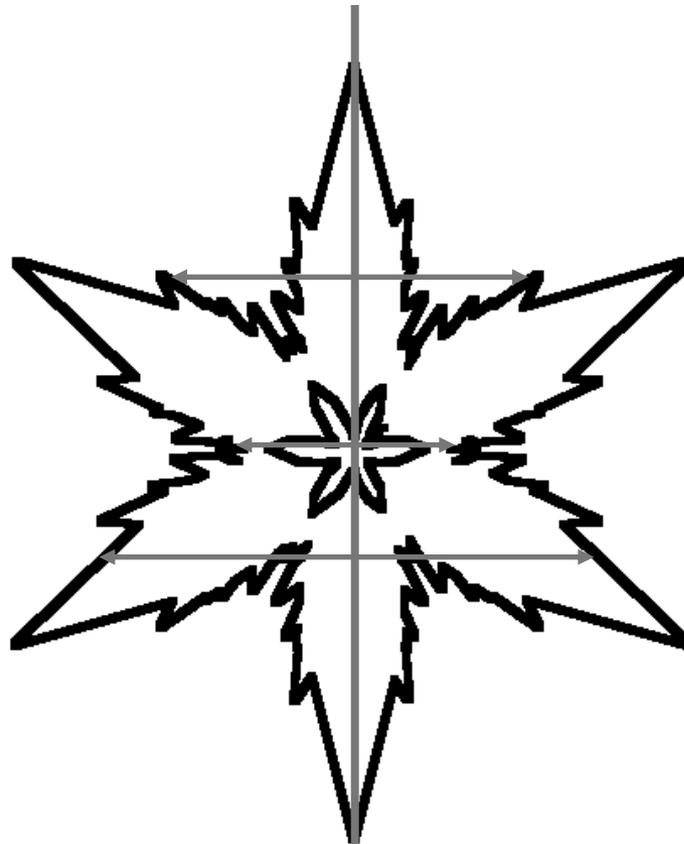


Rotational



Symmetry Detection (1D)

A shape is *symmetric* if there exists a group of transformations that leaves the shape unchanged.

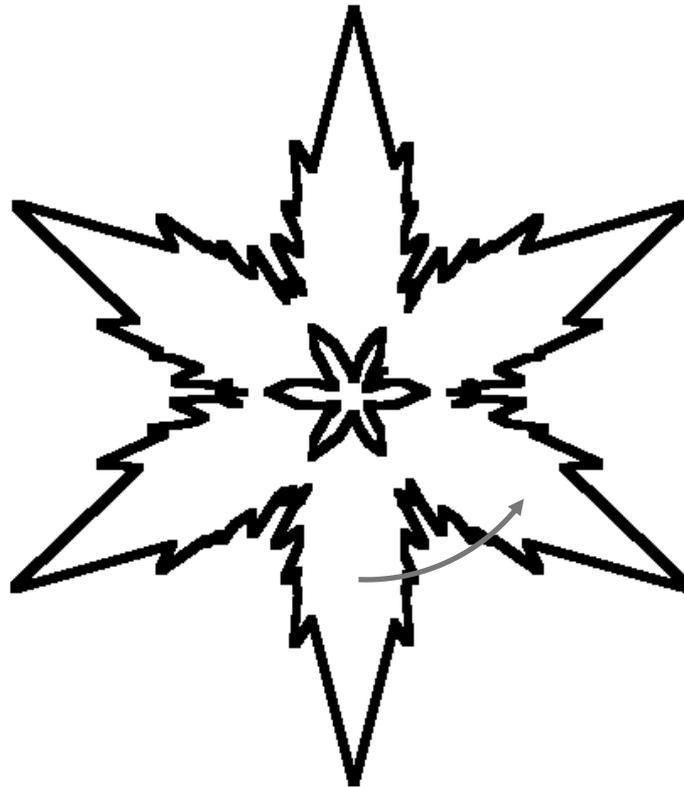


Group: {Identity, Reflection about the vertical axis}



Symmetry Detection (1D)

A shape is *symmetric* if there exists a group of transformations that leaves the shape unchanged.



Group: {Identity, 60° Rotation, 120° Rotation, 180° Rotation, 240° Rotation, 300° Rotation}



Symmetry Detection (1D)

A shape is *symmetric* if there exists a group of transformations that leaves the shape unchanged.

For a reflectively symmetric shape, the group is defined by the axis of reflective symmetry.

For a rotationally symmetric shape, the group is defined by the order of the rotational symmetry:

k -fold \Leftrightarrow unchanged by $n \cdot (360^\circ/k)$ rotations

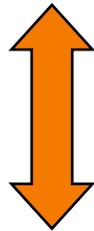


Symmetry Detection (1D)

Approach:

1. By considering a representation of a shape by a circular function, we transform the problem:

Does the shape have symmetries?



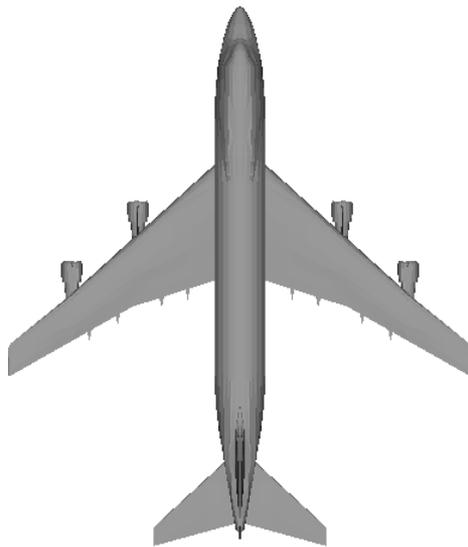
Does the function have symmetries?



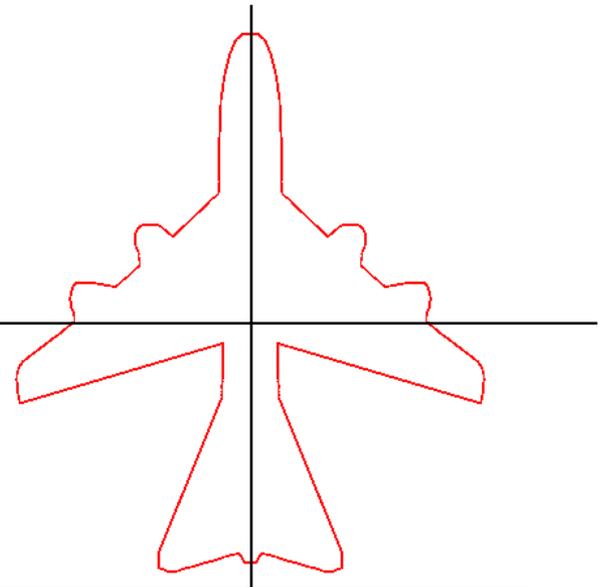
Symmetry Detection (1D)

Approach:

1. By considering a representation of a shape by a circular function, we transform the problem:



Shape



Circular Extent Function



Symmetry Detection (1D)

Approach:

1. By considering a representation of a shape by a circular function, we transform the problem to the challenge of detecting the symmetries of a circular function.
2. To be robust to noise, sampling error, etc., we will focus on answering the question:

How much of each type of symmetry does the shape have?



