Lecture 9: Disjoint Sets / Union-Find

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September 28, 2021

601.433/633 Introduction to Algorithms

Introduction

Informal: Universe of elements, want to maintain *disjoint sets*.

Slightly more formally:

- Make-Set($\mathbf{x}$): create a new set containing just $\mathbf{x}$ (i.e., $\{\mathbf{x}\}$)
- Union($\mathbf{x}, \mathbf{y}$): Replace set containing $\mathbf{x}$ ($\mathbf{S}$) and set containing $\mathbf{y}$ ($\mathbf{T}$) with single set $\mathbf{S} \cup \mathbf{T}$
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Rules: every set has a *unique* representative.

- If $\mathbf{x}$ and $\mathbf{y}$ are in same set, Find($\mathbf{x}$) = Find($\mathbf{y}$)
- If $\mathbf{x}$ and $\mathbf{y}$ are in different sets, then Find($\mathbf{x}$) $\neq$ Find($\mathbf{y}$)
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Note: disjoint (and partition) by construction!

Introduction (II)

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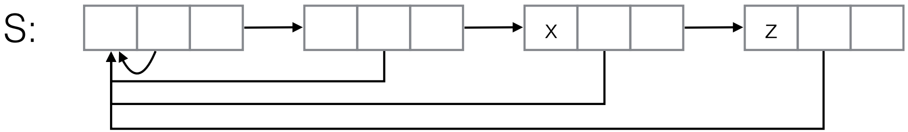
Notation and Notes:

- **m** operations total
- **n** of which are Make-Sets (so **n** elements)
- Assume have pointer/access to elements we care about (like last class)

First Approach: Lists

Linked list for each set.

- Representative of set is head (first element on list)
- Each element has pointer to head and to next element, so stored as triple: (element, head, next)



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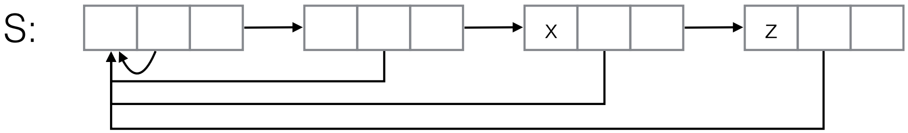
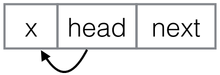


Make-Set($\mathbf{x}$):



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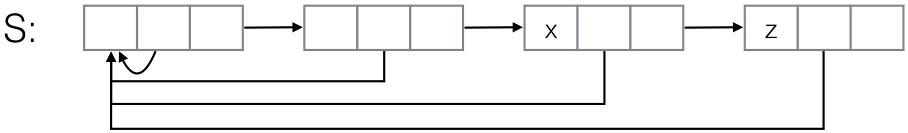
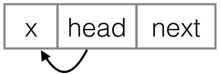
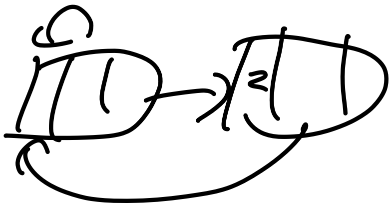


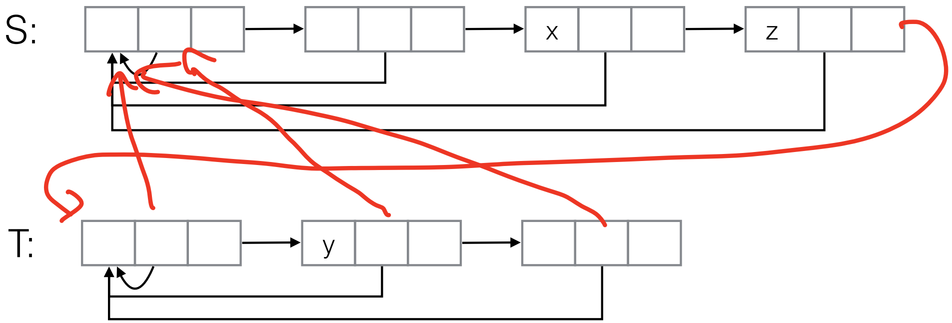
Make-Set(**x**):





Find(**x**): return **x** → head

Union($\mathbf{x}, \mathbf{y}$)



Obvious approach:

- Walk down **S** to final element **z** (starting from **x**)
- Set **z** → next = **y** → head
- Walk down **T**, set every elements head pointer to **x** → head

