Enumeration of Balanced Tournament Designs on 10 points

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September 12, 2003

Abstract

We enumerate the balanced tournament designs on 10 points (BTD(5)) and find that there are exactly 30,220,557 nonisomorphic designs. We also find that there are exactly two nonisomorphic partitioned BTD(5)'s and 8,081,114 factored BTD(5)'s on 10 points. We enumerate other classes of balanced tournament designs on 10 points and give examples of some of the more interesting ones. In 1988 Corriveau [3] enumerated the nonisomorphic BTD(4)'s finding that there are 47 of them. This paper enumerates the next case and provides another good example of the combinatorial explosion phenomenom.

1 Introduction

A *balanced tournament design* of order n, BTD(n), defined on a $2n$-set V is an arrangement of the $\binom{2n}{2}$ distinct unordered pairs of the elements of V into an $n \times (2n-1)$ array such that

1. every element of V is contained in precisely one cell of each column, and
2. every element of V is contained in at most two cells of any row.

Notice that in each row there are precisely two symbols that occur exactly once, termed the *deficient* symbols of the row. The deficient symbols of two different rows are necessarily disjoint. A brief survey of results on BTD's can be found in [7] an earlier more extensive survey is [9].

Example 1.1 *A balanced tournament design of order 5.*

0 1	4 6	0 3	7 8	2 9	5 9	3 5	2 7	1 8
2 3	0 2	6 8	3 9	4 8	1 7	1 4	5 6	0 9
4 5	7 9	1 2	1 5	3 7	2 4	6 9	0 8	3 6
6 7	5 8	4 9	0 4	1 6	3 8	0 7	1 9	2 5
8 9	1 3	5 7	2 6	0 5	0 6	2 8	3 4	4 7

#198271

invariant: 9 1 0 1 0 1 0 1 0 1 0 0 36 175 76 66 7 0

Intuitively, a BTD(n) is formed by finding a one-factorization of K_{2n} (the columns) that is orthogonal to a "near two-factorization" of K_{2n} (the rows). Gelling and Odeh introduced balanced tournament designs in 1973 in [4]. The spectrum was completed by Schellenberg, van Rees and Vanstone [10] in 1977. They proved the following.

Theorem 1.2 [10] *There exists a balanced tournament design of order n if and only if n is a positive integer, $n \neq 2$.*

Two BTD(n) are *isomorphic* if one can be obtained from the other by permuting the rows, columns or elements of the array. It is easy to see that there is a unique BTD(3) up to isomorphism. In 1987, Corriveau enumerated the nonisomorphic BTD(4)'s in [2, 3]; there are precisely 47. He also showed for each of the 396 nonisomorphic one-factorizations of K_{10} that there is a BTD(5) having the given one-factorization as its columns. In this paper we will enumerate the BTD(5)'s completely, and show that not only does each one of the 396 nonisomorphic one-factorizations of K_{10} admit one BTD(5), but each admits on average about 90,000 nonisomorphic BTD(5)'s.

Balanced tournament designs satisfying certain extra conditions have been of considerable interest to researchers since 1977. In this paper we enumerate some of these as well, including partitioned, hamiltonian and factored balanced tournament designs. We also introduce the notions of uniform and doubly uniform balanced tournament designs.

A BTD(n) is said to be *partitioned* (PBTD) if the $2n-1$ columns can be partitioned into three sets A, B, and C of 1, $n-1$ and $n-1$ columns respectively, in such a way that the $n \times n$ array formed by the columns in $A \cup B$ is a Howell design H$(n, 2n)$, as is the $n \times n$ array formed by the columns in $A \cup C$. Hence each of the $2n$ symbols occur exactly once in each row and in each column of $A \cup B$ and of $A \cup C$. Example 3.7 gives two examples of partitioned BTD(5)'s. Note that the deficient symbols of each row of a partitioned BTD are necessarily in the first column. There exists a partitioned BTD(n) for n a positive integer, $n \geq 5$, except possibly for $n \in \{9, 11, 15\}$[6]. There does not exist a PBTD(3) or PBTD(4).