Symmetry Detection (1D)

Goal:

Given a circular function and a symmetry group, we would like to determine how symmetric the function is.



Symmetry Detection (1D)

Goal:

Given a circular function and a symmetry group, we would like to determine how symmetric the function is.

We have:

- A vector space V (the space of circular functions)
- A group G acting on V (the symmetry group)

We want the size of the projection of a vector on the G -invariant subspace V_G :

$$\text{Sym}^2(v, G) = \|\text{Average}(v, G)\|^2$$



Outline

Representation Theory

Symmetry Detection (1D)

- Rotations
- Reflections

Symmetry Detection (2D)

Rotational Symmetry Detection (1D)



Given the group of k -fold rotational symmetries:

$$G_k = \left\{ \text{Identity, Rotation by } \frac{2\pi}{k}, \dots, \text{Rotation by } \frac{(k-1) \cdot 2\pi}{k} \right\}$$

And given a circular array $f[]$, we would like to compute:

$$\text{Sym}^2(f[], G_k) = \left\| \text{Average}(f[], G_k) \right\|^2$$



Rotational Symmetry Detection (1D)

Let's assume that only the l -th Fourier coefficient of $f[\]$ is non-zero:

$$f[\] = \hat{f}[l]v_l[\]$$

where $v_l[\]$ are the discrete samples of the complex exponentials at n regularly spaced positions:

$$v_l[\] = \sqrt{\frac{1}{n}} \left(e^{il2\pi 0/n}, e^{il2\pi 1/n}, \dots, e^{il2\pi(n-2)/n}, e^{il2\pi(n-1)/n} \right)$$

Rotational Symmetry Detection (1D)



Let's assume that only the l -th Fourier coefficient of $f[\]$ is non-zero:

$$f[\] = \hat{f}[l]v_l[\]$$

What is the average of $f[\]$ under the action of G_k ?



Rotational Symmetry Detection (1D)

Let's assume that only the l -th Fourier coefficient of $f[x]$ is non-zero:

$$f[x] = \hat{f}[l]v_l[x]$$

What is the average of $f[x]$ under the action of G_k ?

Recall that rotating the l -th Fourier basis vector by α is equivalent to multiplying it by $e^{-il\alpha}$, so the j -th element of G acts by multiplication by $e^{-ilj2\pi/k}$.

Rotational Symmetry Detection (1D)



Thus, we can write out the average of $f[l]$ under the action of G_k as:

$$\text{Average}(f[l], G_k) = \frac{1}{k} \sum_{j=0}^{k-1} e^{-ilj2\pi/k} \hat{f}[l] v_l[l]$$

Rotational Symmetry Detection (1D)



$$\text{Average}(f[], G_k) = \left(\frac{1}{k} \sum_{j=0}^{k-1} e^{-ilj2\pi/k} \right) \hat{f}[l]v_l[]$$

What is the value of the sum:

$$\frac{1}{k} \sum_{j=0}^{k-1} e^{-ilj2\pi/k}$$

Rotational Symmetry Detection (1D)



$$\text{Average}(f[], G_k) = \left(\frac{1}{k} \sum_{j=0}^{k-1} e^{-ilj2\pi/k} \right) \hat{f}[l]v_l[]$$

What is the value of the sum:

$$\frac{1}{k} \sum_{j=0}^{k-1} e^{-ilj2\pi/k}$$

If we set α to be the angle:

$$\alpha = l2\pi / k$$

Then this is just the sum of the unit-norm complex numbers with angles $\{0, \alpha, \dots, (k-1)\alpha\}$.

Rotational Symmetry Detection (1D)

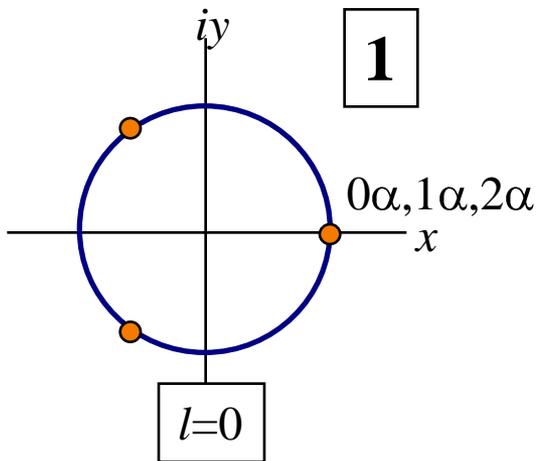


Example ($k=3$): $\frac{1}{3} \sum_{j=0}^2 e^{-i(j\alpha)}$ $\alpha = 12\pi / 3$



Rotational Symmetry Detection (1D)