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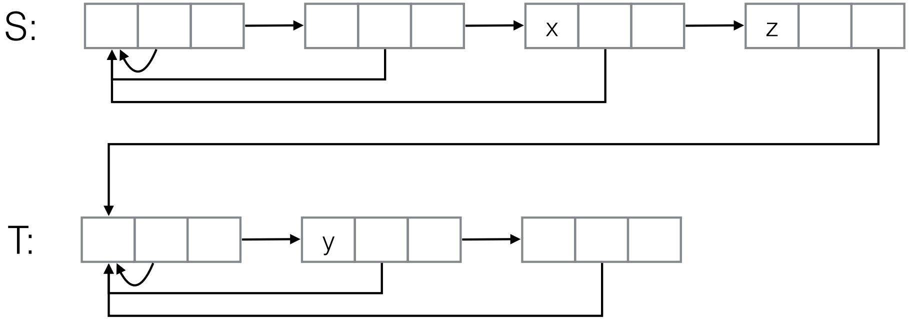

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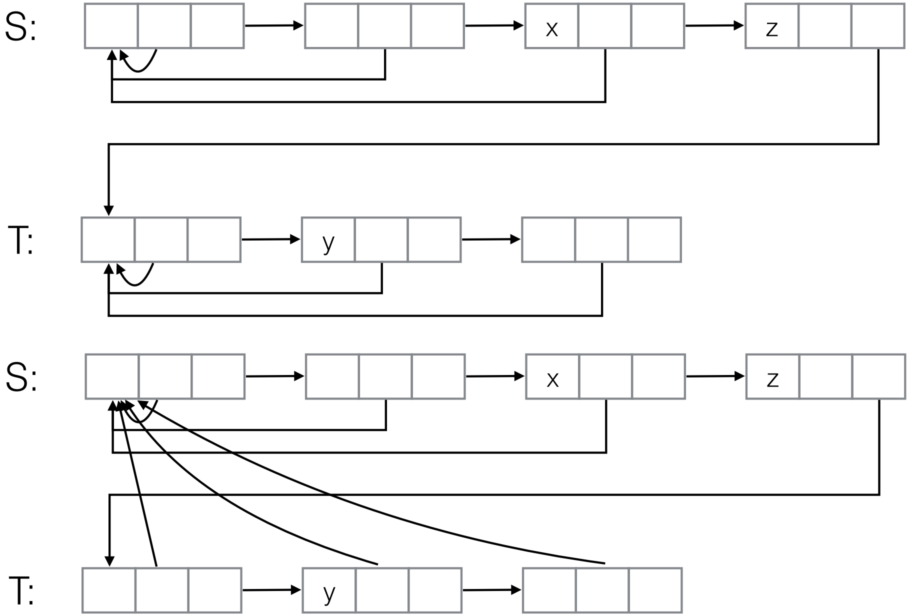
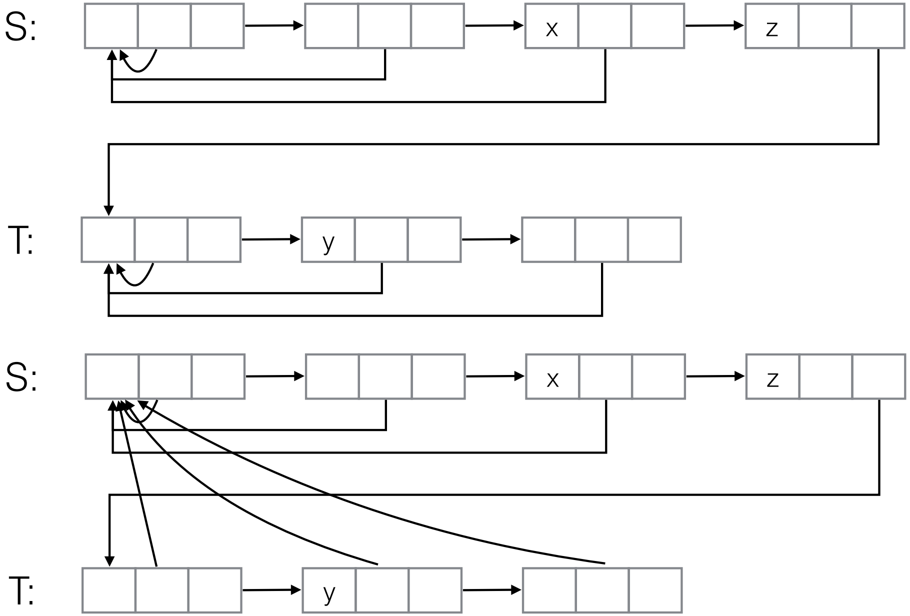
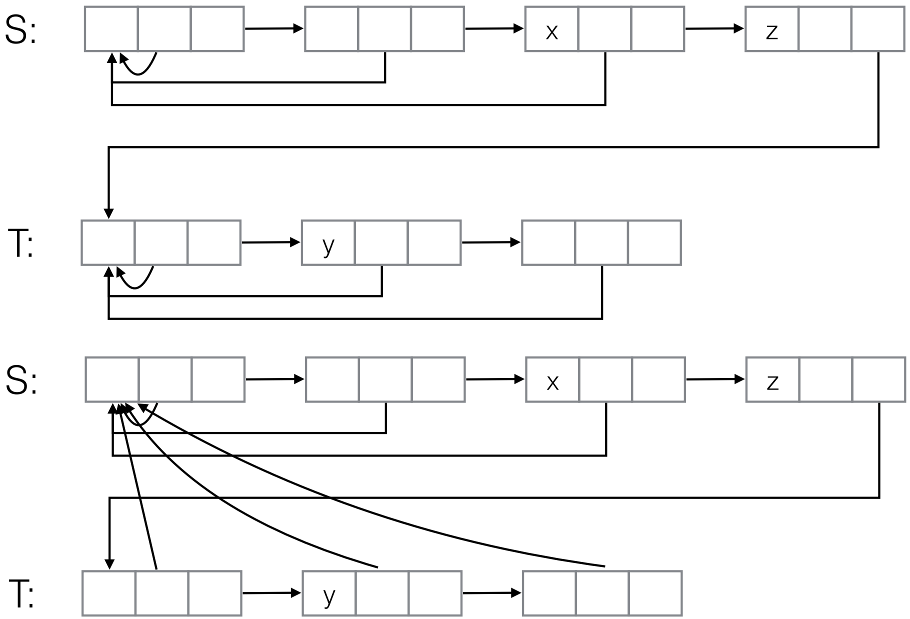


