Lecture 8: Priority Queues and Heaps

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601.433/633 Introduction to Algorithms

Introduction

Priority Queues / Heaps: Like a queue/stack, but instead of FIFO/LIFO, by priority

- Insert($\mathbf{H}, \mathbf{x}$): insert element $\mathbf{x}$ into heap $\mathbf{H}$.
- Extract-Min($\mathbf{H}$): remove and return an element with smallest key
- Decrease-Key($\mathbf{H}, \mathbf{x}, \mathbf{k}$): decrease the key of $\mathbf{x}$ to $\mathbf{k}$.
- Meld($\mathbf{H_1}, \mathbf{H_2}$): replace heaps $\mathbf{H_1}$ and $\mathbf{H_2}$ with their union

Extra Operations:

- Find-Min($\mathbf{H}$): return the element with smallest key
- Delete($\mathbf{H}, \mathbf{x}$): delete element $\mathbf{x}$ from heap $\mathbf{H}$

Min-Heap, but can also do Max-Heap.

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Note: $\mathbf{x}$ is a *pointer* to an element. No way to lookup, so need a pointer to an element to change it.

Obvious Approaches

	Insert	Extract-Min	Decrease-Key	Meld
Linked List				

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No! Sorting lower bound. But maybe can make one $\mathbf{O(1)}$, other $\mathbf{O(\log n)}$?

Today and State of the Art

State of the art: *strict Fibonacci Heaps*.

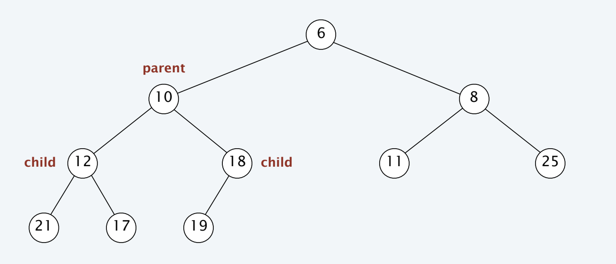
- Too complicated for this class, not practical. See CLRS 19 for Fibonacci Heaps.

Today: binary heaps (should be review), then binomial heaps

- Binomial heaps not quite as complicated as Fibonacci heaps, many of same core ideas

Binary Heaps

- Complete binary tree, except possibly at bottom level.
- Heap order: key of any node no larger than key of its children.



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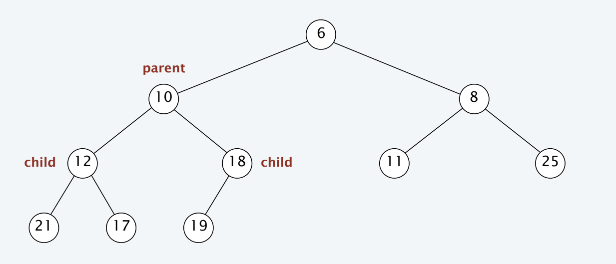


Properties:

- Since (almost) complete binary tree, depth $\Theta(\log n)$
- Min must be at root

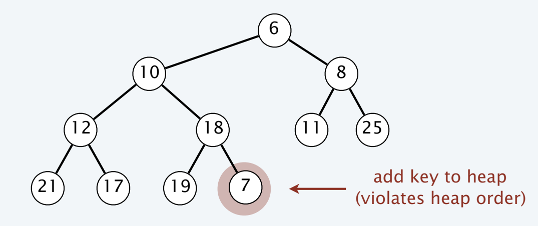
Representation:

- Pointers to root and rightmost leaf
- Every node has pointers to parent and children

Insert($\mathbf{H}, \mathbf{x}$)

Preserve heap *structure*: insert **x** into next open spot (bottom right, or left of new level if bottom level full)

- Might violate heap *order*!