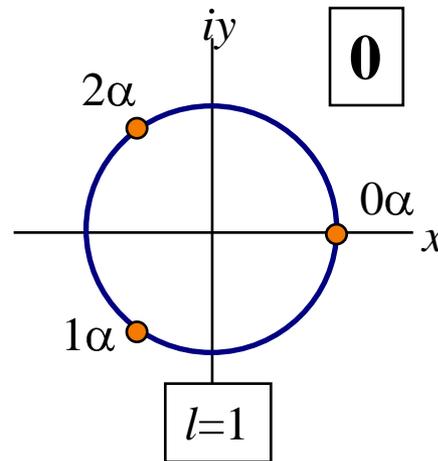
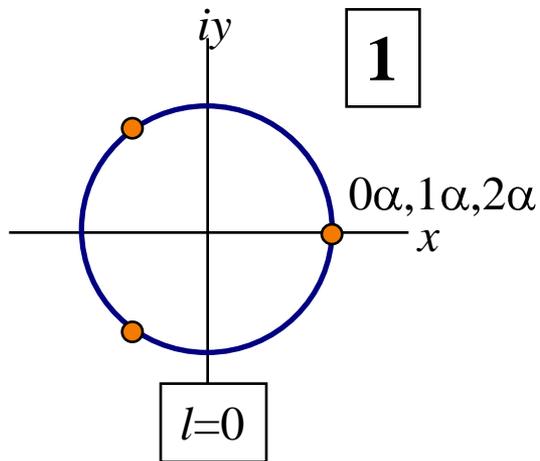
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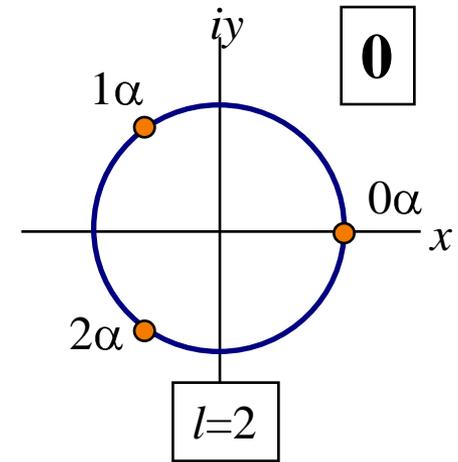
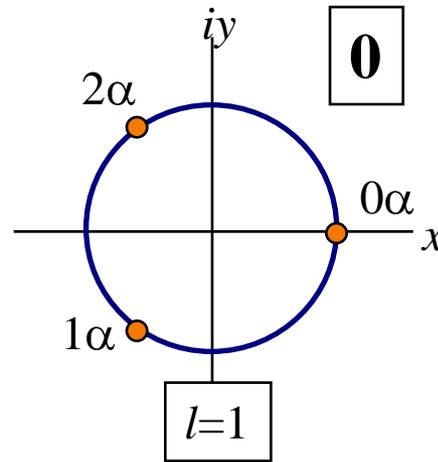
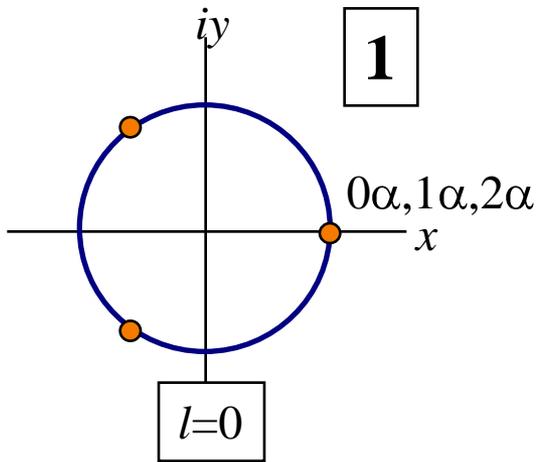
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Rotational Symmetry Detection (1D)

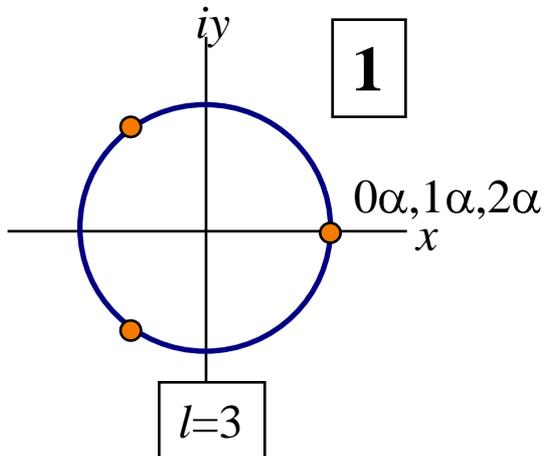
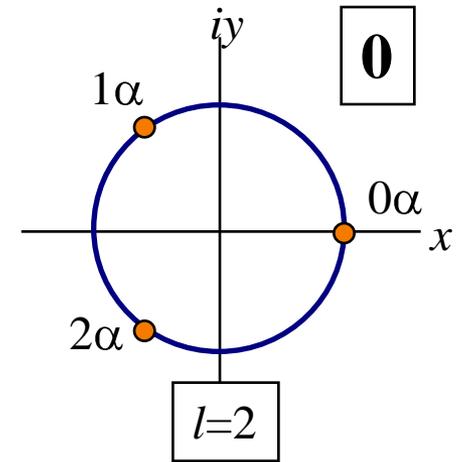
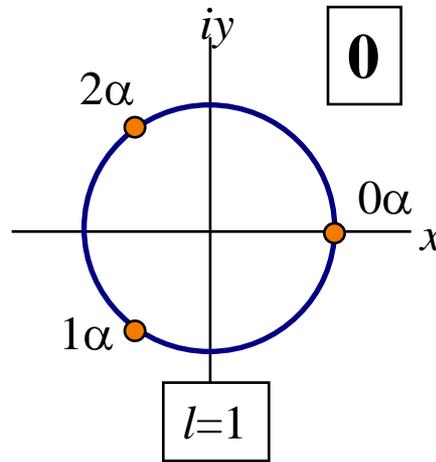
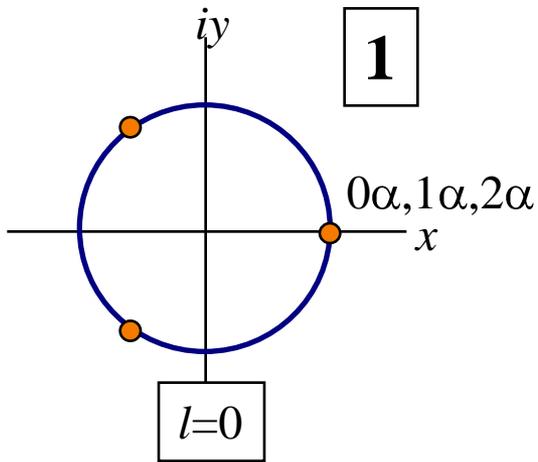
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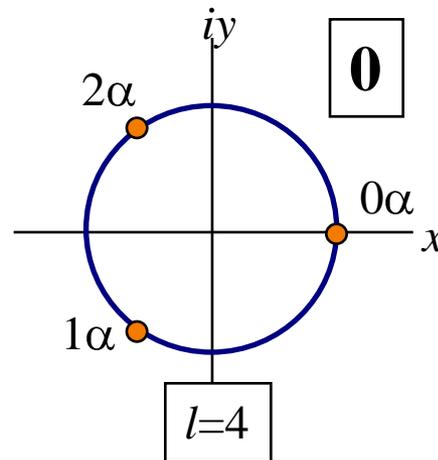
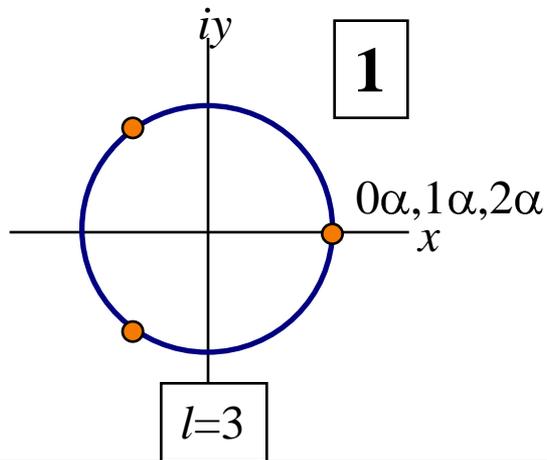
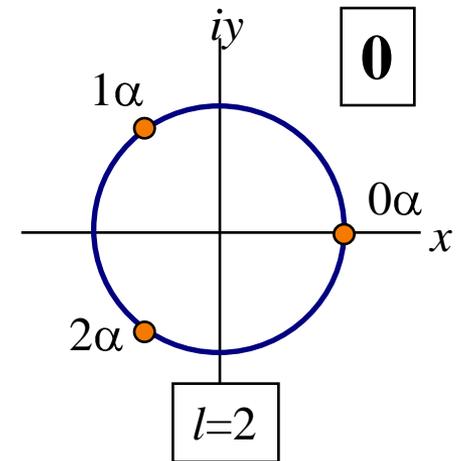
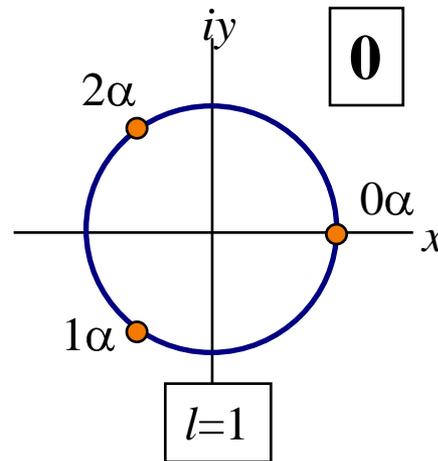
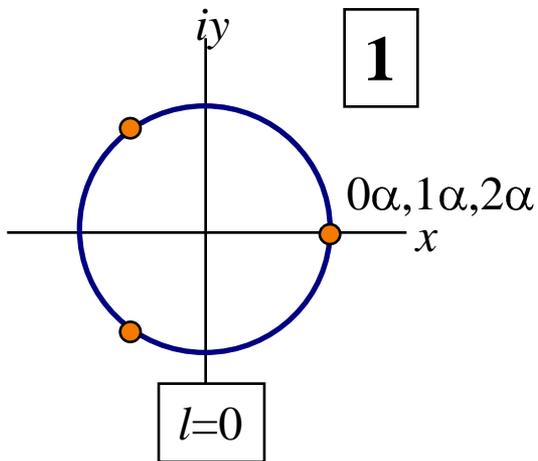
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Rotational Symmetry Detection (1D)