Union(**x**, **y**)



Running time:

Union(x, y)



Running time: $\mathbf{O}(|\mathbf{S}| + |\mathbf{T}|)$

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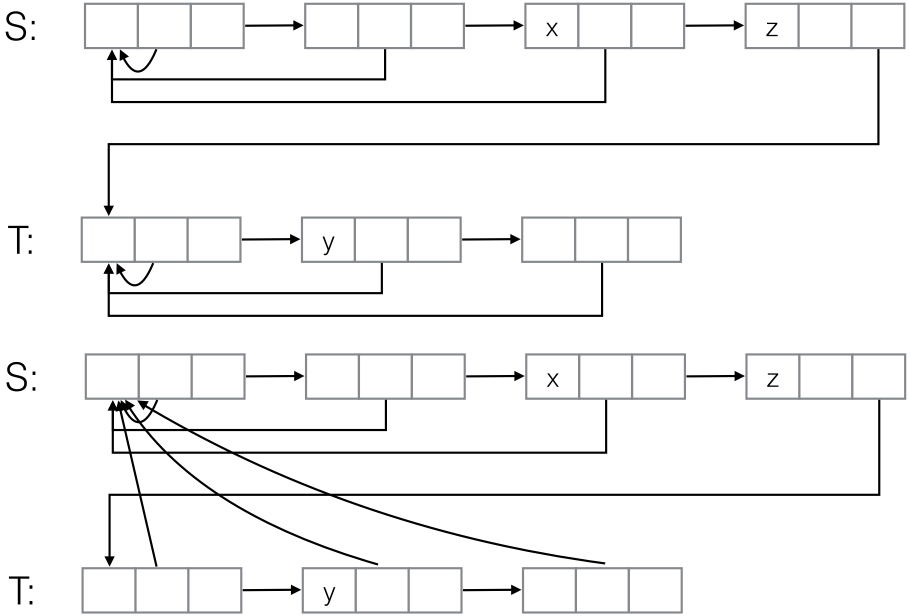
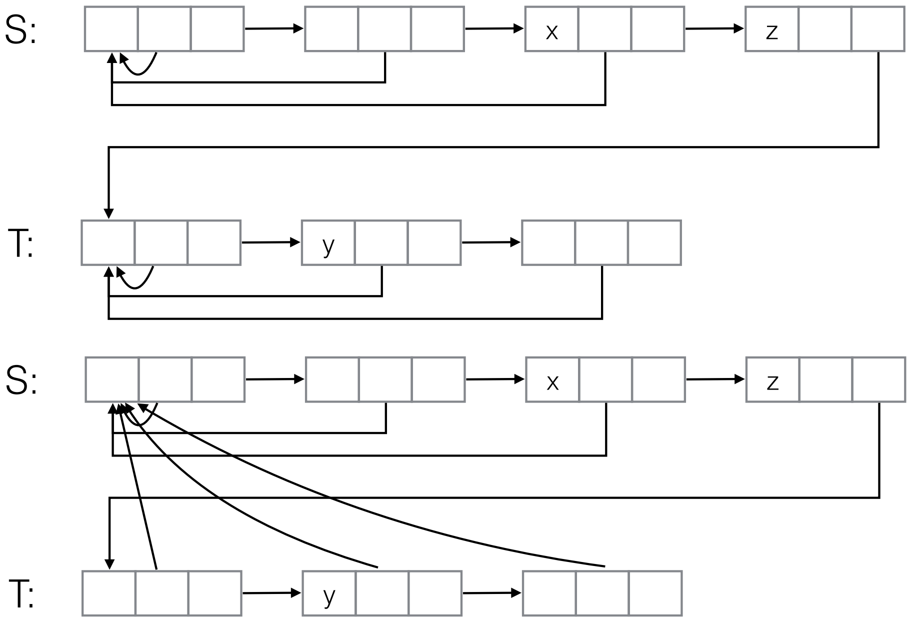