A *factored balanced tournament design* of order n, FBTD(n), is a BTD(n) with the property that in each row there exist n cells, a *factor*, which contain all $2n$ elements of V. Note that any partitioned balanced tournament design is also a factored balanced tournament designs, but not the other way around. We will in fact see that factored BTD's are much more numerous than partitioned BTD's (of order 5). Factored BTD(n) exist for all positive integers $n \neq 2$ [8]. The unique BTD(3) is factored, while 29 out of the 47 BTD(4) are factored BTD's [3].

In any row of a BTD(n), every symbol occurs in two cells, except for two symbols that each occur in only one cell. Hence, it is conceivable that the graph formed by the pairs of symbols occuring in the cells of a given row (the *row graph*) could be a path of length $2n$ (vertices), i.e. a Hamiltonian path. A BTD(n) is said to be a *hamiltonian* BTD(n) (HBTD) if this is true for every row of the array. It is known [5] that there exist HBTD(n) for $n = 1, 4, 5$; there do not exist HBTD(n) for $n = 2, 3$; and there exist HBTD(n) for all positive integers n not divisible by 2, 3 or 5. Out of a total of 47 BTD(4)'s exactly 18 are hamiltonian [3]. A BTD(n) is said to be a *nonhamiltonian* BTD(n) (NHBTD) if no row of the BTD is a hamiltonian path. There are precisely 5 NHBTD(4) [2].

A BTD is called *uniform* if all of the row graphs of the BTD are isomorphic; so hamiltonian is a special case of uniform. When the underlying graphs of the union of any two columns are all isomorphic two-regular graphs, the underlying (column) one-factorization of the BTD is called a *uniform* one-factorization. If both the rows and columns are uniform, the BTD is called *doubly uniform.*

In Section 2 we will discuss the algorithm that we employed to enumerate the BTD(5). Section 3 we summarize our main results give some interesting examples of BTD that were found in our enumeration.

2 The Algorithm

In this section, we discuss the algorithm that we employed to enumerate and construct the nonisomorphic BTD(5)'s. The algorithm consists of a list processing algorithm to find all distinct BTD that can be constructed from a particular one-factorization of K_{10}. We then used invariants to greatly reduce the number of tests needed to check the distinct BTD for isomorphism. Finally if two distinct BTD's had the same invariant, we attempted to construct an isomorphism between them.

A BTD is *generated* from a one-factorization in the obvious way: place the edges of each one-factor in a column and then order each column so that in each row every symbol occurs at most twice. So the columns of any two BTD's generated from a given one-factorization contain all the same pairs of symbols. Clearly then, BTD's generated from two nonisomorphic one-factorizations must be nonisomorphic. We also see that two distinct BTD's generated from a one-factorization with automorphism group of order 1 must also be nonisomorphic. It is well known that there are 396 nonisomorphic one-factorizations of the complete graph on 10 points K_{10} (see [1] for the complete list). Our algorithm begins by inputting one of these one-factorizations and proceeds to generate all possible BTD's from this one-factorization. We use list processing to generate the BTD's.

First we use a procedure called *legal*. This procedure constructs all potential rows for proported BTD. It searches for a set of nine edges, one from each one-factor, that has the property that each symbol occurs at most twice. (These are the legal rows for the BTD). Two legal rows are termed *compatible* if they have no edges in common and if the deficient symbols in these two rows are disjoint. Using a standard backtracking procedure, the program then constructs all sets of 5 compatible rows for each given one-factorization, these will all be distinct BTD's generated from the one-factorization. So if D is a BTD generated in this manner, then the *ith* column of D will be precisely the *ith* one-factor in the one-factorization. In general we found about 90,000 distinct BTD's for each one-factorization of K_{10} and this took roughly 30 seconds of CPU time (per one-factorization) on a PC running at 1.3 GHz.