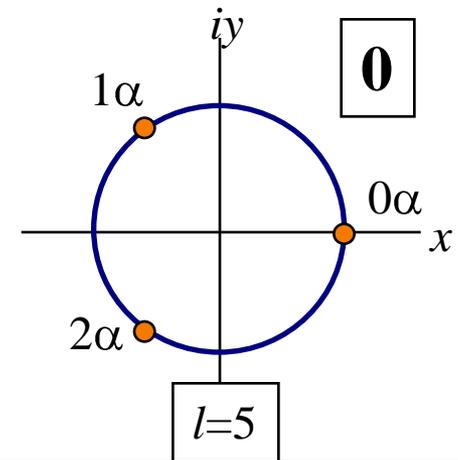
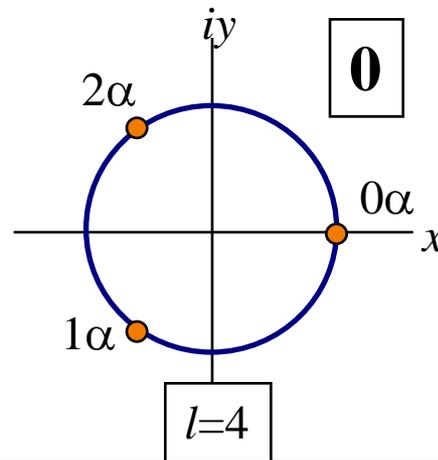
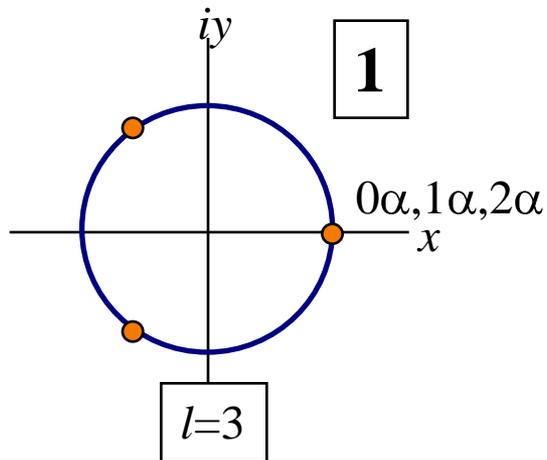
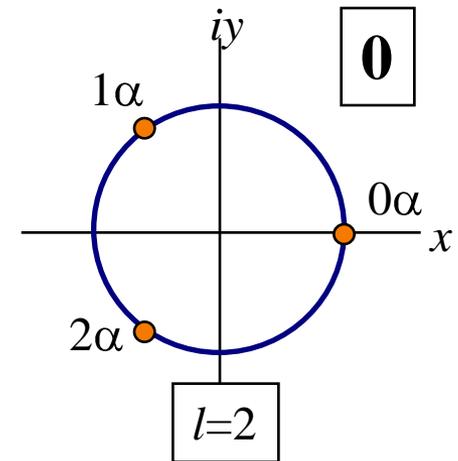
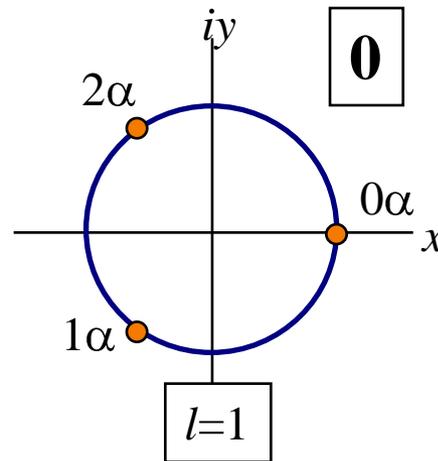
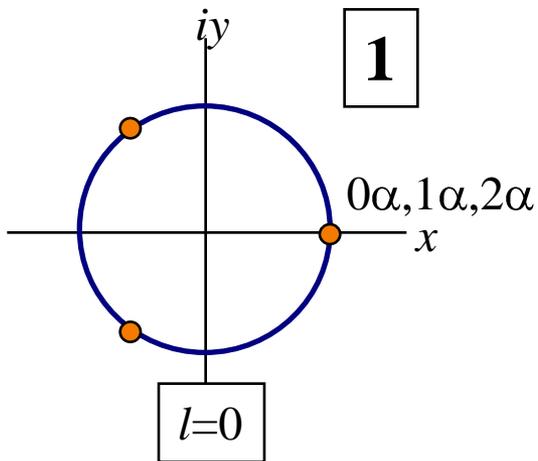
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Rotational Symmetry Detection (1D)

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Rotational Symmetry Detection (1D)



What is the value of the sum:

$$\frac{1}{k} \sum_{j=0}^{k-1} e^{-ilj2\pi/k}$$

When l is a multiple of k , the sum is equal to 1, otherwise, it is equal to zero.