Running time: $\mathbf{O}(|\mathbf{S}| + |\mathbf{T}|)$

- $|\mathbf{S}|$ to walk down $\mathbf{S}$ to final element
- $|\mathbf{T}|$ to walk down $\mathbf{T}$ resetting head pointers

Union($\mathbf{x}, \mathbf{y}$)



Running time: $\mathbf{O(|S| + |T|)}$

- $|\mathbf{S}|$ to walk down $\mathbf{S}$ to final element
- $|\mathbf{T}|$ to walk down $\mathbf{T}$ resetting head pointers

Since $|\mathbf{S}|, |\mathbf{T}|$ could be $\mathbf{\Theta(n)}$, can only say $\mathbf{O(n)}$ for Unions

Improved Union($\mathbf{x}, \mathbf{y}$)

Observation: don't need to preserve ordering inside the Union!

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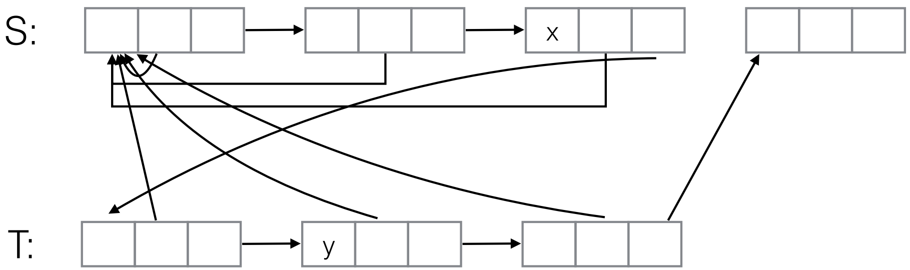

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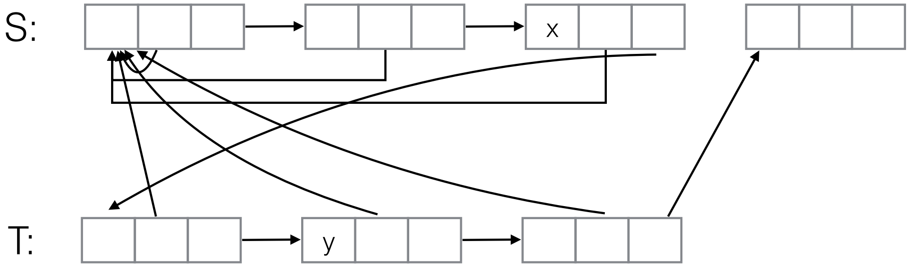

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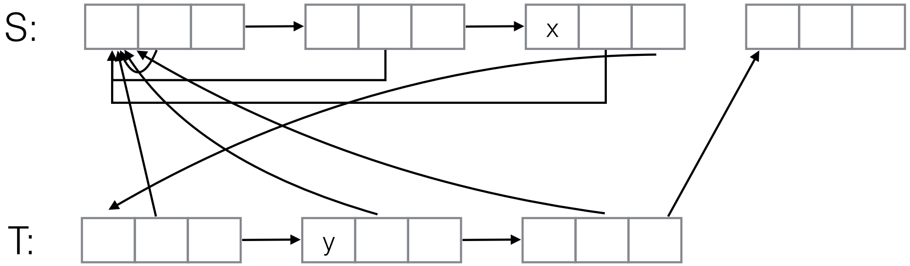

Running time: $\mathbf{O(|T|)}$

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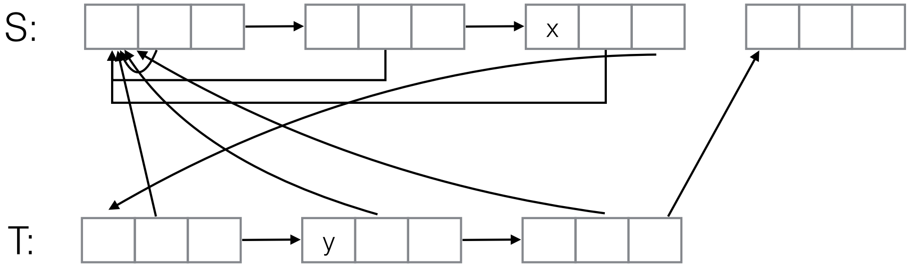

Running time: $\mathbf{O(|T|)}$

- Still can't say anything better than $\mathbf{O(n)}$

Even more improved Union($\mathbf{x}, \mathbf{y}$)

Observation: Why splice $\mathbf{T}$ into $\mathbf{S}$? Could also splice $\mathbf{S}$ into $\mathbf{T}$.