To distinguish nonisomorphic BTD(5)'s we used a concatenated string of three invariants. These invariants are called the *one-factorization invariant*, the *row invariant* and the *4-cell invariant*. The *one-factorization invariant* of a BTD is n if it is generated from one-factorization $\#n$, for $1 \leq n \leq 396$.

Each row graph of a BTD consists of a path and a two-factor; the endpoints of the path are the two deficient symbols in that row. We denote by P_n the path of length n (vertices) and C_n the cycle of length n. In the case of BTD(5) there are 11 possible row graphs. The row graph types are:

1: P_{10} 2: $P_7 \cup C_3$ 3: $P_6 \cup C_4$ 4: $P_5 \cup C_5$ 5: $P_4 \cup C_6$ 6: $P_3 \cup C_7$
7: $P_2 \cup C_8$ 8: $P_4 \cup C_3 \cup C_3$ 9: $P_2 \cup C_4 \cup C_4$ 10: $P_3 \cup C_4 \cup C_3$ 11: $P_2 \cup C_5 \cup C_3$

The *row invariant* of a balanced tournament design D is a vector $(a_1, a_2, \ldots, a_{11})$ of length 11 where D has a_i row graphs of type i. Thus for any BTD(5), $\sum a_i = 5$.

Given any two rows and any two columns, the union of the four cells in those two rows and columns forms a graph with four edges. There are six possibilities for these 4-cell graphs. These 4-cell graph types are:

1: $P_2 \cup P_2 \cup P_2 \cup P_2$ 2: $P_3 \cup P_2 \cup P_2$ 3: $P_3 \cup P_3$ 4: $P_4 \cup P_2$ 5: P_5 6: C_4

The *4-cell invariant* of D is a vector $(f_1, f_2, \ldots, f_6)$ of length 6 where D has f_i 4-cell graphs of type i. For any BTD(5), $\sum f_i = \binom{9}{2} \times \binom{5}{2} = 360$.

It is clear that any two BTD's that have either different one-factorization or row or 4-cell invariants can not be isomorphic. We concatenate all three of these invariants to obtain a vector of length 18 that we use as our *BTD-invariant* or just *invariant* for short.

If two BTD's have the same BTD-invariant, they are then tested for isomorphism. Knowing the row graph structure greatly speeds up the search for the isomorphism. If D_1 and D_2 are BTD's with the same row invariant, then clearly any isomorphism from D_1 to D_2 must take a row of D_1 to a row of D_2 with the same row graph type. Hence it must take the path in that row of D_1 to the path in the row of D_2. Also the cycle(s) must map to the cycles. This greatly reduces the number of possible mappings that need to be tested to see if they are indeed isomorphisms. In general, if two BTD's have the same BTD-invariant, they are probably isomorphic, so we were looking to find the isomorphism as opposed to showing that none existed.

When a new BTD is found to be nonisomorphic to all the BTD's previously constructed, it is added to the list and is stored. Hence at any point in the running of the algorithm, the list contains a set of nonisomorphic BTD's.

Constructed in this manner, the BTD's have a natural order which we now describe. Given a BTD D constructed from a one-factorization F of K_{10}, each column c gives a permutation ρ_c of the set $\{1,2,3,4,5\}$ where the pair in row r of the D is the $\rho_c(r)th$ pair given in the $c'th$ one factor of F in the listing of F presented in [1] (pages 655 – 660). Now from D construct a new 5×9 array D', by putting $\rho_c(r)$ in cell (r, c) of D'. These D' arrays are ordered lexicographically by rows. Given two BTD's D_1 and D_2 constructed from one-factorizations F_i and F_j, respectively, then say that $D_1 < D_2$ if $i < j$ or if $i = j$ and $D_1' < D_2'$. This gives an ordering of all the BTD's that we construct. For each of the BTD's given as examples in this paper, we first give the number of that BTD, then we give the BTD invariant.