Rotational Symmetry Detection (1D)



Thus, given a circular array $f[l]$, and given the group of k -fold symmetries G_k , if we express $f[l]$ in terms of its Fourier decomposition as:

$$f[l] = \sum_{l=0}^{n-1} \hat{f}[l] v_l[l]$$



Rotational Symmetry Detection (1D)

Thus, given a circular array $f[l]$, and given the group of k -fold symmetries G_k , if we express $f[l]$ in terms of its Fourier decomposition as:

$$f[l] = \sum_{l=0}^{n-1} \hat{f}[l] v_l[l]$$

the average of $f[l]$ under the action of G_k , can be obtained by zeroing out all the Fourier coefficients of $f[l]$ whose index is not a multiple of k :

$$\text{Average}(f[l], G_k) = \sum_{l=0}^{\lfloor (n-1)/k \rfloor} \hat{f}[k \cdot l] v_{k \cdot l}[l]$$



Rotational Symmetry Detection (1D)

This implies that we can compute the measure of k -fold symmetry of $f[l]$ by summing the square norms of the Fourier coefficients of $f[l]$ that are multiples of k :

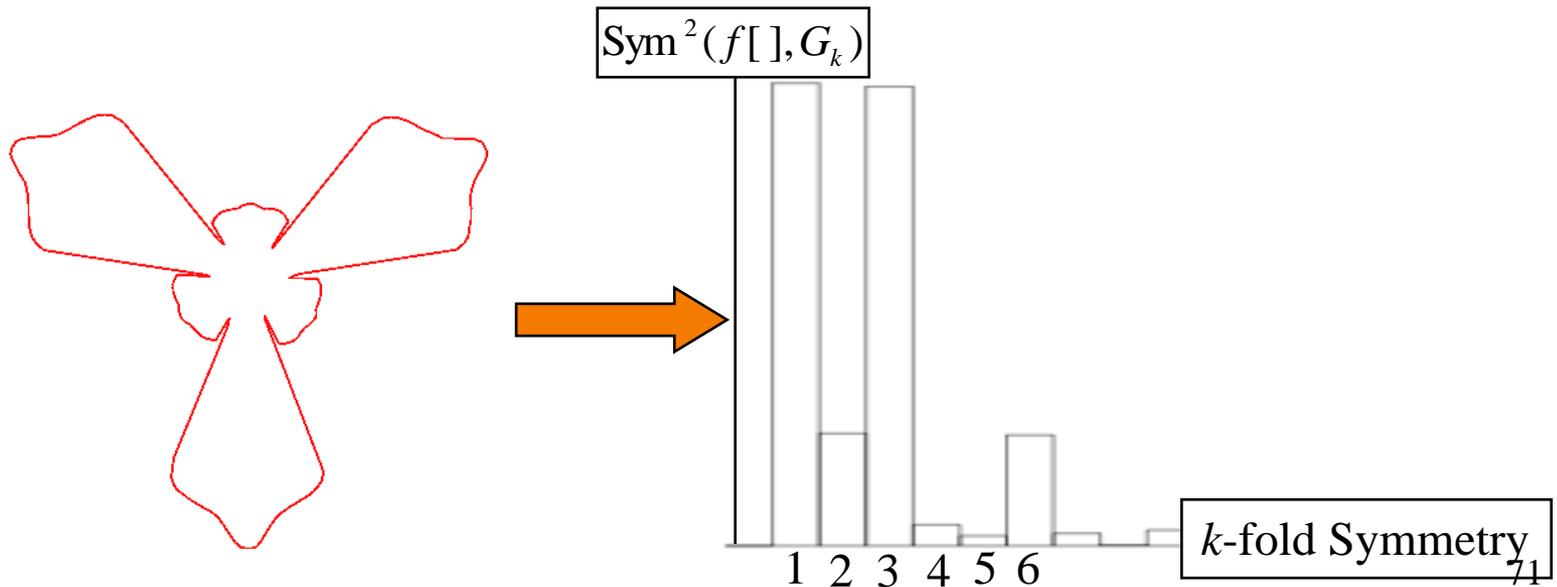
$$\text{Sym}^2(f[l], G_k) = \sum_{l=0}^{\lfloor (n-1)/k \rfloor} \left\| \hat{f}[k \cdot l] \right\|^2$$



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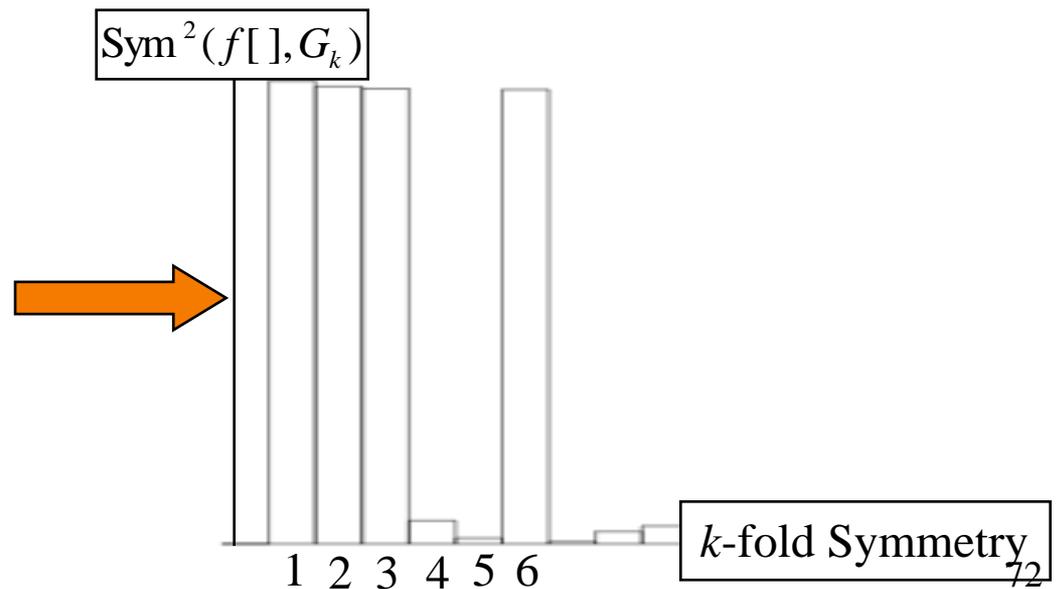
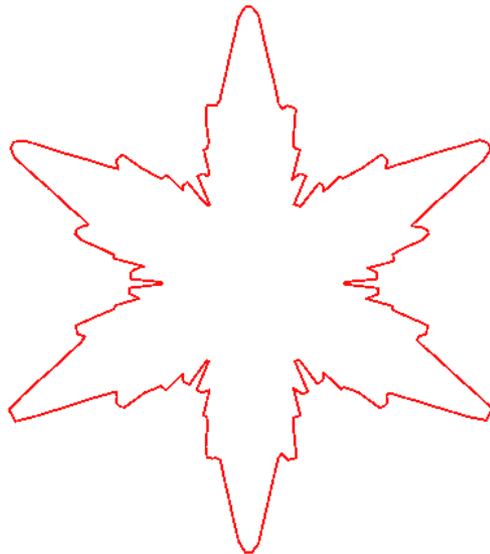