- Time $\mathbf{O}(|\mathbf{S}|)$

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- Time $\mathbf{O(|S|)}$

Splice smaller into bigger!

- Store size of set in head node.
- Splice smaller into bigger: time $\mathbf{O(\min(|S|, |T|))}$
- *Still* only $\mathbf{O(n)}$. But now can make stronger amortized guarantee!

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Theorem

The amortized cost of Find and Union is $\mathbf{O(1)}$*, and the amortized cost of Make-Set is* $\mathbf{O(\log n)}$.

Corollary

The total running time is $\mathbf{O(m + n \log n)}$.

Amortized Analysis of List Algorithm

Banking/accounting argument: bank for every element

- When an element is created (via Make-Set), add $\mathbf{\log n}$ tokens to its bank
- Find does not affect banks
- When doing Union, take token from bank of each element in smaller set.

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- Can only happen at most $\log n$ times. $\square$

Amortized Analysis of List Algorithm (cont'd)

Make-Set:

- True cost: $O(1)$
- Change in banks: $\log n$

$\implies$ Amortized cost: $O(1) + O(\log n) = O(\log n)$

Find:

- True cost: $O(1)$
- Change in banks: 0

$\implies$ Amortized cost: $O(1) + 0 = O(1)$

Union:

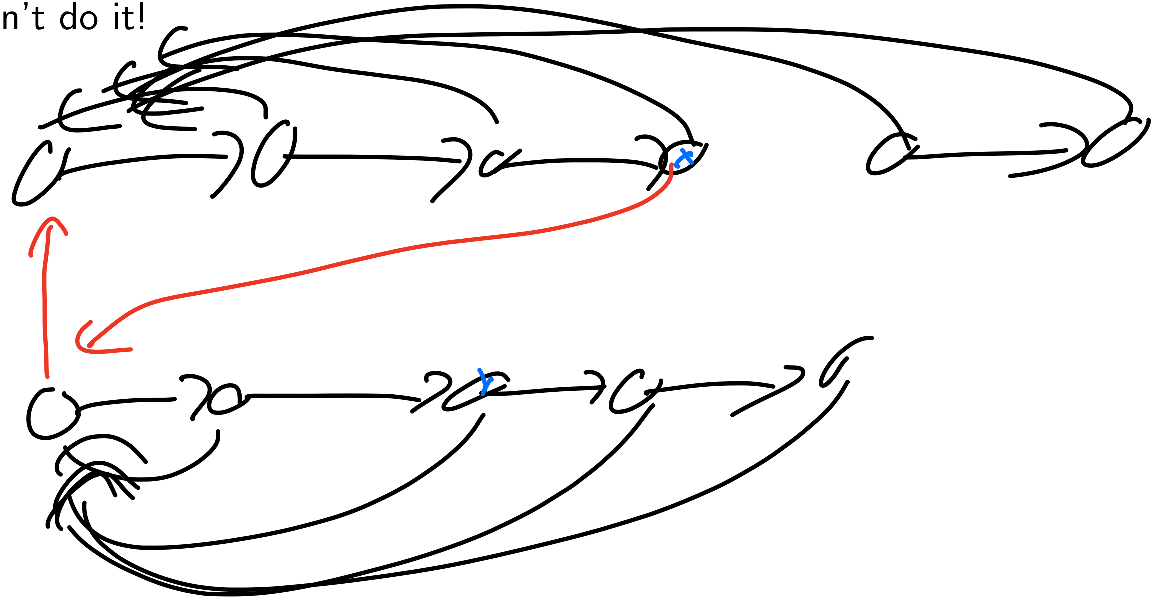
- True cost: $\min(|S|, |T|)$
- Change in banks: $-\min(|S|, |T|)$

$\implies$ Amortized cost: $\min(|S|, |T|) - \min(|S|, |T|) = 0 = O(1)$.

Even Better

Starting idea: want to make Unions faster, willing to make Finds a little slower.

- Slow part of Union: updating all head pointers in smaller list.
- Don't do it!