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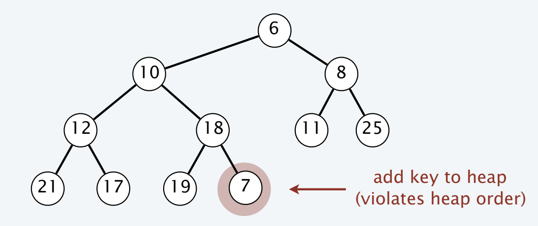
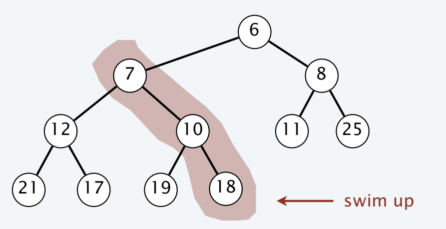


"Swim up": as long as **x** smaller than its parent, swap with parent



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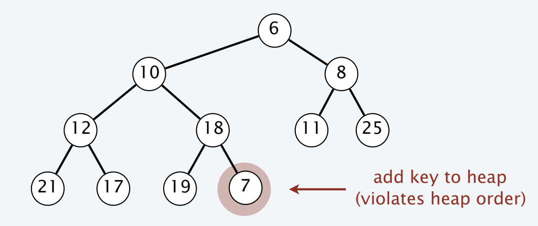

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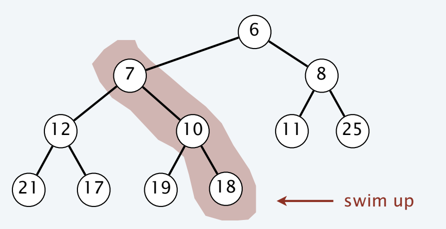

Running time: $\mathbf{O(\log n)}$ worst case (also amortized) via depth

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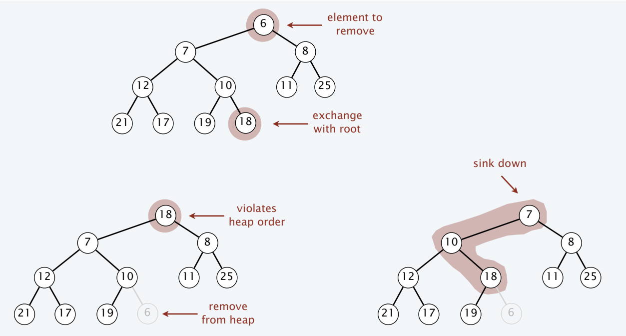

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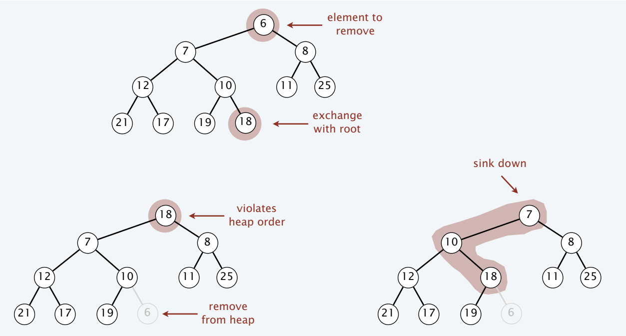

Running time: $\mathbf{O(\log n)}$ worst case (via depth). Amortized: $\mathbf{O(1)}$ (not obvious)

Decrease-Key($\mathbf{H}, \mathbf{x}, \mathbf{k}$)

Decrease key of $\mathbf{x}$ to $\mathbf{k}$, "swim up" until heap order restored.

Running time: $\mathbf{O(\log n)}$ (depth)

Meld($\mathbf{H_1}, \mathbf{H_2}$)

Assume both heaps have size $\mathbf{n}$.

- Obvious approach: insert each element of $\mathbf{H_2}$ into $\mathbf{H_1}$. Time: $\mathbf{O(n \log n)}$

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  - At most $\mathbf{n/2^h}$ nodes at height $\mathbf{h}$

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$$\sum_{h=0}^{\log n} h\left(\frac{n}{2^h}\right) = n\sum_{h=0}^{\log n} \frac{h}{2^h} \leq O(n)$$

Amortized Extract-Min

Weights: $\mathbf{w(x)}$ = depth of $\mathbf{x}$

- Root has weight $\mathbf{0}$, its children have weight $\mathbf{1}$, etc.

Potential: $\mathbf{\Phi(H)} = \sum_x \mathbf{w(x)}$

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- True cost: height $\mathbf{h = \Theta(\log n)}$ of tree, plus $\mathbf{O(1)}$ (for initial swap).
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Uses Inserts to "pay for" Extract-Mins.