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Outline

Representation Theory

Symmetry Detection (1D)

- Rotations
- Reflections

Symmetry Detection (2D)



Reflective Symmetry Detection (1D)

Given the group of reflections about an axis with angle α :

$$G_\alpha = \{ \text{Identity, Reflection about } \alpha \}$$

And given a circular array $f[\]$, we would like to compute:

$$\text{Sym}^2(f[\], G_\alpha) = \|\text{Average}(f[\], G_\alpha)\|^2$$

Reflective Symmetry Detection (1D)



To do this we need to know how the group elements act on the circular array $f[]$:

Reflective Symmetry Detection (1D)



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- The identity element acts trivially

$$f[l] \rightarrow f[l]$$



Reflective Symmetry Detection (1D)

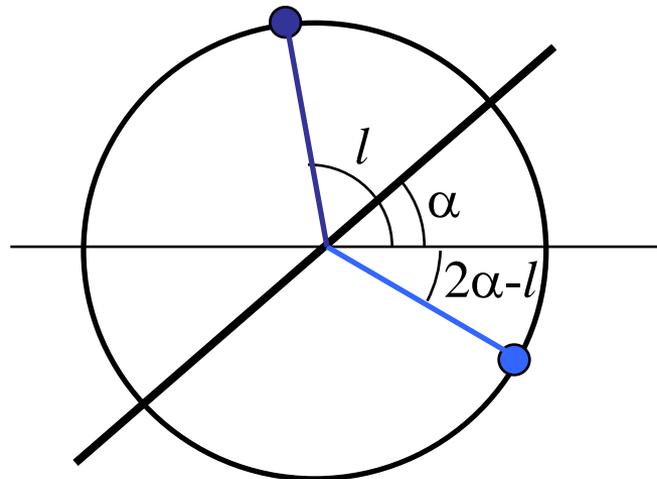
To do this we need to know how the group elements act on the circular array $f[l]$:

- The identity element acts trivially

$$f[l] \rightarrow f[l]$$

- The reflection about the line with angle α acts by:

$$f[l] \rightarrow f[2\alpha - l]$$





Reflective Symmetry Detection (1D)

Setting $g[l]$ to be the reflection of $f[l]$ about the origin:

$$g[l] = f[-l]$$

we can express the reflection of $f[l]$ about the line with angle α as:

$$\rho_{2\alpha} \left[\overset{\sim}{g[l]} \right]$$



Reflective Symmetry Detection (1D)

Now we can express the average of $f[x]$ over the reflection group G_α as:

$$\text{Average}(f[x], G_\alpha) = \frac{1}{2} (f[x] + \rho_{2\alpha}(g[x]))$$



Reflective Symmetry Detection (1D)

Now we can express the average of $f[\]$ over the reflection group G_α as:

$$\text{Average}(f[\], G_\alpha) = \frac{1}{2} (f[\] + \rho_{2\alpha}(g[\]))$$

Computing the measure of reflective symmetry we get:

$$\begin{aligned} \text{Sym}^2(f[\], G_\alpha) &= \|\text{Average}(f[\], G_\alpha)\|^2 \\ &= \left\| \frac{1}{2} (f[\] + \rho_{2\alpha}(g[\])) \right\|^2 \end{aligned}$$



Reflective Symmetry Detection (1D)

$$\text{Sym}^2(f[], G_\alpha) = \left\| \frac{1}{2} (f[] + \rho_{2\alpha}(g[])) \right\|^2$$

Expanding this in terms of dot-products, we get:

$$\text{Sym}^2(f[], G_\alpha) = \frac{1}{4} (\|f[]\|^2 + \|\rho_{2\alpha}(g[])\|^2 + 2\langle f[], \rho_{2\alpha}(g[]) \rangle)$$

(where we use the fact that $f[]$ is real-valued to lose the complex conjugation).



Reflective Symmetry Detection (1D)

$$\text{Sym}^2(f[], G_\alpha) = \frac{1}{4} \left(\|f[]\|^2 + \|\rho_{2\alpha}(g[]) \|^2 + 2\langle f[], \rho_{2\alpha}(g[]) \rangle \right)$$

Using the fact that the representation is unitary and that reflecting about the origin does not change the size of $f[]$, we get:

$$\text{Sym}^2(f[], G_\alpha) = \frac{1}{4} \left(\|f[]\|^2 + 2\langle f[], \rho_{2\alpha}(g[]) \rangle \right)$$



Reflective Symmetry Detection (1D)

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Allowing us to express the measure of reflective symmetry in terms of a cross-correlation:

$$\text{Sym}^2(f[], G_\alpha) = \frac{1}{2} \left(\|f[]\|^2 + \langle g[]^* f[] \rangle_{[2\alpha]} \right)$$