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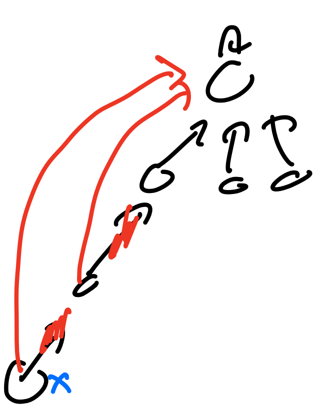

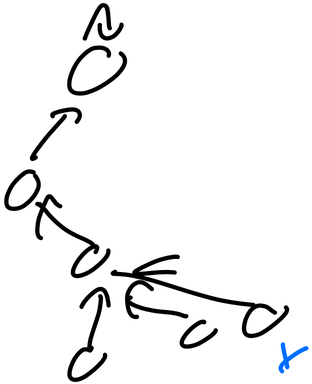

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Idea 2: *Union By Rank*

- Size of set was important for lists, less important for trees.
- Choose which set to splice into which by *rank* of trees (related to height)

Main Result

Theorem

When using Path Compression and Union By Rank, total time at most $\mathbf{O(m \log^* n)}$.

$\mathbf{\log^*}$: iterated $\mathbf{\log_2}$.

- $\mathbf{\log^* n}$ = # times apply $\mathbf{\log_2}$ until get to $\mathbf{1}$

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- Basically $\mathbf{\log^* n}$ always $\leq \mathbf{5}$.

Stronger theorem: total time at most $\mathbf{O(m \cdot \alpha(m, n))}$.

- $\mathbf{\alpha(m, n)}$: inverse Ackermann function. Grows even slower than $\mathbf{\log^*}$.
- See CLRS for details

Formal Procedures: Make-Set and Find

Make-Set($\mathbf{x}$): Set $\mathbf{x} \rightarrow \textbf{rank} = \mathbf{0}$ and $\mathbf{x} \rightarrow \textbf{parent} = \mathbf{x}$

- Running time: $\mathbf{O(1)}$.

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Find($\mathbf{x}$): Walk from $\mathbf{x}$ to root, and return root. Set parent pointers of $\mathbf{x}$ and all ancestors to root.

- If $\mathbf{x} \to \mathbf{parent} = \mathbf{x}$ then return $\mathbf{x}$
- $\mathbf{x} \to \mathbf{parent} = \mathbf{Find(x \to parent)}$
- Return $\mathbf{x} \to \mathbf{parent}$

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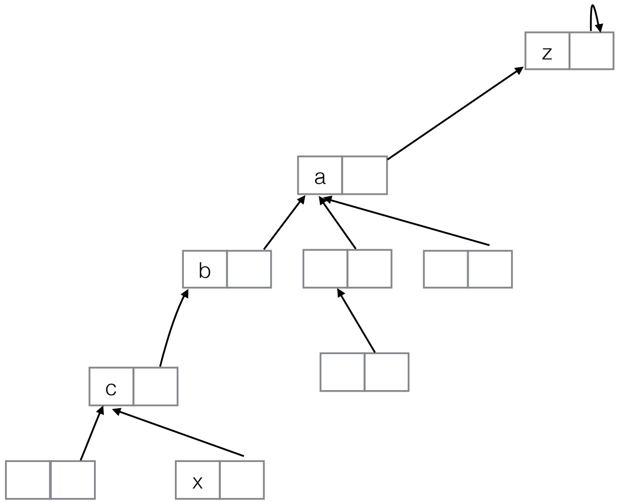
Running time of Find: depth of $\mathbf{x}$ (distance to root)

Find example



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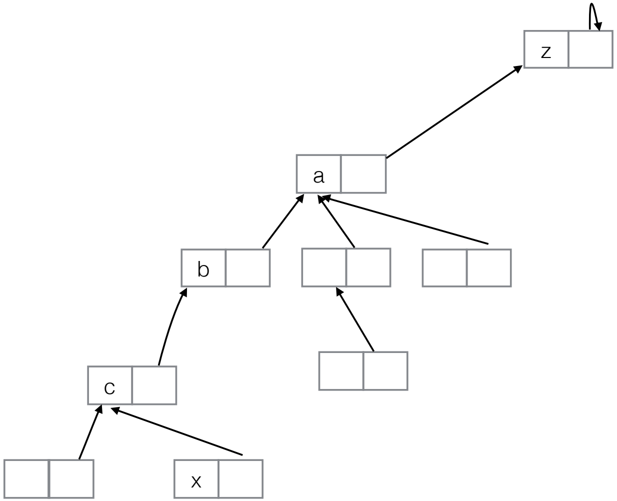