Improvements

Downsides of binary heaps:

- Do at least as many Inserts as Extract-Mins! Want $\mathbf{O(1)}$ Insert, $\mathbf{O(\log n)}$ Extract-Min
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Binomial Heaps:

- Get Insert down to $\mathbf{O(1)}$ (amortized)
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- Downside: $\mathbf{O(\log n)}$ Extract-Min, $\mathbf{O(\log n)}$ Find-Min

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Fibonacci Heaps:

- Everything $\mathbf{O(1)}$ (amortized) except $\mathbf{O(\log n)}$ Extract-Min (amortized)

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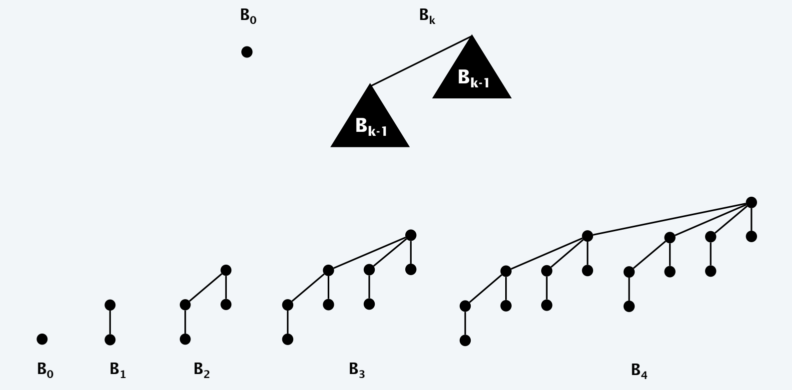
- $\mathbf{B_0}$ = single node.
- $\mathbf{B_k}$ = one $\mathbf{B_{k-1}}$ linked to another $\mathbf{B_{k-1}}$.

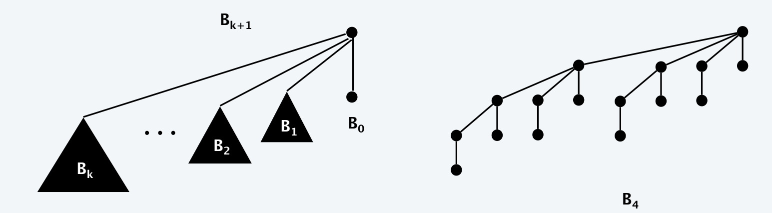


Structure Lemma

Lemma

The order $\mathbf{k}$ *binomial tree* $\mathbf{B_k}$ *has the following properties:*

1. *Its height is* $\mathbf{k}$.
2. *It has* $\mathbf{2^k}$ *nodes*
3. *The degree of the root is* $\mathbf{k}$
4. *If we delete the root, we get* $\mathbf{k}$ *binomial trees* $\mathbf{B_{k-1}, \dots, B_0}$.

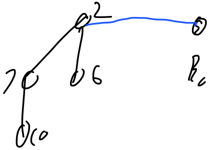


Binomial Heap

Definition

A *binomial heap* is a collection of binomial trees so that each tree is heap ordered, and there is exactly **0** or **1** tree of order **k** for each integer **k**.

Keep roots of trees in linked list, from smallest order (not key!) to largest



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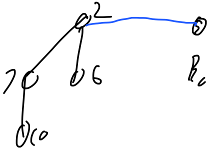

With $\mathbf{n}$ items, no choices about which binomial trees exist in heap!

- Write $\mathbf{n}$ in binary: $\mathbf{b_a b_{a-1} \ldots b_1 b_0}$.
- Tree $\mathbf{B_k}$ exists if and only if $\mathbf{b_k = 1}$

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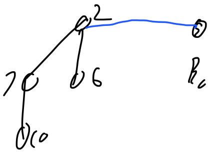

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$\Longrightarrow$ at most $\mathbf{\log n}$ trees, and by lemma each has height $\leq \mathbf{\log n}$

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Analyze all operations both worst-case and amortized.

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- Worst-case: $\mathbf{O(\log n)}$
- Amortized: doesn't change potential, also $\mathbf{O(\log n)}$.

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Key operation: we'll use Meld to do Insert and Extract-Min

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Warmup: $\mathbf{H_1}, \mathbf{H_2}$ both single trees of same order $\mathbf{k}$.