The program halts when all 396 nonisomorphic one-factorizations of K_{10} have been inputted.

3 Results, Observations and Some Interesting Examples

After running the program for roughly 5 hours on a PC running at 1.3 GHz, we arrived at our main result.

Theorem 3.1 *There are exactly 30,220,557 nonisomorphic balanced tournament designs of order 5.*

Example 3.2 *The following are balanced tournament designs #1 and #30,220,557 and their BTD invariant.*

0 1	0 2	1 2	3 9	3 7	5 9	4 8	4 7	5 6
2 3	4 6	4 9	1 5	6 9	3 8	0 7	0 8	2 7
4 5	1 3	5 7	7 8	2 8	2 4	1 6	3 6	0 9
6 7	5 8	6 8	0 4	0 5	1 7	2 9	1 9	3 4
8 9	7 9	0 3	2 6	1 4	0 6	3 5	2 5	1 8

#1

1 1 1 0 1 0 0 0 0 0 0 2 42 170 83 52 12 1

0 1	7 9	6 8	0 4	3 9	2 4	3 5	1 7	2 5
2 3	5 8	2 7	6 9	4 6	1 8	0 7	3 4	0 9
4 5	3 6	1 5	2 8	7 8	0 6	1 9	2 9	4 7
6 7	1 4	4 9	1 3	0 5	5 9	2 6	0 8	3 8
8 9	0 2	0 3	5 7	1 2	3 7	4 8	5 6	1 6

#30,220,557

396 0 0 0 0 0 4 1 0 0 0 0 44 168 84 48 16 0

It is interesting to look at the number of distinct and nonisomorphic BTD's that were generated from each of the 396 one-factorizations of K_{10}. Remember that each of the distinct BTD's generated from a single one-factorization has the pairs in one-factor i of the one-factorization in column i. The greatest number of distinct BTD's from any one-factorization was 123,876 from one-factorization #1, the least was 63,504 from one-factorization #290 (this is the one factorization GK_{10}, [1]) while the average was 89,998.8. The total number of distinct BTD's that were generated was 35,639,544

The greatest number of nonisomorphic BTD's from any one factorization was 103,912 from one-factorization #48; the least was 293 from one-factorization #1.

Table 3.3 *Number of distinct BTD's generated per one-factorization (in thousands)*

number of distinct BTD's	60-70	70-80	80-90	90-100	100-110	110-120	120-130
number of OFs	1	11	222	132	21	8	1

Table 3.4 *Number of nonisomorphic BTD's generated per one-factorization (in thousands)*

number of nonisomorphic.	< 1	1-2	2-5	5-10	10-20	20-30	30-40
number of OFs	1	1	1	5	10	9	8

number of nonisomorphic	40-50	50-60	60-70	70-80	80-90	90-100	100-110
number of OFs	55	7	1	3	185	100	10

We also computed the order of the automorphism group of each of the 30,220,557 BTD(5)'s. We found that there were 30,202,632 with automorphism group of order 1 and 26 with the largest automorphism group of order 8. The following table describes our findings.

Table 3.5 *The frequency distrubution of the automorphism groups of BTD(5)'s*

group order	1	2	4	5	8
number of BTD's	30,202,632	17,681	213	5	26

Of course when generating all the BTD(5)'s we were concerned with counting ones with special properties. We have

Theorem 3.6 *There are exactly 2 nonisomorphic partitioned BTD(5)'s, 8,081,144 factored BTD(5)'s, 2,236,254 hamiltonian BTD(5)'s and 339,354 nonhamiltonian BTD(5)'s. Every one-factorization of K_{10} admits at least one factored BTD and at least one nonhamiltonian BTD.*

These values were all easy to compute using the row invariant of the BTD's. A BTD is factored if and only if every row graph consists of a path of even length and all the cycles are also even. To find the partitioned BTD's we could narrow the search to those BTDs where every row graph consists of a path of length 2, all the cycles are of even length and the deficient pair of points in each row are all in the same column. We found exactly 54 BTD's satisfying all these conditions, but only two of them were indeed partitioned BTD's and both of these come from one-factorization #378. Hamiltonian BTD's are particularly easy to spot as the row invariant of a hamiltonian BTD must be (5 0 0 0 0 0 0 0 0 0 0), i.e. 5 rows all of type 1. Nonhamiltonian BTD's are precisely those which have $a_1 = 0$ in their row invariant.