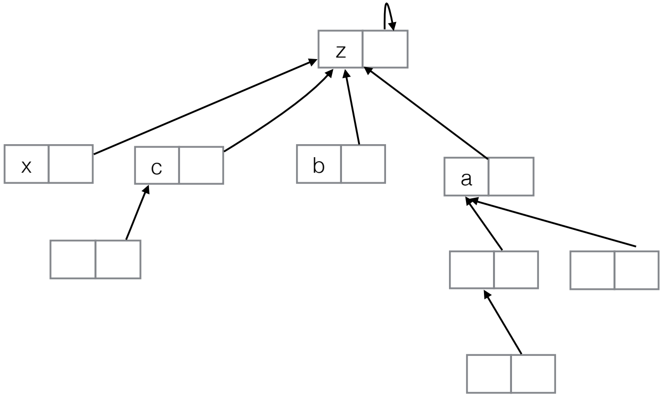

Formal Procedure: Union

Link($\mathbf{r_1}, \mathbf{r_2}$): Only applied to root nodes

- If $\mathbf{r_1} \rightarrow \mathbf{rank} > \mathbf{r_2} \rightarrow \mathbf{rank}$, set $\mathbf{r_2} \rightarrow \mathbf{parent} = \mathbf{r_1}$
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Union($\mathbf{x}, \mathbf{y}$): Link(Find($\mathbf{x}$), Find($\mathbf{y}$))

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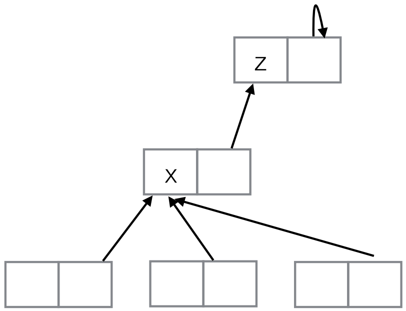
Running time of Link: $\mathbf{O(1)}$

Union($\mathbf{x, y}$): Link(Find($\mathbf{x}$), Find($\mathbf{y}$))

- Running time: depth($\mathbf{x}$) + depth($\mathbf{y}$)

Union example



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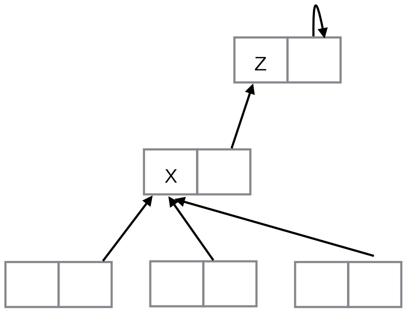

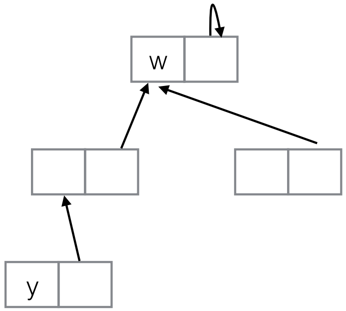

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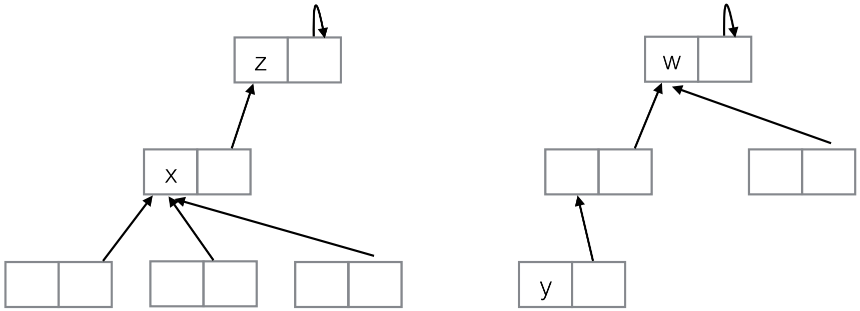
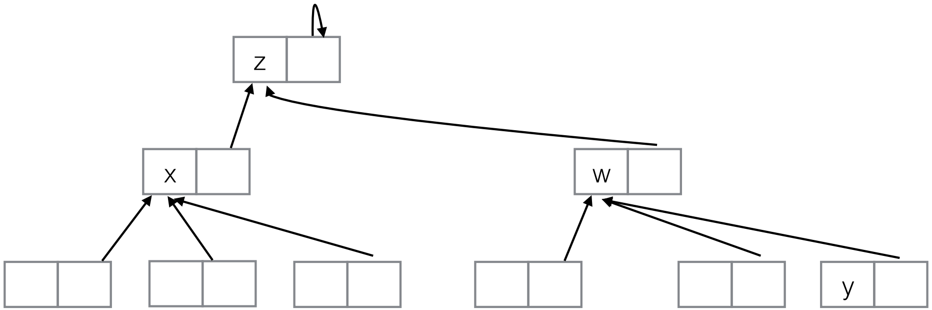
Union example