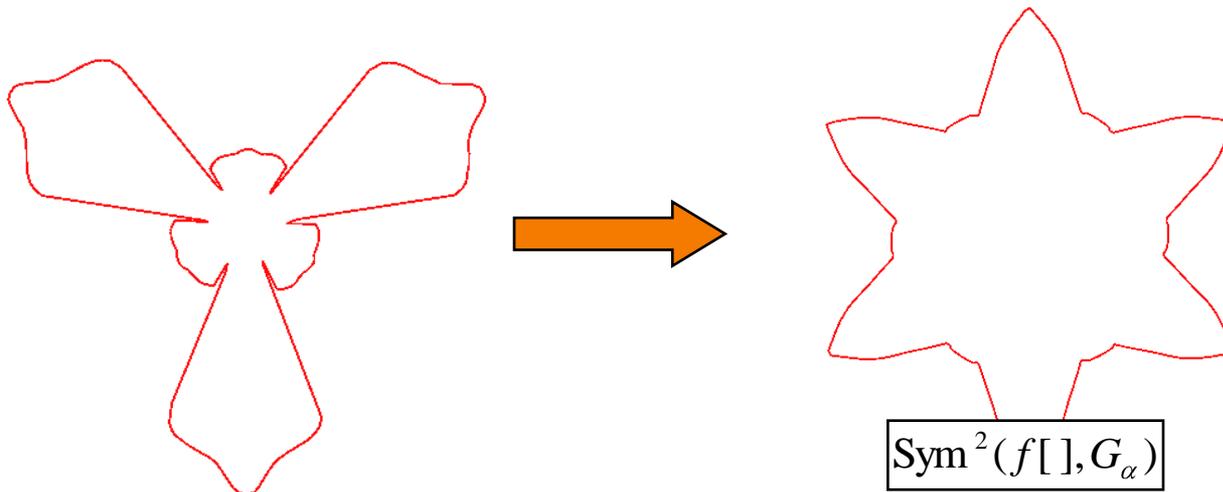


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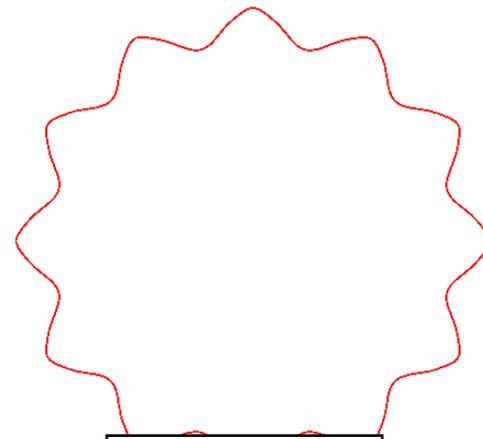
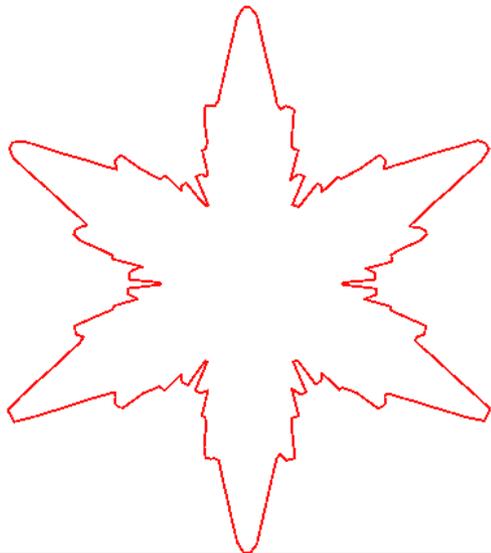


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$$\text{Sym}^2(f[], G_\alpha)$$



Outline

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Symmetry Detection (2D)



Symmetry Detection (2D)

What about the case when we would like to compute the rotational and reflective symmetries of a 2D grid about some point?

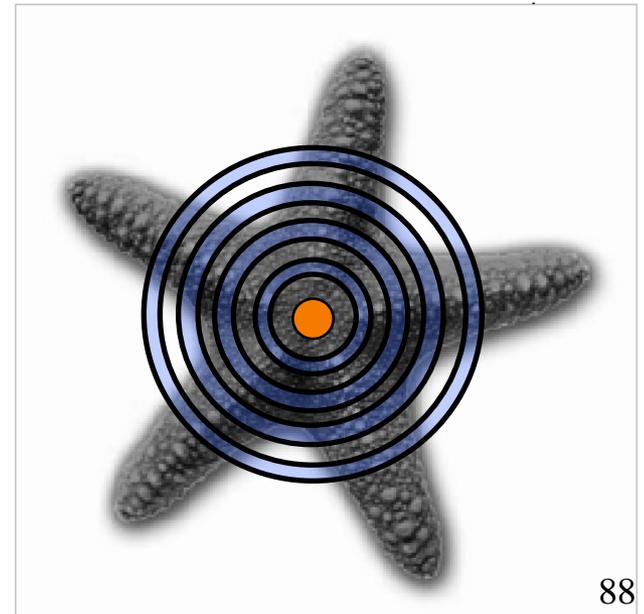




Symmetry Detection (2D)

What about the case when we would like to compute the rotational and reflective symmetries of a 2D grid about some point?

We can use the fact that rotations and reflections will map concentric circles back into themselves.



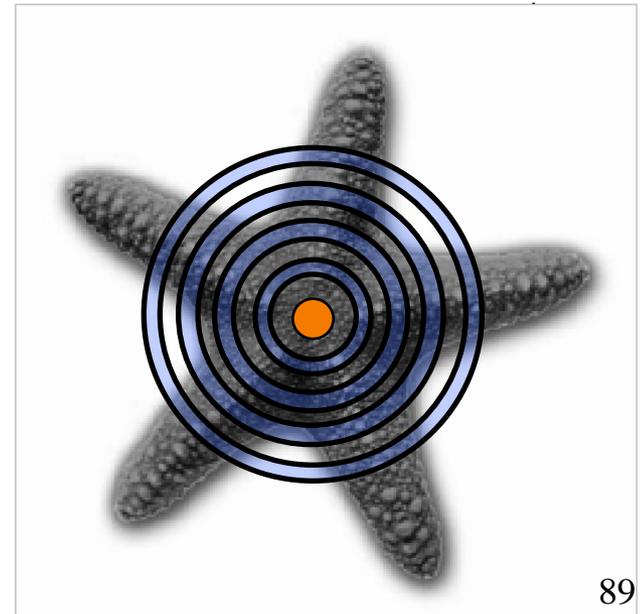


Symmetry Detection (2D)

What about the case when we would like to compute the rotational and reflective symmetries of a 2D grid about some point?

We can use the fact that rotations and reflections will map concentric circles back into themselves.

So when we compute the average over the symmetry group, we can consider the different radii independently.

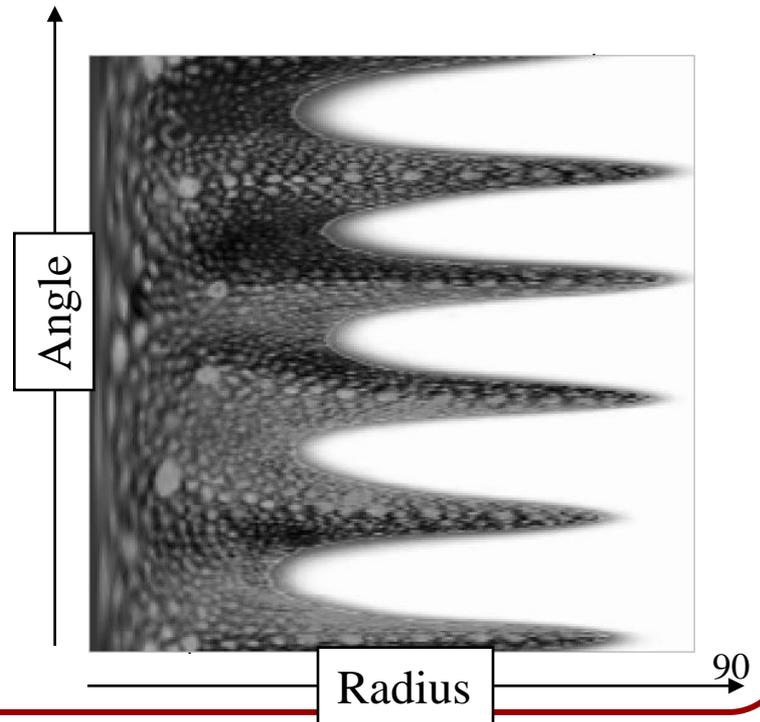




Symmetry Detection (2D)