- Union has size $\mathbf{2^k + 2^k = 2^{k+1}}$: just a single $\mathbf{B_{k+1}}$
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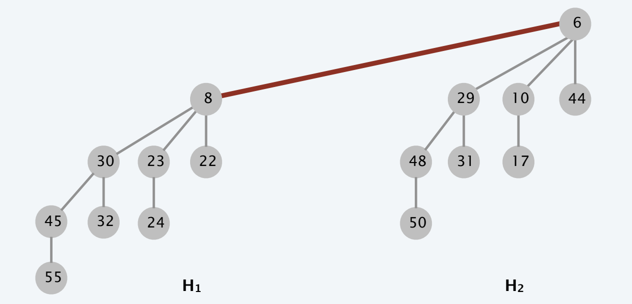

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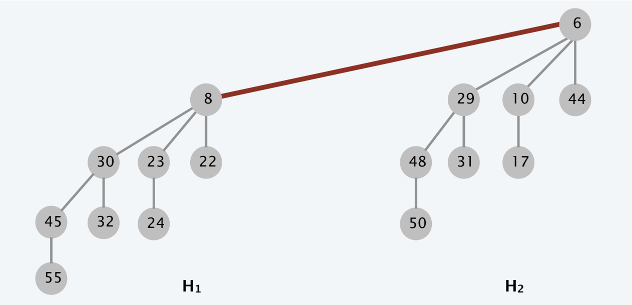
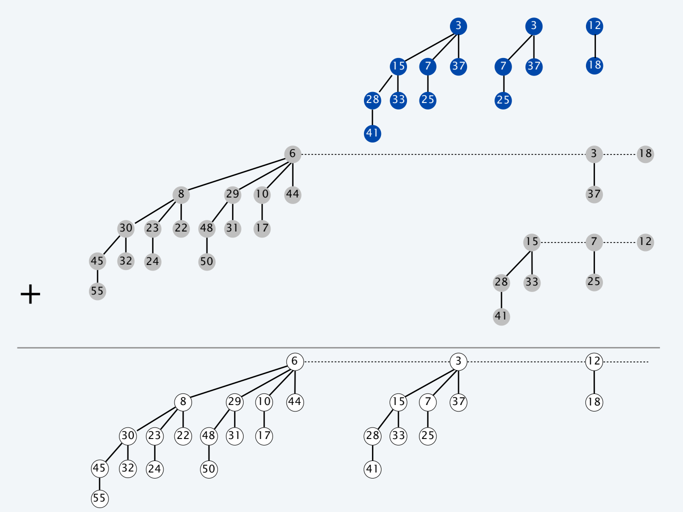


Link of two trees.

- Worst-case time: $\mathbf{O(1)}$ (create a single link). Normalize: call $\mathbf{1}$
- $\mathbf{\Delta\Phi}$: two trees to one: $\mathbf{-1}$
- Amortized cost: $\mathbf{1 - 1 = 0 = O(1)}$.

Meld($\mathbf{H_1}, \mathbf{H_2}$): General Case

(Almost) just like binary addition!



Meld($\mathbf{H_1}, \mathbf{H_2}$): Analysis

Easy to prove correct (exercise for home).

Running time:

- Worst case: $\mathbf{O(1)}$ per "order" $\mathbf{k} \implies \leq \mathbf{O(\log n)}$
- Amortized: Potential does not go up, but could stay the same $\implies$ $\mathbf{O(\log n)}$ amortized

Insert($\mathbf{H}, \mathbf{x}$)

Use Meld:

- Create new heap $\mathbf{H}'$ with one $\mathbf{B_0}$ consisting of just $\mathbf{x}$
- Meld($\mathbf{H}, \mathbf{H}'$)

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Running Time:

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- Amortized:
  - Like incrementing a binary counter!
  - If we link $\mathbf{k}$ trees, potential goes down by $\mathbf{k - 1}$
  - Cost $=$ # links plus $\mathbf{1}$ (for making new heap)
  - Amortized cost $= \mathbf{k + 1 + \Delta\Phi = k + 1 - (k - 1) = 2 = O(1)}$

Extract-Min($\mathbf{H}$)

Use Meld again!

- $\mathbf{O(\log n)}$ to Find-Min: one of the roots.
- Delete and return this root: tree turns into a new heap!
- Meld with original heap (minus the tree)

Correctness: Obvious

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Correctness: Obvious

Running Time:

- Worst-Case: $\mathbf{O(\log n)}$ from creating new heap, Meld
- Amortized:
  - Potential can go up! But by at most $\mathbf{\log n}$
  - Amortized time at most $\mathbf{O(\log n) + \log n = O(\log n)}$