If **z → rank ≥ w → rank**



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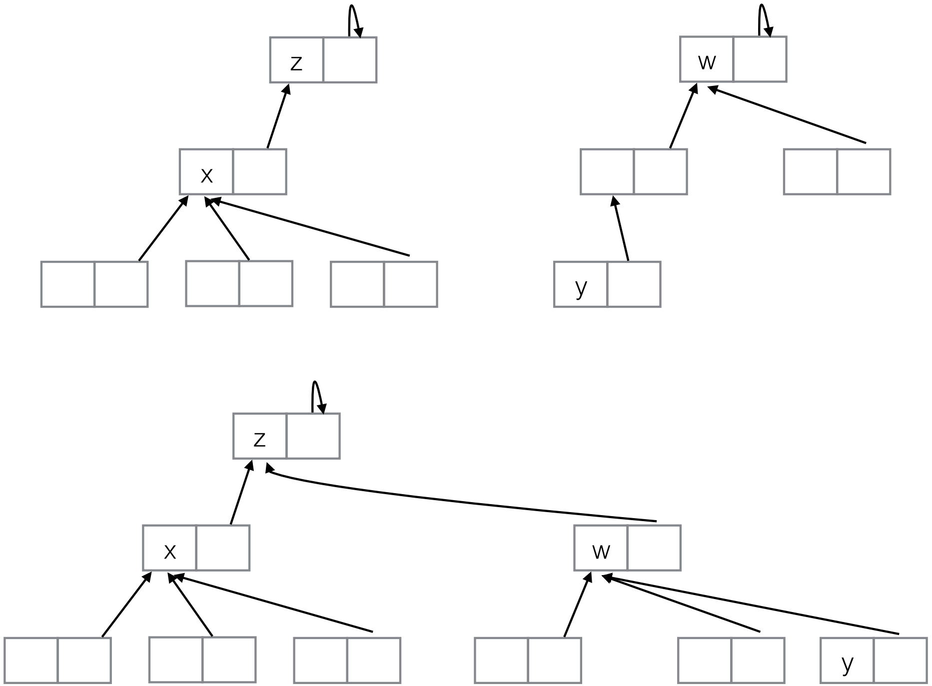

If $\mathbf{z \to rank \geq w \to rank}$

If $\mathbf{z \to rank = w \to rank}$,
then $\mathbf{(z \to rank)}++$

Properties of Ranks

1. If **x** not a root, then $(\mathbf{x} \to \mathbf{rank}) < (\mathbf{x} \to \mathbf{parent} \to \mathbf{rank})$
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$\Longrightarrow$ At least $\mathbf{2^{r-1}} + \mathbf{2^{r-1}} = \mathbf{2^r}$ nodes in combined tree. □

Nodes of rank $\mathbf{r}$

Lemma

There are at most $\mathbf{n/2^r}$ *nodes of rank at least* $\mathbf{r}$*.*

Proof.

Let $\mathbf{x}$ node of rank at least $\mathbf{r}$. Let $\mathbf{S_x}$ be descendants of $\mathbf{x}$ when it first got rank $\mathbf{r}$.
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So at most $\mathbf{2m}$ Finds, want to bound total # parent pointers followed.

- At most one parent pointer to root per Find $\implies$ at most $\mathbf{O(m)}$ parent pointers to roots.
- So only need to worry about parent pointers to non-roots.

Main Result II: Buckets

Put elements in buckets according to rank (only in analysis).

Notation: $\mathbf{2 \uparrow i}$ denote a tower of $\mathbf{i}$ $\mathbf{2}$'s

- $\mathbf{2 \uparrow 1 = 2, \quad 2 \uparrow 2 = 2^2 = 4, \quad 2 \uparrow 3 = 2^{2^2} = 2^4 = 16, \quad 2 \uparrow 4 = 2^{2^{2^2}} = 2^{16} = 65536}$
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From Lemma: at most $\mathbf{n/(2^{2\uparrow(i-1)}) = n/(2 \uparrow i)}$ elements in bucket $\mathbf{i}$.

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$$\sum_{\mathbf{x}} \alpha(\mathbf{x}) = \sum_{\mathbf{i=0}}^{\mathbf{O(\log^* n)}} \sum_{\mathbf{x\in B(i)}} \alpha(\mathbf{x}) \leq \sum_{\mathbf{i=0}}^{\mathbf{O(\log^* n)}} \sum_{\mathbf{x\in B(i)}} (\mathbf{2\uparrow i}) \leq \sum_{\mathbf{i=0}}^{\mathbf{O(\log^* n)}} \frac{\mathbf{n}}{\mathbf{2\uparrow i}}(\mathbf{2\uparrow i}) = \mathbf{O(n\log^* n)}$$

$$\leq \mathbf{O(m\log^* n)},$$