To implement this, we:

- Parameterize the grid in polar coordinates



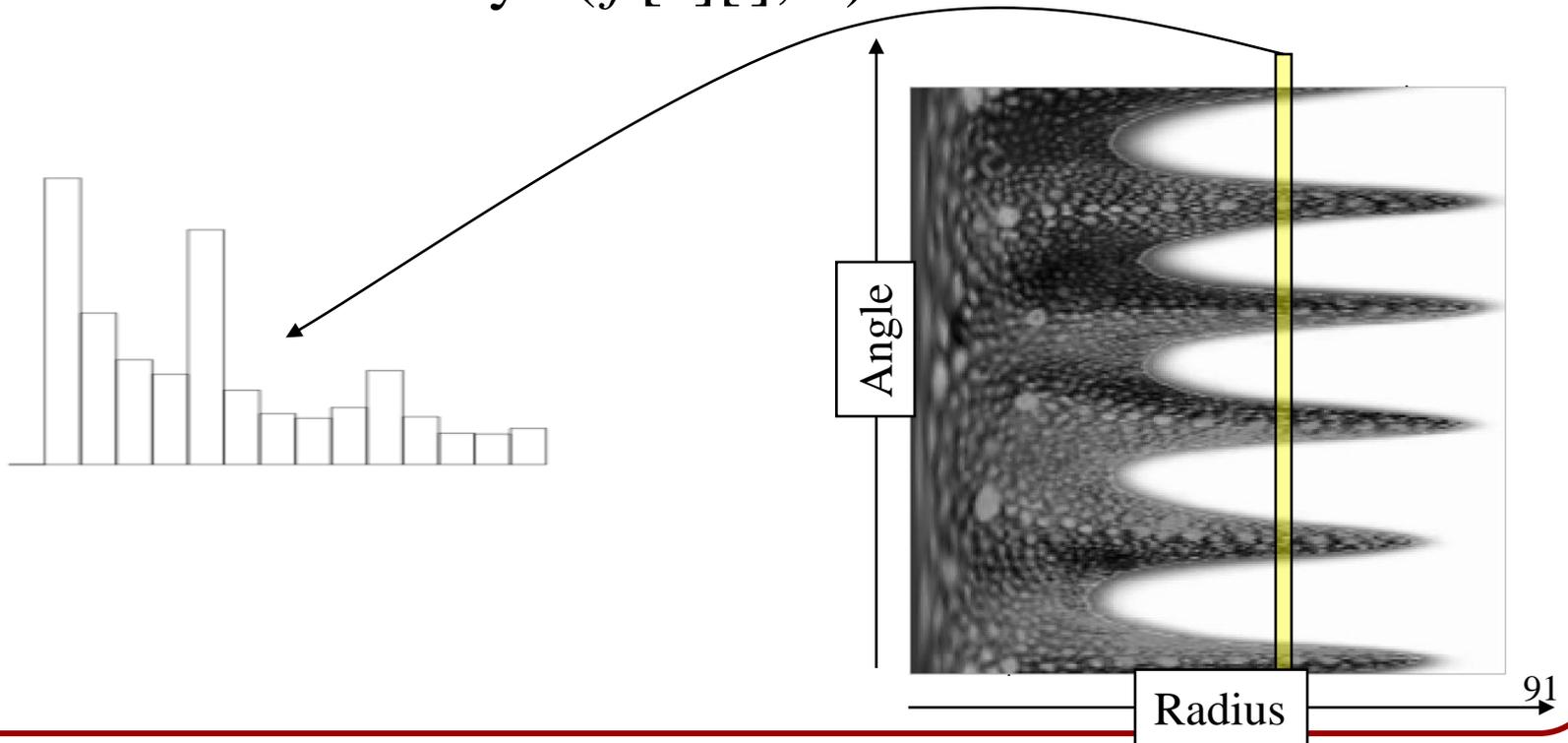


Symmetry Detection (2D)

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- Parameterize the grid in polar coordinates
- We compute the (square) measure of symmetry for each radius independently:

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- And we sum the symmetry measures over the different radii:

$$\text{Sym}^2(f[][], G) = \sum_r \text{Sym}^2(f[r][], G) \cdot r$$



Symmetry Detection (2D)

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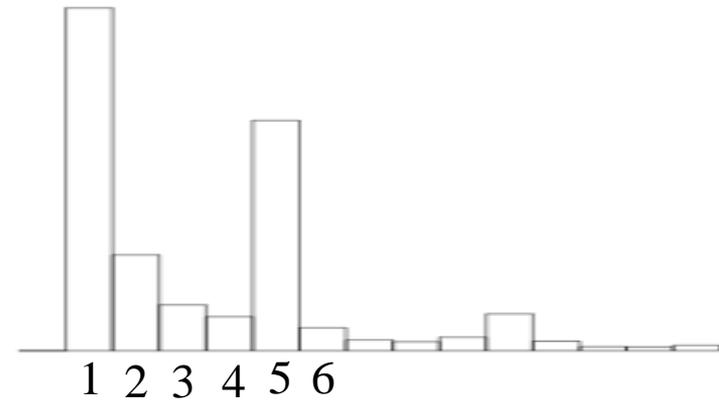
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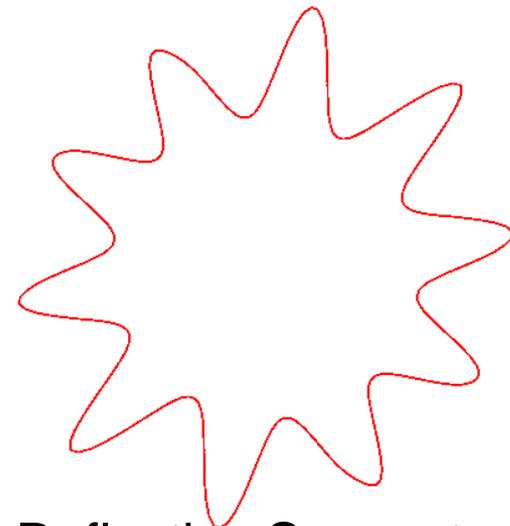
The r term is needed to account for the change of variables:

$$\int_{x^2+y^2 \leq 1} f(x, y) dx dy = \int_0^1 \int_0^{2\pi} f(r, \theta) d\theta \cdot r dr$$

Symmetry Detection (2D)



Rotational Symmetry



Reflective Symmetry