Example 3.7 *The two nonisomorphic partitioned balanced tournament designs.*

0 1	7 9	6 9	7 8	6 8	2 4	2 5	3 4	3 5
2 3	5 8	4 7	5 6	4 9	0 6	1 9	0 8	1 7
4 5	3 6	2 8	3 9	2 7	1 8	0 7	1 6	0 9
6 7	0 2	1 5	0 4	1 3	5 9	3 8	2 9	4 8
8 9	1 4	0 3	1 2	0 5	3 7	4 6	5 7	2 6

#29,245,581

378 0 0 0 0 0 0 2 0 3 0 0 68 132 112 24 20 4

0 1	7 9	6 9	7 8	6 8	2 4	2 5	3 4	3 5
2 3	5 8	4 7	5 6	4 9	1 8	0 7	1 6	0 9
4 5	3 6	2 8	3 9	2 7	0 6	1 9	0 8	1 7
6 7	1 4	0 3	1 2	0 5	5 9	3 8	2 9	4 8
8 9	0 2	1 5	0 4	1 3	3 7	4 6	5 7	2 6

#29,245,598

378 0 0 0 0 0 0 2 0 3 0 0 86 98 130 20 22 4

Both of these partitioned BTD's have underlying one-factorization #378. The one on the right is isomorphic to the PBTD(5) given in [7] (page 239, Example 3.7). The isomorphism σ from the BTD in the Handbook to the one above is

$$\sigma = \begin{pmatrix} \alpha & \infty & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 6 & 3 & 5 & 0 & 4 & 1 & 7 & 2 \end{pmatrix}$$

Every partitioned balanced tournament designs and every hamiltonian BTD is also a factored balanced tournament designs (FBTD), so Example 3.7 provides two examples of factored BTD(5)'s. Example 1.1 is a FBTD that is not a partitioned or a hamiltonian BTD.

We also enumerated the uniform BTD(5)'s. A BTD(5) is called *uniform of type t* if all five of the row graphs are of type t. So a uniform BTD of type 1 is a hamiltonian BTD. Notice again that these are particularly easy to identify as their row invariant will consist of a 5 in position t and a 0 in all others.

Theorem 3.8 *There are 2,236,723 uniform BTD(5)'s. There are exactly 2,236,254 of type 1, 330 of type 2, 75 type 3, 20 type 4, 19 type 5, 3 type 6, 10 type 7, 0 type 8, 12 type 9, and none of type 10 or 11.*

Example 3.9 *A uniform BTD of each possible type. The one of type 1 is a hamiltonian BTD, the ones of type 1, 3, 5, 7 and 9 are also factored BTD's.*

Type 1

0 1	3 6	4 9	0 4	7 8	5 9	2 6	1 7	3 8
2 3	1 4	6 8	6 9	0 5	3 7	1 9	0 8	2 5
4 5	7 9	0 3	5 7	1 2	0 6	4 8	2 9	1 6
6 7	5 8	1 5	2 8	3 9	2 4	0 7	3 4	0 9
8 9	0 2	2 7	1 3	4 6	1 8	3 5	5 6	4 7

#30,220,422

396 5 0 0 0 0 0 0 0 0 0 0 0 30 205 70 30 25 0

Type 2

0 1	0 2	1 2	3 9	6 9	3 8	4 8	5 6	4 7
2 3	5 8	4 9	1 5	3 7	0 6	1 6	2 7	0 9
4 5	7 9	6 8	0 4	2 8	1 7	3 5	1 9	3 6
6 7	4 6	0 3	7 8	1 4	5 9	2 9	0 8	2 5
8 9	1 3	5 7	2 6	0 5	2 4	0 7	3 4	1 8

#439

2 0 5 0 0 0 0 0 0 0 0 0 0 34 180 74 64 8 0

Type 3

0 1	7 9	0 3	6 8	4 7	3 5	2 4	1 5	2 6
2 3	4 6	7 8	1 2	3 9	0 6	1 9	0 8	5 7
4 5	0 2	6 9	3 7	0 5	1 7	3 8	4 9	1 8
6 7	5 8	1 4	5 9	2 8	2 9	0 7	3 6	3 4
8 9	1 3	2 5	0 4	1 6	4 8	5 6	2 7	0 9

#2,737,235

45 0 0 5 0 0 0 0 0 0 0 0 0 52 148 93 56 10 1

Type 4

0 1	0 2	4 9	7 8	3 4	3 8	2 5	1 6	5 6
2 3	5 8	6 8	0 4	1 7	0 6	1 9	2 9	3 7
4 5	4 6	1 2	3 9	6 9	2 7	0 7	3 5	1 8
6 7	1 3	0 3	1 5	2 8	5 9	4 8	4 7	0 9
8 9	7 9	5 7	2 6	0 5	1 4	3 6	0 8	2 4

#2,132,559

38 0 0 0 5 0 0 0 0 0 0 0 0 47 161 87 50 15 0

Type 5

0 1	7 9	7 8	6 8	1 3	2 9	4 6	2 4	3 5
2 3	5 8	0 3	5 9	4 7	0 6	1 9	1 7	4 8
4 5	1 4	2 6	3 7	2 8	1 8	0 7	5 6	0 9
6 7	0 2	4 9	1 2	0 5	5 7	3 8	3 9	1 6
8 9	3 6	1 5	0 4	6 9	3 4	2 5	0 8	2 7

#13,170,633

173 0 0 0 0 5 0 0 0 0 0 0 0 47 158 87 56 12 0

Type 6

0 1	5 8	4 7	2 6	3 7	4 8	2 5	3 6	0 9
2 3	0 2	6 8	1 5	4 9	1 7	6 9	0 8	3 4
4 5	1 3	0 3	7 8	2 8	2 9	0 7	1 9	5 6
6 7	7 9	5 9	0 4	1 6	3 5	3 8	2 4	1 8
8 9	4 6	1 2	3 9	0 5	0 6	1 4	5 7	2 7

#442,244

15 0 0 0 0 0 5 0 0 0 0 0 0 36 186 76 44 18 0

Type 7

0 1	3 6	4 9	5 9	2 7	5 7	3 8	4 6	2 8
2 3	7 9	1 5	0 4	6 9	1 8	5 6	0 8	4 7
4 5	0 2	7 8	6 8	1 3	2 9	0 7	3 9	1 6
6 7	5 8	0 3	1 2	4 8	3 4	1 9	2 5	0 9
8 9	1 4	2 6	3 7	0 5	0 6	2 4	1 7	3 5

#893,192

22 0 0 0 0 0 0 5 0 0 0 0 0 20 216 60 48 16 0

Type 9

0 1	7 9	7 8	6 8	6 9	3 4	2 4	2 5	3 5
2 3	5 8	4 9	5 9	4 8	0 6	0 7	1 7	1 6
4 5	3 6	2 6	3 7	2 7	1 8	1 9	0 8	0 9
6 7	1 4	1 5	0 4	0 5	2 9	3 8	3 9	2 8
8 9	0 2	0 3	1 2	1 3	5 7	5 6	4 6	4 7

#893,360

22 0 0 0 0 0 0 0 0 5 0 0 84 112 132 0 24 8

In [3] it was conjectured that the one-factorization GK_{2n} never generates a hamiltonian BTD (except for $n = 1$). We find that this is indeed the case for $n = 5$. In this case the one-factorization #290 is GK_{10} and it does indeed generate no hamiltonian BTD's. In fact it generates no uniform BTD's at all. This is unusual as only two one-factorizations (out of 396) fail to generate any uniform BTD's (the other is #1). Hence this adds significant evidence to the conjecture. The next table gives the number of one-factorizations of K_{10} that admit a uniform BTD(5) of each of the 11 types.

Table 3.10 *The number of one-factorizations admitting each type of uniform BTD(5).*

Uniform of type	1	2	3	4	5	6	7	8	9	10	11
number of OF's	394	201	65	18	16	3	8	0	12	0	0

It is possible that all five row-graphs of a BTD could be nonisomorphic (*heterogeneous BTD's*), in fact this is quite common. We find that there are exactly 833,525 nonisomorphic BTD(5)'s with this property. Since there are exactly five row graph types that can underly a factored BTD (they must be a path of even length in addition to cycle(s) of even length), it is possible that there may be examples of heterogeneous FBTD. Indeed there are exactly 1647 nonisomorphic heterogeneous factored BTD(5)'s. One such example is given in Example 1.1. In general a typical BTD is composed of 3 different row graphs. The exact breakdown is given in the next table.

Table 3.11 *The number of nonisomorphic row graphs in a BTD(5).*

number of nonisomorphic rows	1	2	3	4	5
number of BTD's	2,236,723	9,561,055	12,034,066	5,555,188	833,525

A one-factorization is *perfect* if the union of any pair of one factors is a hamiltonian cycle. One-factorization #396 is the only perfect one-factorization of K_{10}. This one-factorization generates exactly 122 uniform BTD's, 120 of type 1 (hamiltonian) and two of type 5. These are examples of doubly uniform BTD's. A doubly uniform BTD of type 1 is given in Example 3.9, both doubly uniform BTD's of type 5 are given in Example 3.12. The only other uniform one-factorization is #1. It has the property that the union of any pair of one-factor is $C_4 \cup C_6$. We found that this one-factorization fails to generate a uniform BTD (out of 293 nonisomorphic ones). Thus the only doubly uniform BTD's are those 122 designs generated one-factorization #396.

Example 3.12 *The two doubly uniform balanced tournament designs of type 5.*

0 1	3 6	4 9	2 8	0 5	5 9	2 6	1 7	4 7
2 3	1 4	6 8	5 7	4 6	3 7	1 9	0 8	0 9
4 5	7 9	0 3	1 3	7 8	0 6	4 8	2 9	2 5
6 7	5 8	1 5	6 9	1 2	2 4	0 7	3 4	3 8
8 9	0 2	2 7	0 4	3 9	1 8	3 5	5 6	1 6

#30,220,424

396 0 0 0 0 5 0 0 0 0 0 0 30 195 70 50 15 0

0 1	0 2	2 7	5 7	4 6	5 9	1 9	3 4	3 8
2 3	7 9	4 9	2 8	0 5	3 7	4 8	5 6	1 6
4 5	3 6	1 5	1 3	7 8	2 4	2 6	0 8	0 9
6 7	1 4	6 8	0 4	3 9	1 8	0 7	2 9	2 5
8 9	5 8	0 3	6 9	1 2	0 6	3 5	1 7	4 7

#30,218,751

396 0 0 0 0 5 0 0 0 0 0 0 30 200 70 40 20 0

We conclude with a discussion of the invariants. First, there were 14,129,867 different invariant vectors for the 30,220,557 nonisomorphic balanced tournament designs. Thus the invariant had sensitivity of .467. The sensitivity of each of the three invariants individually is 396/30,220,557 = .0000131 for the one-factorization invariant; 2726/30,220,557 = .0000902 for the row invariant and 5467/30,220,557 = .000190 for the 4-cell invariant. We found many examples of two nonisomorphic BTD's that had exactly one, two or all three identical invariants (one-factorization, row or 4-cell).

References

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